CORE STABILITY OF GAMES ON CONVEX GEOMETRIES

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A cooperative game on a convex geometry is introduced by Bilbao [1] as a model of cooperation structures in which only certain coalitions are allowed to form. Let $N = \{1, 2, \dots, n\}$ be a finite set. A *convex geometry* $\mathcal{L}$ on N is a family of subsets of N with the properties: $\emptyset \in \mathcal{L}$; $N \in \mathcal{L}$; $S, T \in \mathcal{L}$ imply $S \cap T \in \mathcal{L}$; $S \in \mathcal{L} \setminus \{N\}$ implies $S \cup \{i\} \in \mathcal{L}$ for some $i \in N \setminus S$. Note that a convex geometry was first introduced by Edelman–Jamison [3] as a combinatorial abstraction of "convexity." A *cooperative game* on a convex geometry $\mathcal{L}$ on N is a function $v : \mathcal{L} \to \mathbb{R}$ with $v(\emptyset) = 0$, where $\mathbb{R}$ is the set of real numbers. The *players* are the elements of N and the *coalitions* are the members of $\mathcal{L}$. For short, we denote $\sum_{i \in S} x_i$ by $x(S)$.

Let $\mathcal{L}$ be a convex geometry on the players N, and $v : \mathcal{L} \to \mathbb{R}$ be a game on $\mathcal{L}$. The set of *imputations* is $I(\mathcal{L}, v) = \{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } \{i\} \in \mathcal{L}\}$, and the *core* of the game v on $\mathcal{L}$ is $\mathrm{Core}(\mathcal{L}, v) = \{x \in I(\mathcal{L}, v) : x(S) \geq v(S) \text{ for all } S \in \mathcal{L}\}$. Let $x, y \in I(\mathcal{L}, v)$ and $S \in \mathcal{L}$. We say that y *dominates* x *via* S if $y(S) \leq v(S)$ and $y_i > x_i$ for all $i \in S$, which is denote by $x \prec_S y$. A *stable set* of a game v on a convex geometry $\mathcal{L}$ is a set $K \subseteq I(\mathcal{L}, v)$ of imputations satisfying the following two conditions: for all $y \in I(\mathcal{L}, v) \setminus K$, there exist $x \in K$ and $S \in \mathcal{L}$ such that $y \prec_S x$; for all $x, y \in K$ and $S \in \mathcal{L}$, it holds that $x \not\prec_S y$ and $y \not\prec_S x$.

A well-known fact on stable sets is that, if $\mathcal{L} = 2^N$ and a game v is convex (or supermodular), then the core is a unique stable set [4], where a game $v : 2^N \to \mathbb{R}$ is *convex* if we have $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \in 2^N$. We extend this result to a general convex geometry, and show that the core is a unique stable set of the game if a game on a convex geometry is quasi-convex (or quasi-supermodular), namely, $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, which may be first defined in [2].

References

[1] J.M. Bilbao, *Cooperative games on combinatorial structures*, Kluwer Academic Publishers, Boston Dordrecht London, 2000.

[2] J.M. Bilbao, E. Lebrón and N. Jiménez, The core of games on convex geometries, *European Journal of Operational Research* **119**, 1999, 365–372.

[3] P. H. Edelman and R. E. Jamison, The theory of convex geometries, *Geometriae Dedicata* **19**, 1985, 247–270.

[4] L.S. Shapley, Cores of convex games, *International Journal of Game Theory* **1**, 1971, 11–26.