Greedy edge-disjoint paths in complete graphs

Yoshio Okamoto

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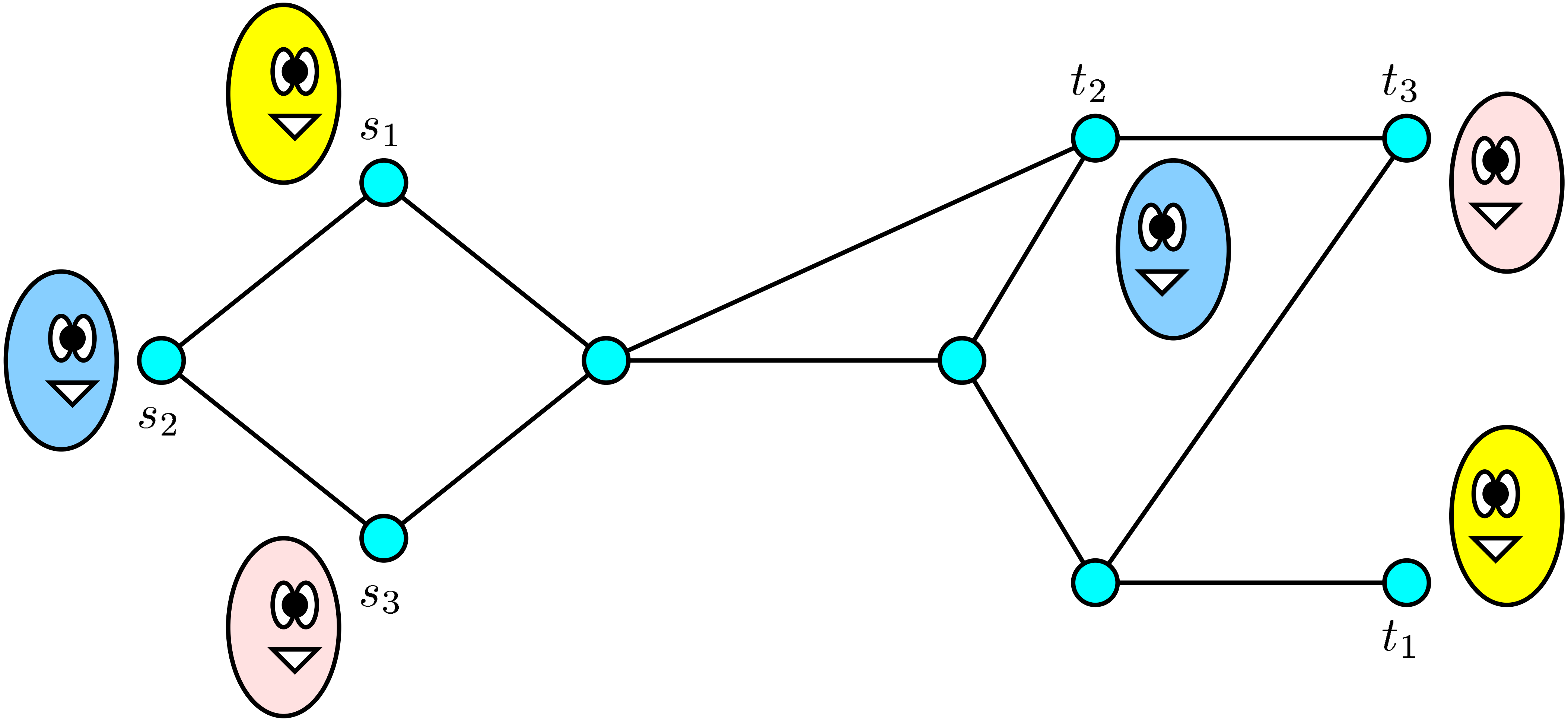
Supervised by Prof. Thomas Erlebach

◀◀ Contents ▶▶

- Preliminaries
  - Maximum edge-disjoint path problem (MEDP)
  - Approximation algorithm, approximation ratio
  - Greedy algorithms for MEDP
- Contributions for MEDP in complete graphs
  - Improved upper bound of the approximation ratio
  - Lower bound of the approximation ratio

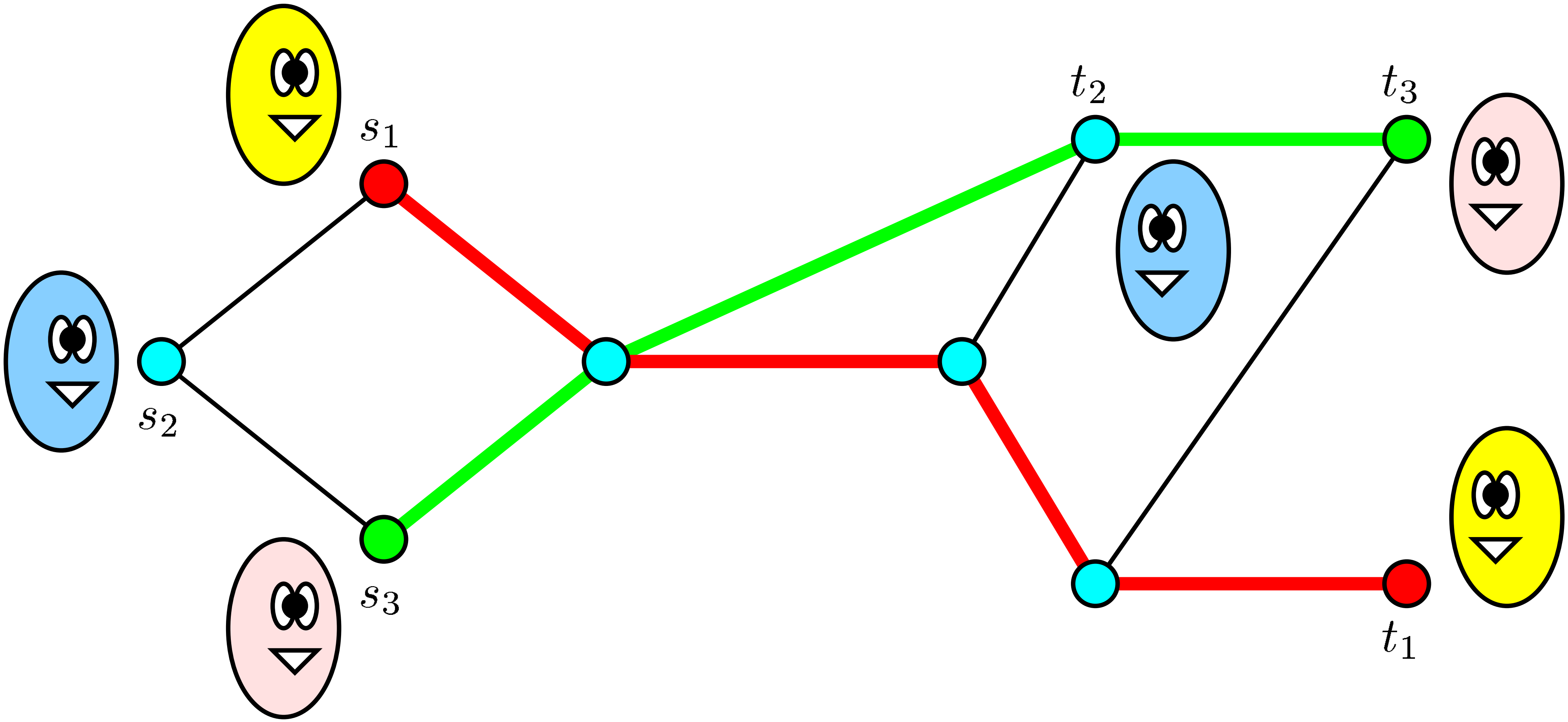
◀◀ A communication network ▶▶

$\mathcal{R} = \{\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}\}$



◀◀ A communication network ▶▶

$\mathcal{A} = \{\{s_1, t_1\}, \{s_3, t_3\}\}$



◀◀ Maximum edge-disjoint path problem ▶▶

Input:

$G = (V, E)$ (undirected) graph

$\mathcal{R} = \left\{ \{s_i, t_i\} \in \binom{V}{2} : i = 1, \ldots, k \right\}$ the multiset of requests

Feasible solution:

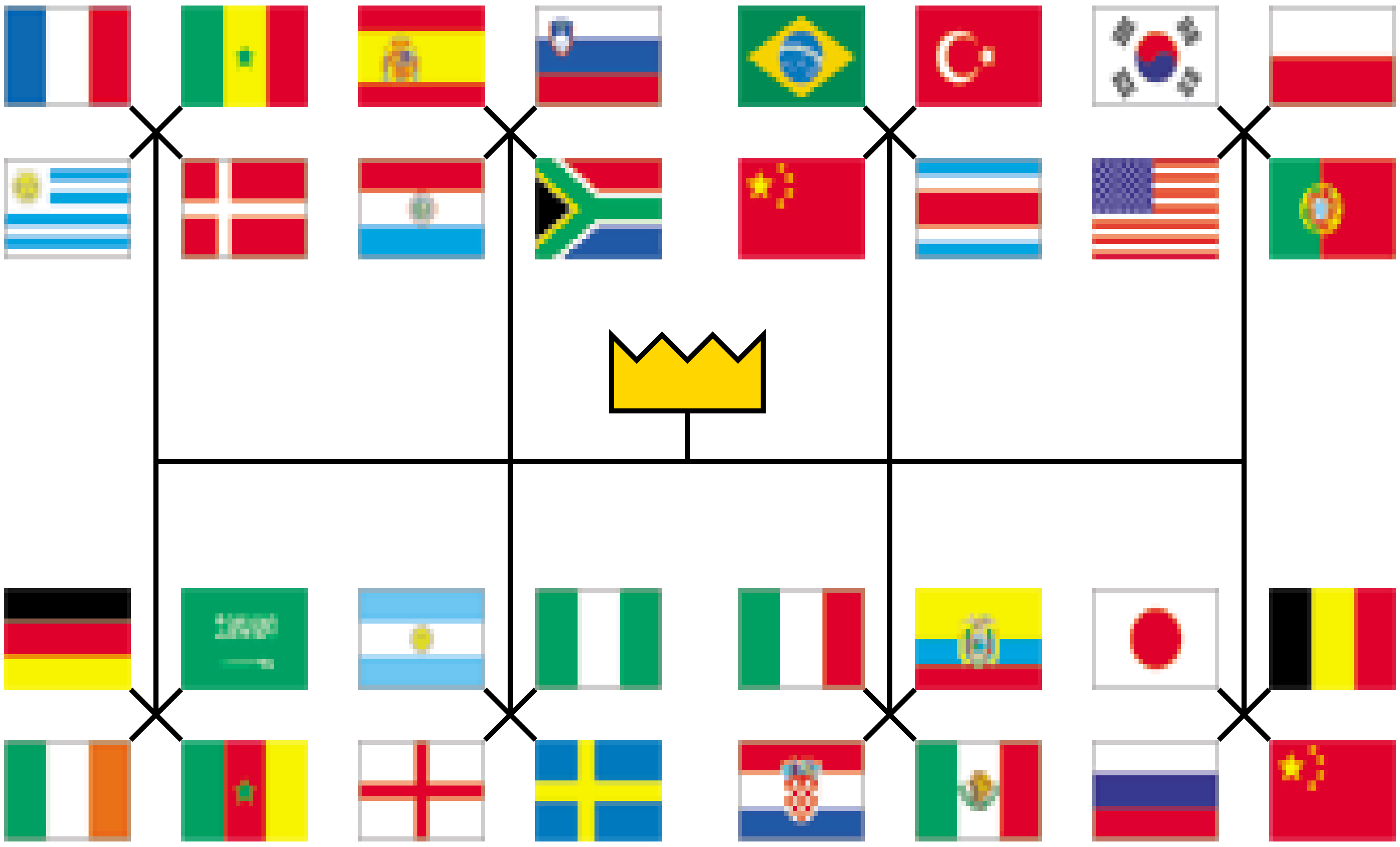
$\mathcal{A} \subseteq \mathcal{R}$ accepted requests

$\mathcal{P}$ set of edge-disjoint paths in G

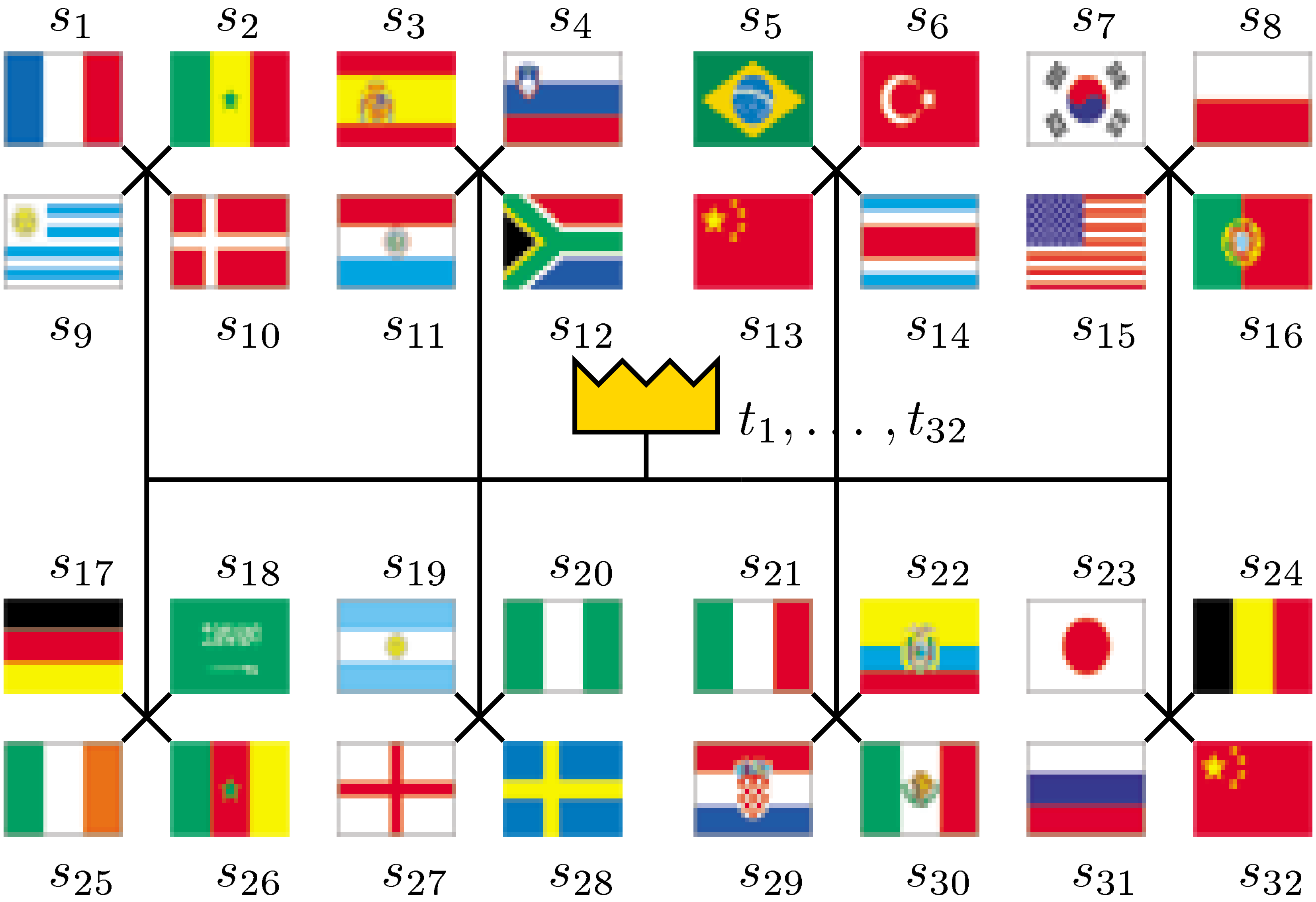
each request $i \in \mathcal{A}$ is connected by a path $\in \mathcal{P}$

Goal: maximize $|\mathcal{A}|$

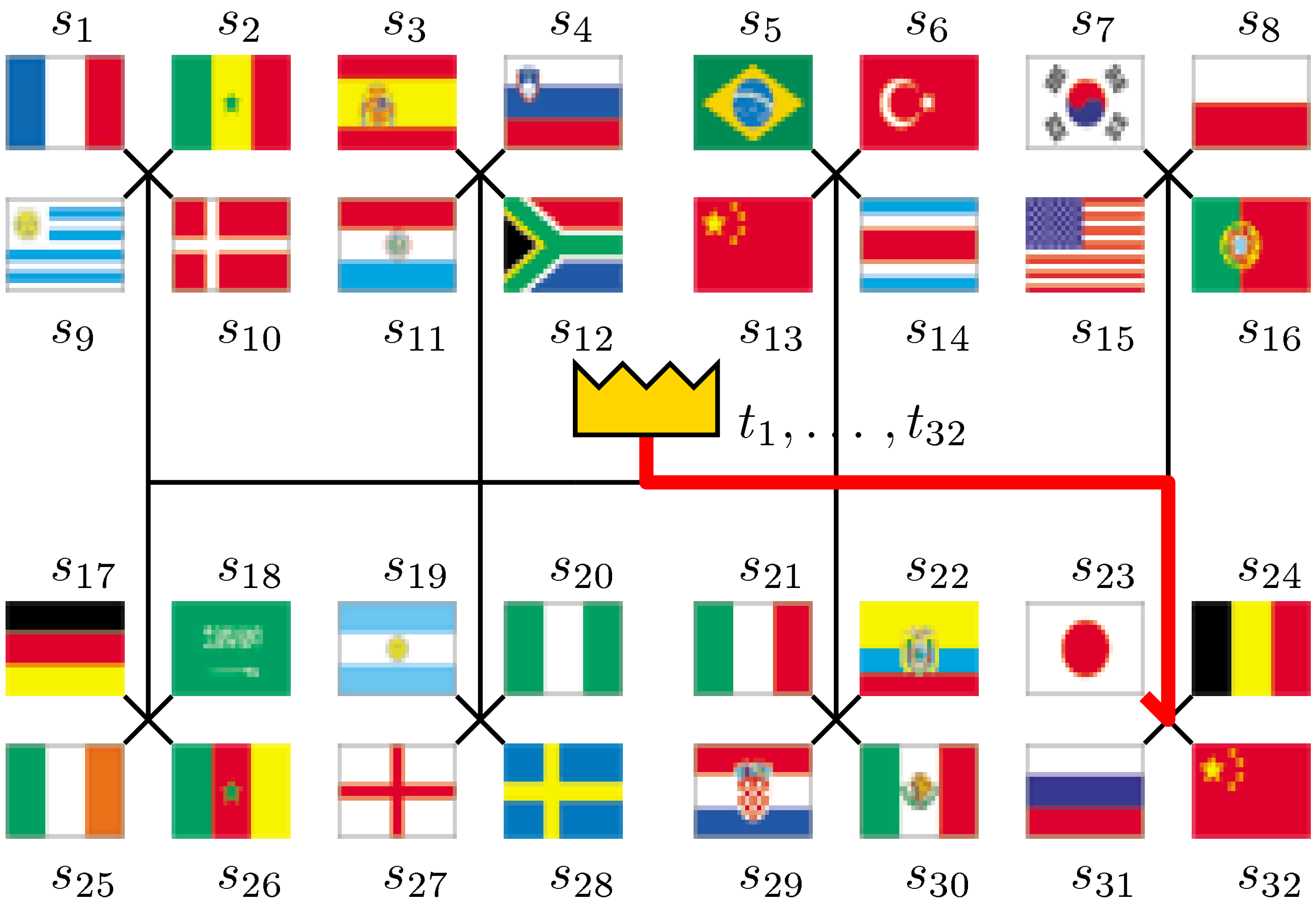
◀◀ FIFA World Cup ▶▶



◀◀ FIFA World Cup ▶▶



◀◀ FIFA World Cup: a possible result ▶▶



◀◀ Hardness of MEDP ▶▶

It is known that MEDP is NP-hard,
even if input graphs are restricted to complete graphs.

(Erlebach–Vukadinović, 2001)

How to cope with the hardness

- Exact algorithm, but not efficient
  - enumeration, branch-and-bound, ...
- Efficient algorithm, but not exact
  - genetic algorithm, simulated annealing, ...
  - approximation algorithm

◀◀ Approximation algorithm ▶▶

$\boxed{\text{Def.}}$ (approximation algorithm, approximation ratio)

An algorithm for a maximization problem is called a ρ-approximation algorithm if for any instance

- it runs in polynomial time,
- it outputs a feasible solution with the value Approx satisfying
  $$\frac{1}{\rho} \times \mathrm{Opt} \leq \mathrm{Approx} \leq \mathrm{Opt}.$$
  (Opt is the optimal value of the instance.)

$\rho \geq 1$ is called the approximation ratio of this algorithm.
(the smaller ρ, the better algorithm.)

◀◀ Approx. alg.s for MEDP in complete graphs ▶▶

Erlebach–Vukadinović (2001) showed that
some approximation algorithms achieve constant approximation ratios.

Algorithm	Approx. Ratio
The shortest-path-first greedy algorithm (SGA)	54
The bounded-length greedy algorithm (BGA)	54
The tripartition algorithm	27

Q.

These values are very large.

What are the ‘real’ approximation ratios?

◀◀ Contributions ▶▶

For MEDP in complete graphs,

1. A result by Kolman–Scheideler (2002) implies that
   BGA is a 17-approximation algorithm.

2. Indeed,
   SGA and BGA are 9-approximation algorithms.

3. There is an instance which shows that
   SGA cannot be better than a 2-approximation algorithm.

Additional contributions:

- There is an instance for arbitrary graphs which shows that
  SGA cannot be better than an $O(\sqrt{m})$-approximation algorithm.
- SGA can be interpreted via the primal-dual framework.

◀◀ Bounded-length greedy algorithm (BGA) ▶▶

$\mathcal{R} = \{r_1, r_2, \ldots, r_k\}$

L: an additional parameter

For $i = 1$ to k do

  Find a shortest path π_i connecting the req. r_i;

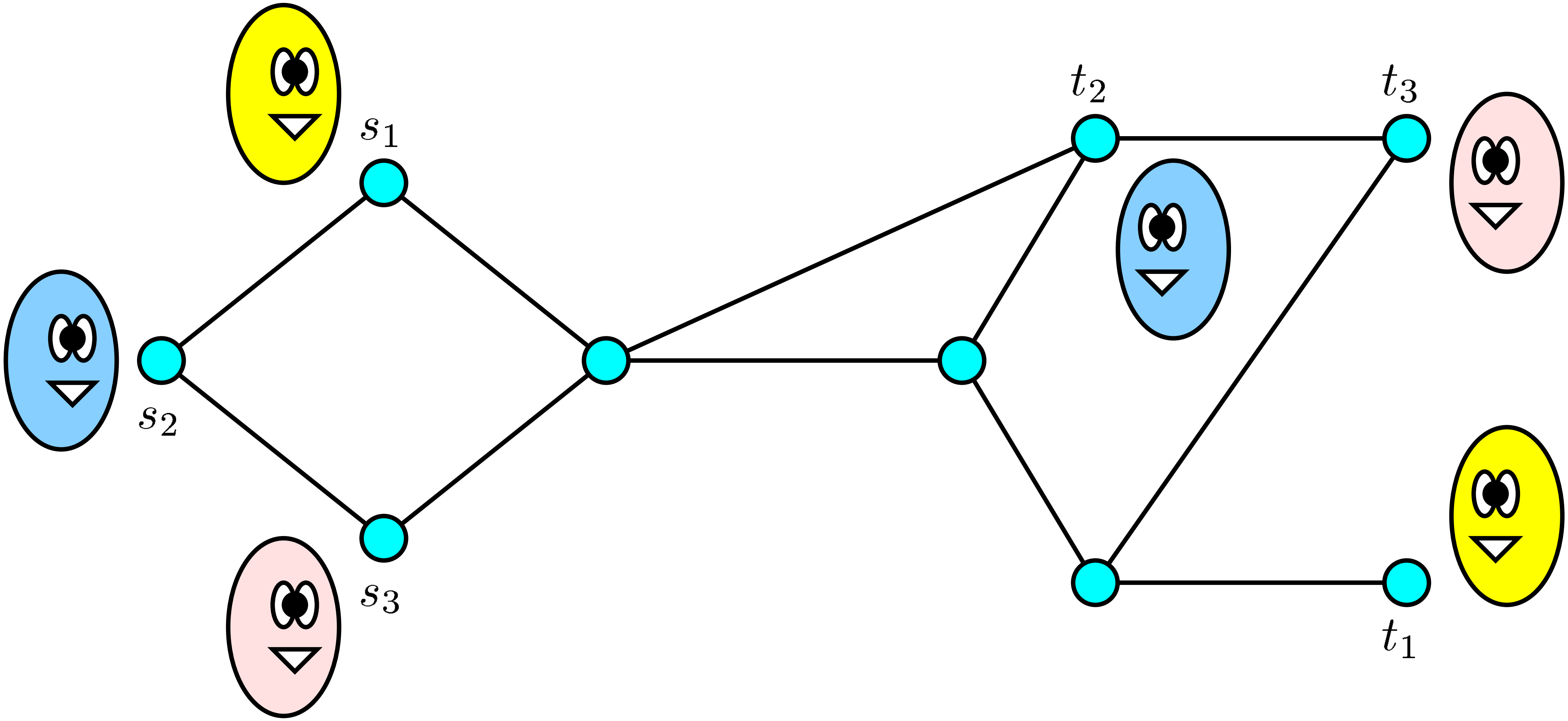
  If the length of $\pi_i \leq L$ then

    Accept r_i and remove all the edges of π_i from G.

(first introduced by Kleinberg (1996))

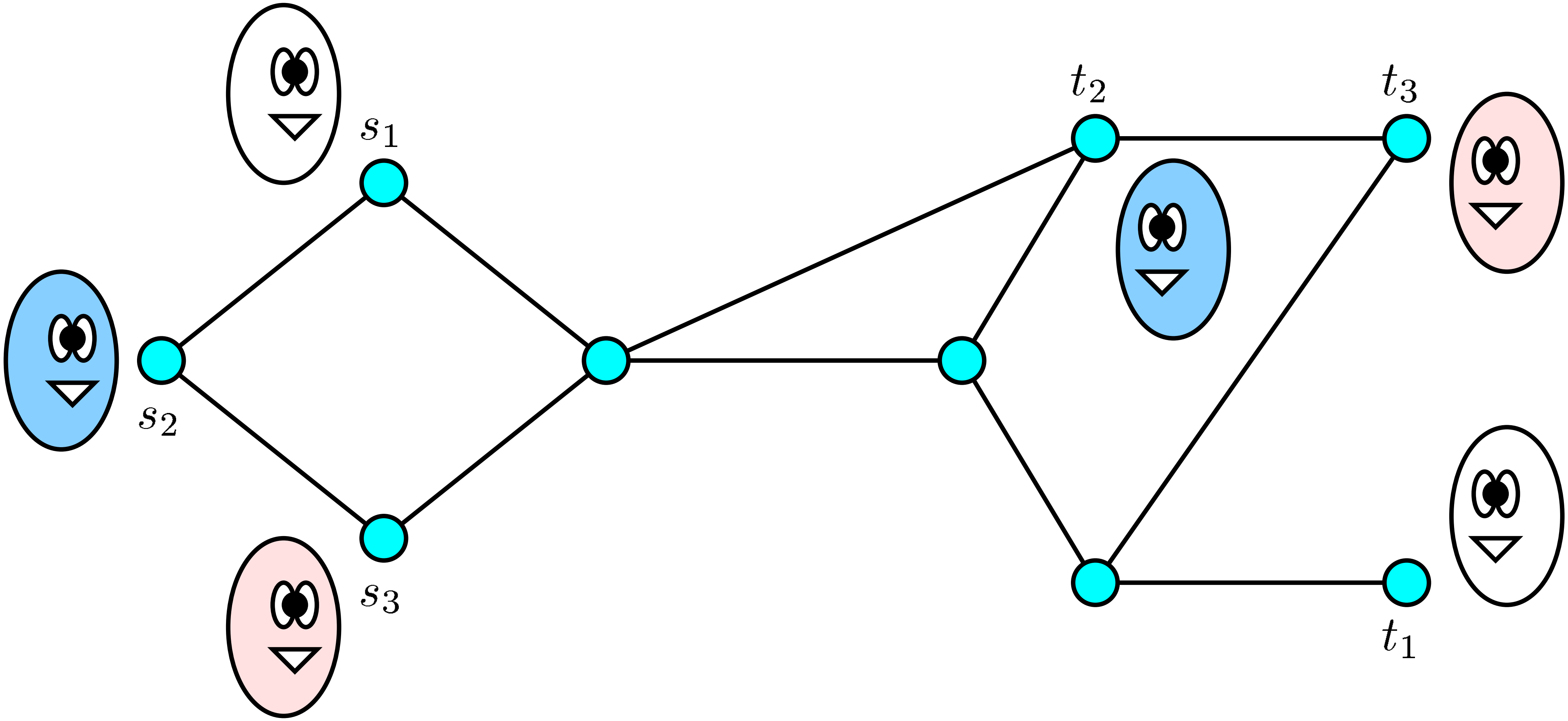
Example: BGA

$L = 3$



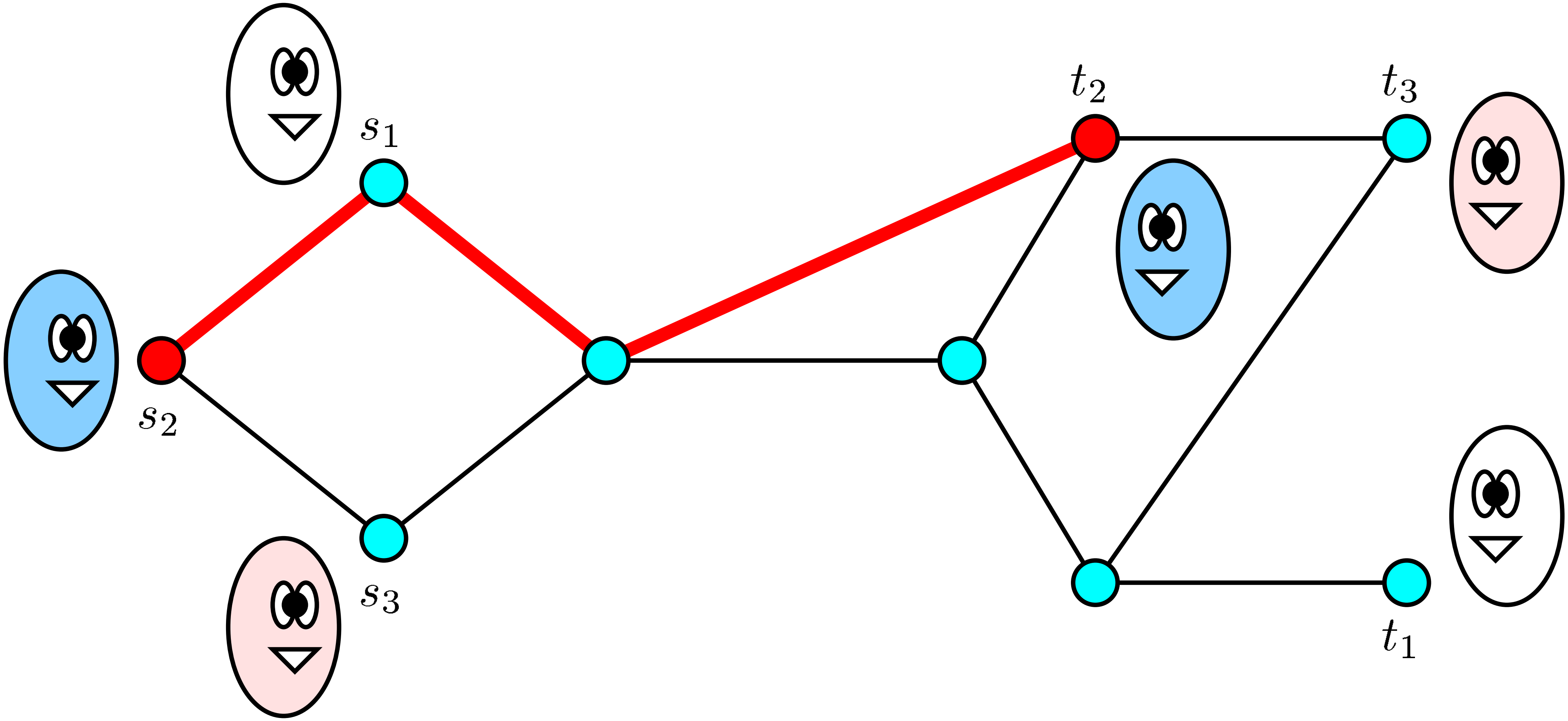
◀◀ Example: BGA ▶▶

$L = 3$



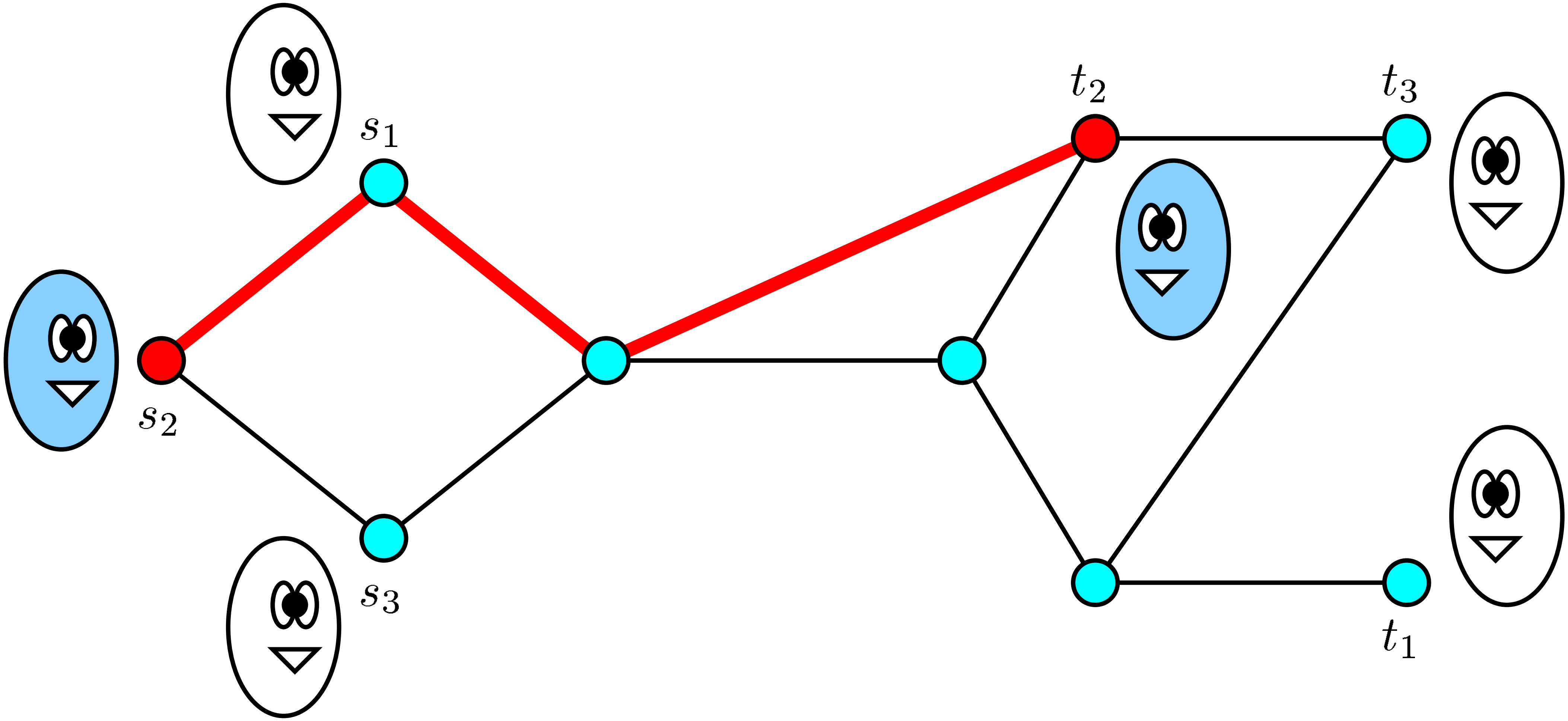
◀◀ Example: BGA ▶▶

$L = 3$



◀◀ Example: BGA ▶▶

$L = 3$



◀◀ Kolman–Scheideler's result ▶▶

Kolman–Scheideler (2002) showed that

> For UFP in arbitrary undirected graphs
>
> if an instance has the flow number F,
>
> then BGA with $L = 4F$ is a $(16F + 1)$-approximation algorithm.

Remarks

- UFP (the unsplittable flow problem) is a generalization of MEDP.
- The flow number is a measure of the communication property of a network.

◀◀ Result 1 ▶▶

Lem.

The flow number of a complete graph with the unit capacity is one. ($F = 1$)

Kolman–Scheideler (2002) (again)

For MEDP, if an instance has the flow number F, then BGA with $L = 4F$ is a $(16F + 1)$-approx. algorithm.

Thm.

BGA with $L = 4$ is a 17-approximation algorithm for MEDP in complete graphs.

◀◀ Contributions ▶▶

For MEDP in complete graphs,

1. A result by Kolman–Scheideler (2002) implies that
   BGA is a 17-approximation algorithm.

2. Indeed,
   SGA and BGA are 9-approximation algorithms.

3. There is an instance which shows that
   SGA cannot be better than a 2-approximation algorithm.

Additional contributions:

- There is an instance for arbitrary graphs which shows that
  SGA cannot be better than an $O(\sqrt{m})$-approximation algorithm.
- SGA can be interpreted via the primal-dual framework.

◀◀ Shortest-path-first greedy algorithm (SGA) ▶▶

While some req. $\in \mathcal{R}$ can be routed in G do

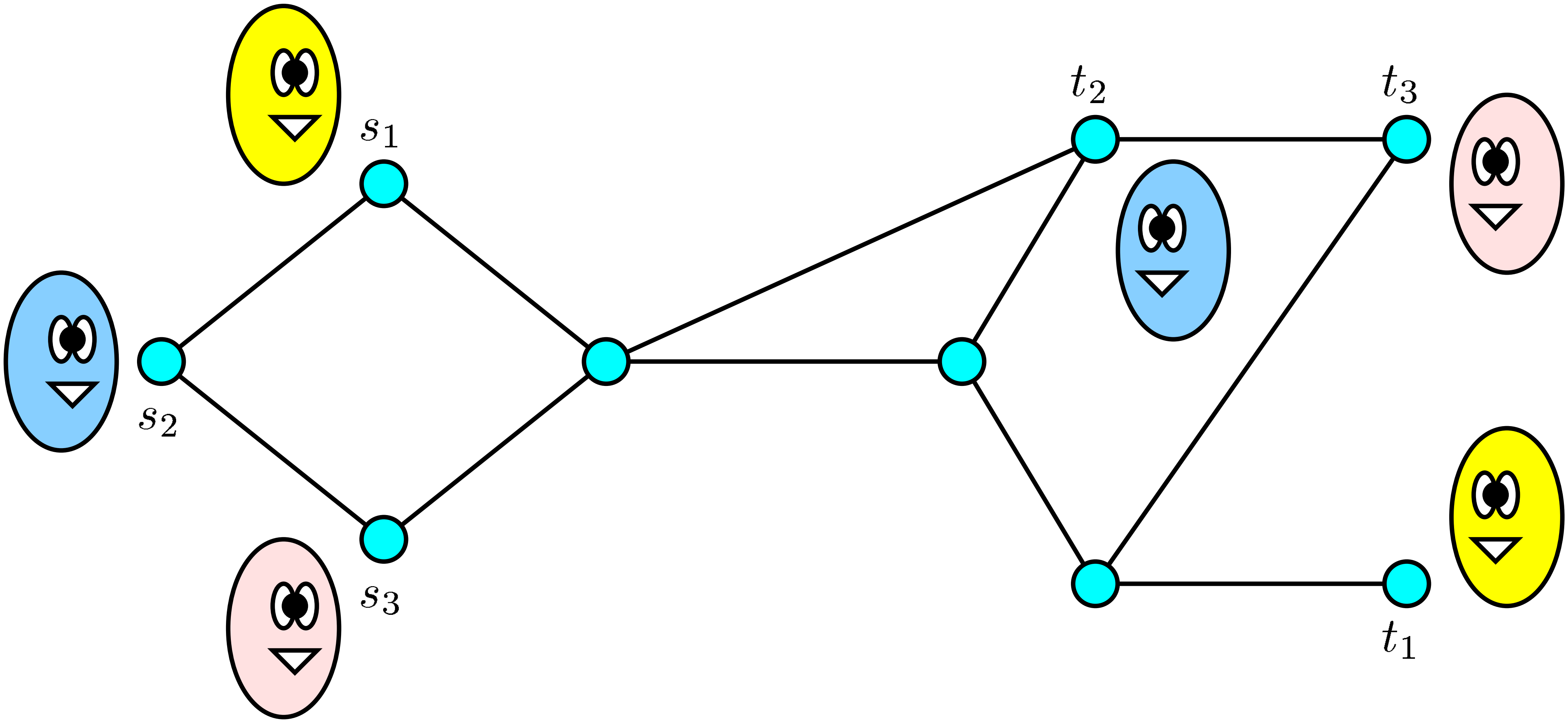
Find a req. r that can be routed using the fewest edges $\{e_1, \ldots, e_l\}$;

Accept r;

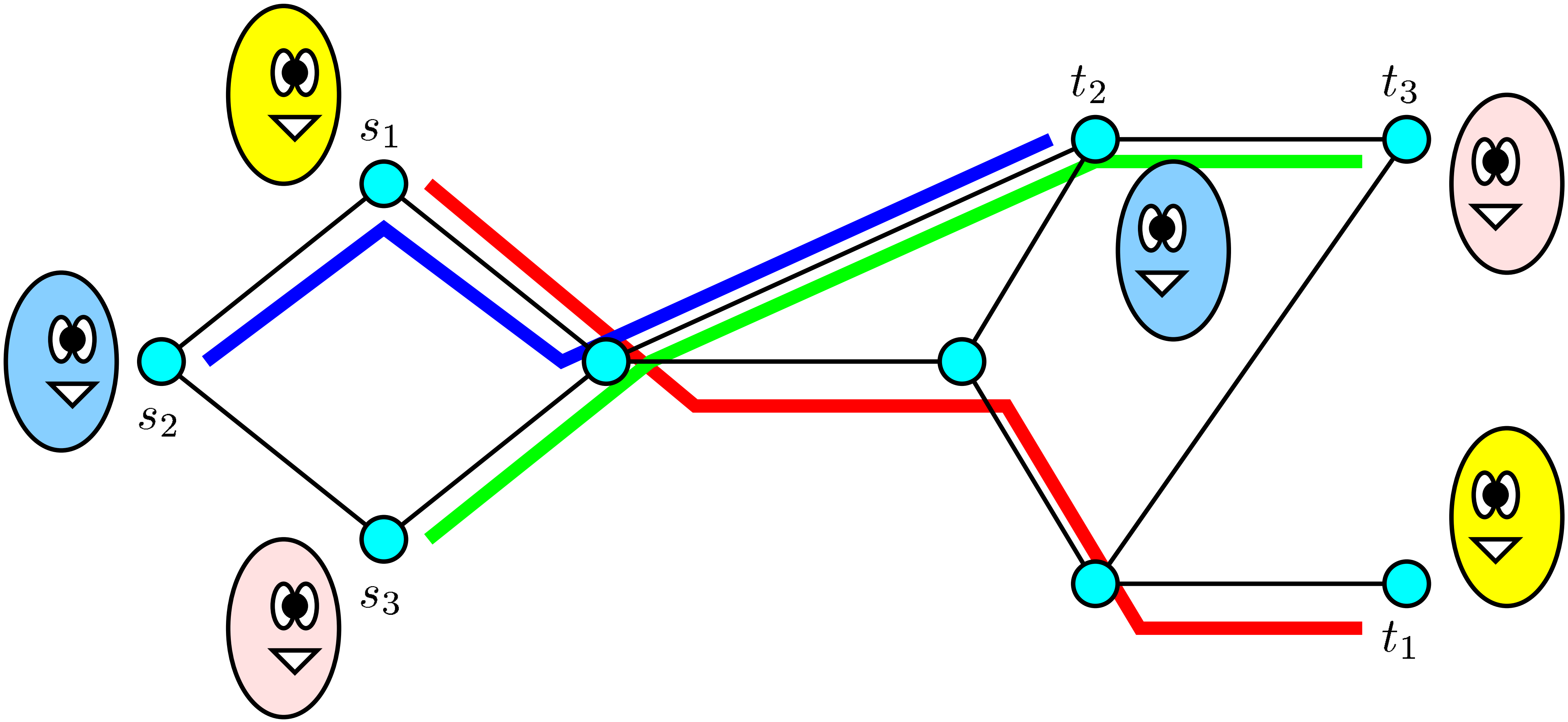
Remove $\{e_1, \ldots, e_l\}$ from G and r from $\mathcal{R}$.

(first introduced by Kolliopoulos–Stein (1998))

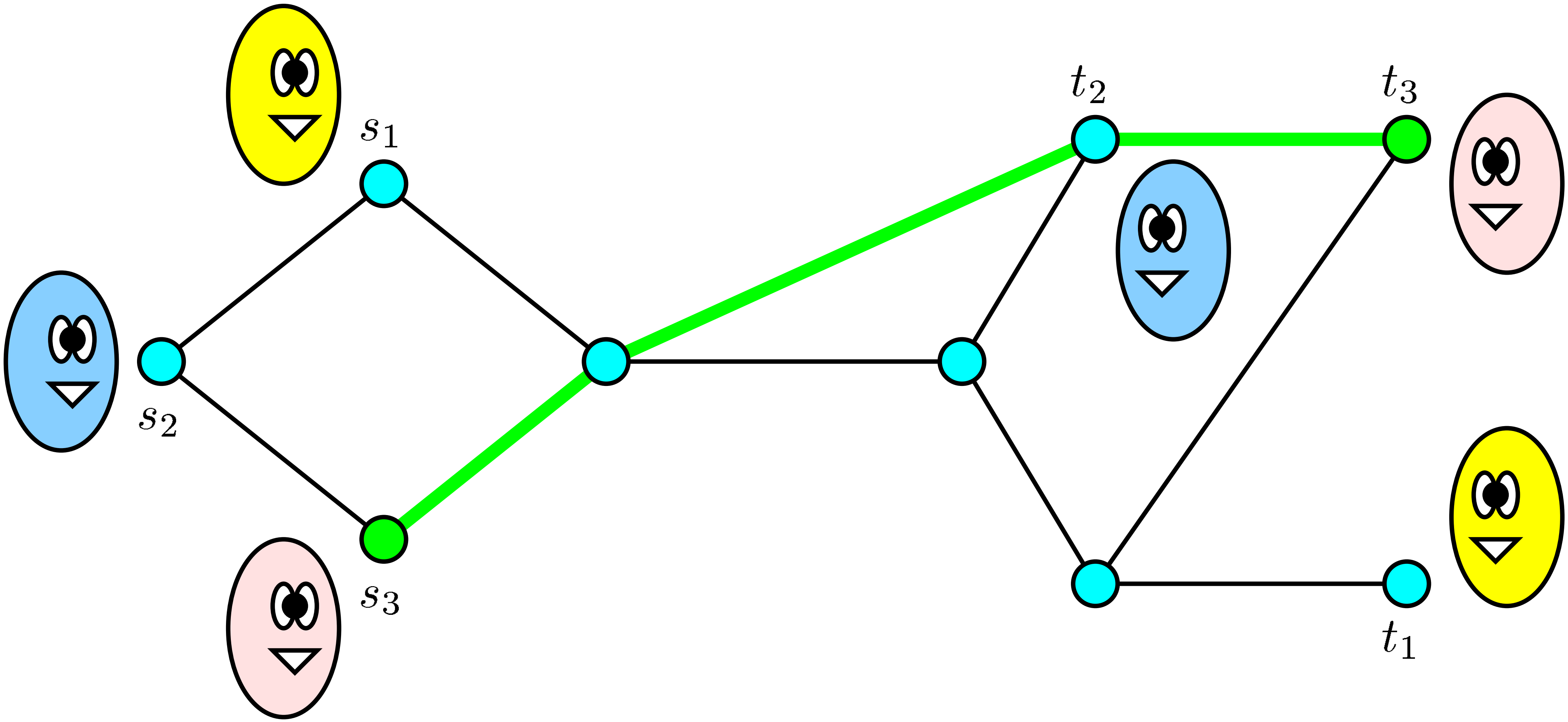
◀◀ Example: SGA ▶▶



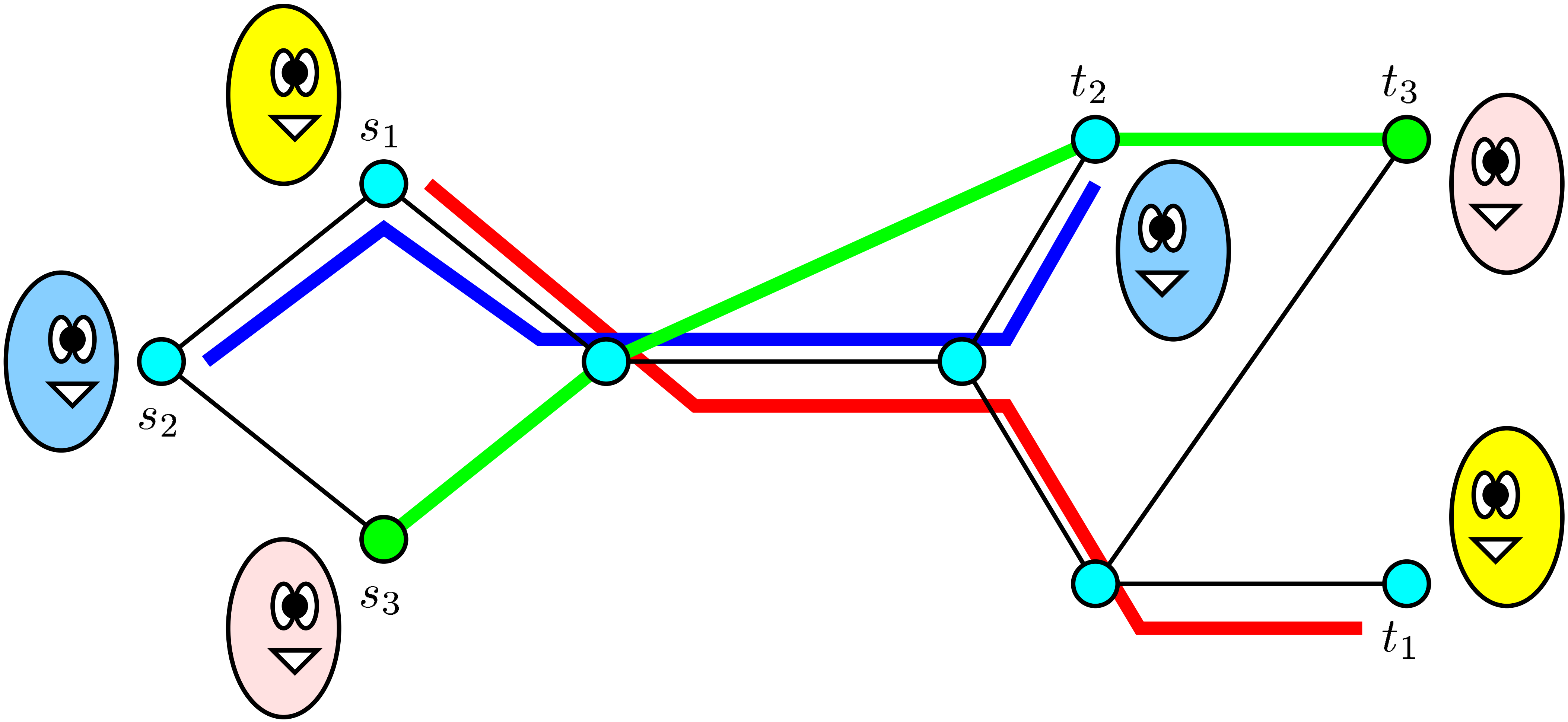
◀◀ Example: SGA ▶▶



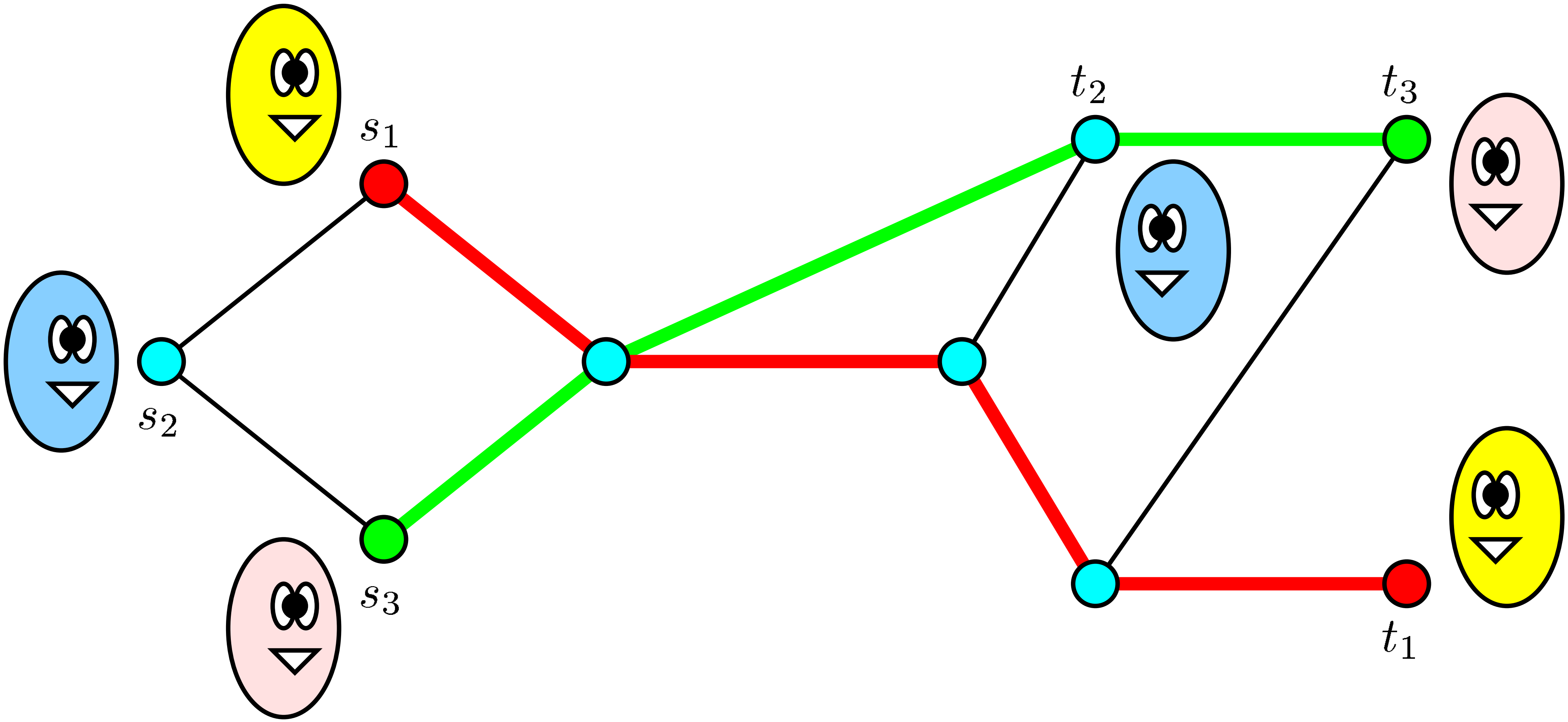
◀◀ Example: SGA ▶▶



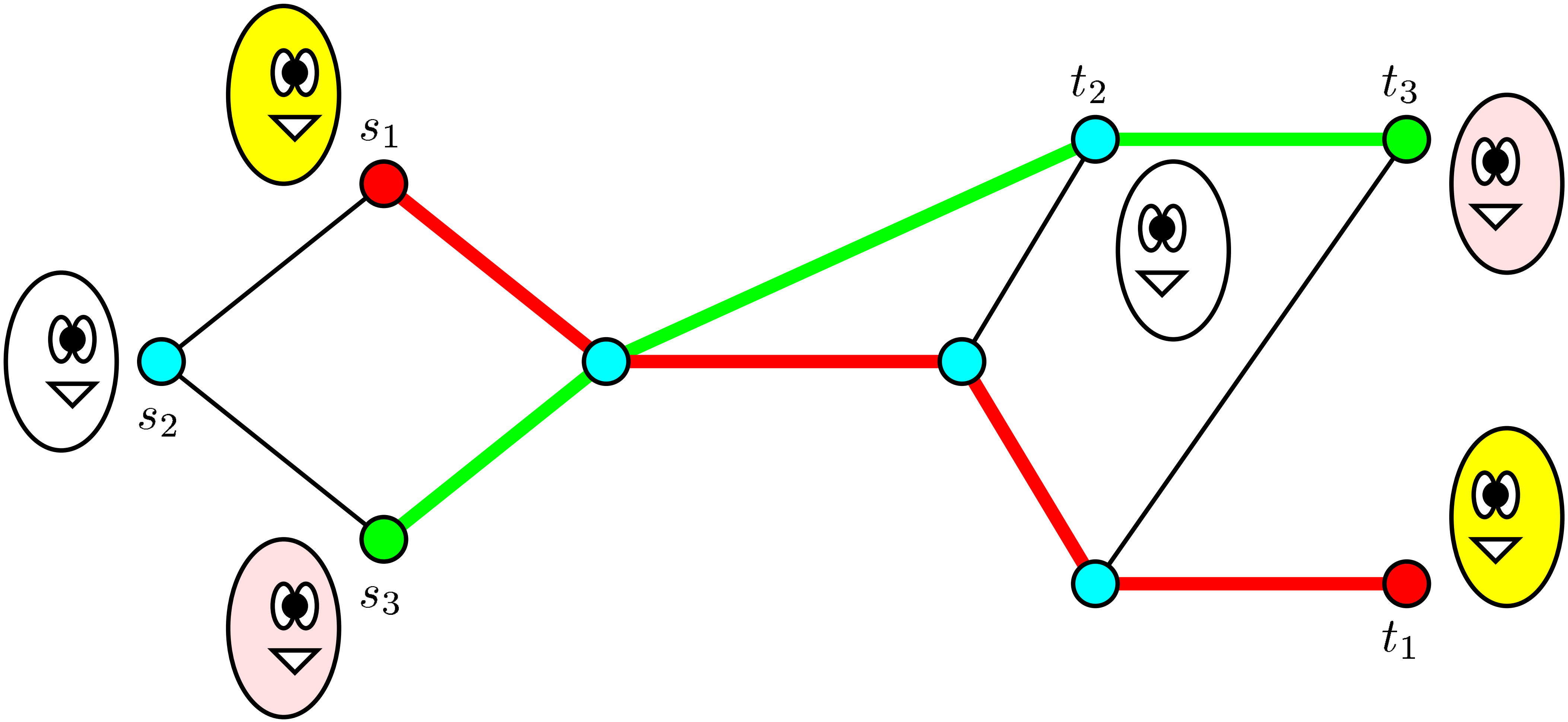
◀◀ Example: SGA ▶▶



◀◀ Example: SGA ▶▶



Example: SGA



◀◀ Result 2 ▶▶

Thm.

SGA and BGA with $L = 4$ are 9-approximation algorithms for MEDP in complete graphs.

Proof

Use

- the fact that

  the flow number of a complete graph with the unit capacity is one

- the shortening lemma (Kolman–Scheideler (2002)).

◀◀ Shortening lemma ▶▶

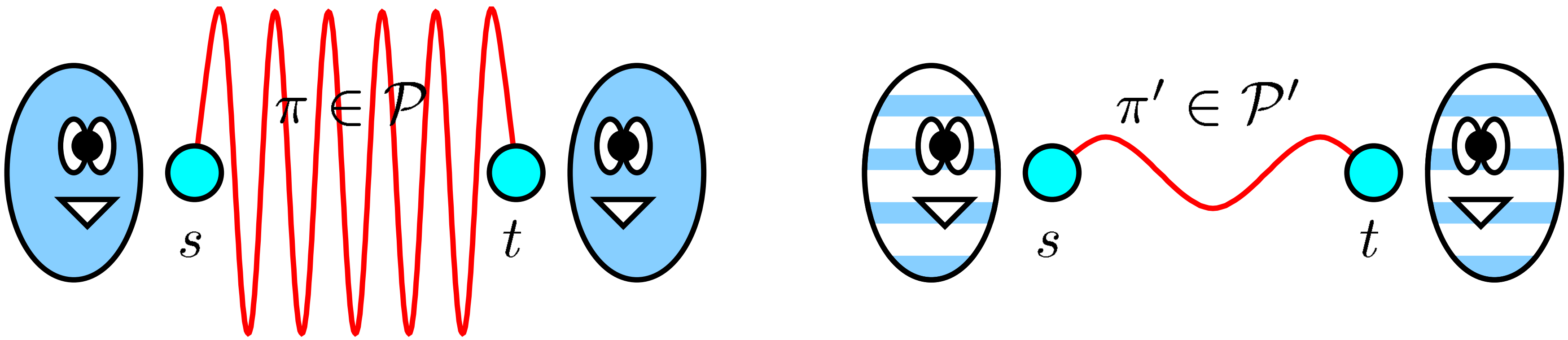
Consider the linear programming relaxation of MEDP.

Lem. (the shortening lemma, Kolman–Scheideler (2002))

$(\mathcal{A}, \mathcal{P})$ a feasible solution of UFP

$\Longrightarrow$ $\exists$ $(\mathcal{A}, \mathcal{P}')$ a fractional solution s.t.

- the longest path of $\mathcal{P}'$ has at most $4F$ edges (F the flow number),
- for each request $\in \mathcal{A}$, the fraction of acceptance is $1/2$.



◀◀ Proof sketch of Result 2 ▶▶

$(\mathcal{A}, \mathcal{P})$ an output of SGA $(\mathrm{Approx} = |\mathcal{A}| = |\mathcal{P}|)$
$(\mathcal{O}, \mathcal{Q})$ an optimal solution $(\mathrm{Opt} = |\mathcal{O}| = |\mathcal{Q}|)$

Apply the shortening lemma to $(\mathcal{O}, \mathcal{Q})$:

$\exists$ a fractional solution $(\mathcal{O}, \mathcal{Q}')$ satisfying

- the longest path of $\mathcal{Q}'$ has ≤ 4 edges
- for each request $\in \mathcal{O}$, the fraction of acceptance is 1/2.

Divide $\mathcal{Q}'$ into two parts:

- $\mathcal{Q}'_1 =$ the set of paths connecting the req. $\notin \mathcal{A}$;
- $\mathcal{Q}'_2 = \mathcal{Q}' \setminus \mathcal{Q}'_1$.

Observation:

(The total fractions of acceptance via $\mathcal{Q}'_2$) $\leq |\mathcal{A}|/2$.

◀◀ Proof sketch of Result 2 ▶▶

Setting:

$(\mathcal{A}, \mathcal{P})$ an output of SGA $(\mathrm{Approx} = |\mathcal{A}| = |\mathcal{P}|)$;

$(\mathcal{O}, \mathcal{Q})$ an optimal solution $(\mathrm{Opt} = |\mathcal{O}| = |\mathcal{Q}|)$;

$(\mathcal{O}, \mathcal{Q}')$ a fractional solution after applying the shortening lemma;

$\mathcal{Q}' = \mathcal{Q}'_1 \cup \mathcal{Q}'_2$; $\mathcal{Q}'_1$ = the set of paths connecting the req. $\notin \mathcal{A}$.

Observations:

- Each path $q' \in \mathcal{Q}'_1$ has to intersect some $p \in \mathcal{P}$ of length ≤ 4.
- The sum of the fractions via the paths intersecting $p \in \mathcal{P}$ is at most the length of p.

$\Longrightarrow$ (The total fractions of acceptance via $\mathcal{Q}'_1$) $\leq 4|\mathcal{A}|$.

$\Longrightarrow$ $\mathrm{Opt} = 2\times$(the total frac. of acceptance via $\mathcal{Q}'$)

$$\leq 2(4|\mathcal{A}| + |\mathcal{A}|/2) \leq 9|\mathcal{A}|.$$

◀◀ Table of the approximation ratios ▶▶

Contributors	Tripartition	BGA	SGA
Erlebach–Vukadinović (2001)	27	54	54
Kolman–Scheideler (2002)		17	
This work		9	9

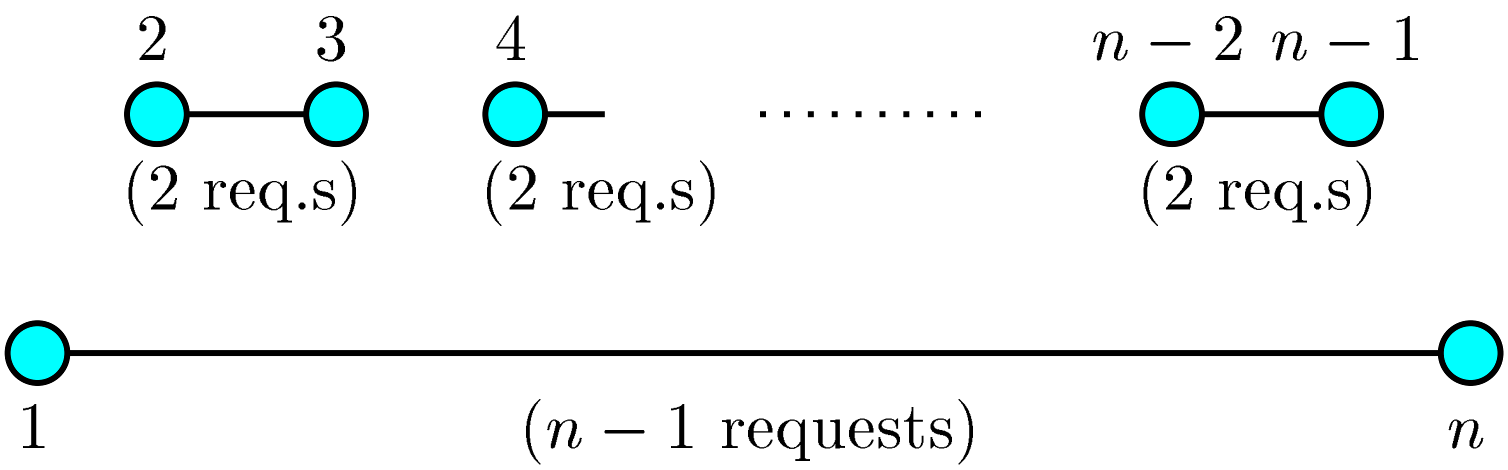
◀◀ Next question ▶▶

Q.

How much can we improve the bound of the approx. ratio?? or
What is the 'real' approximation ratio??

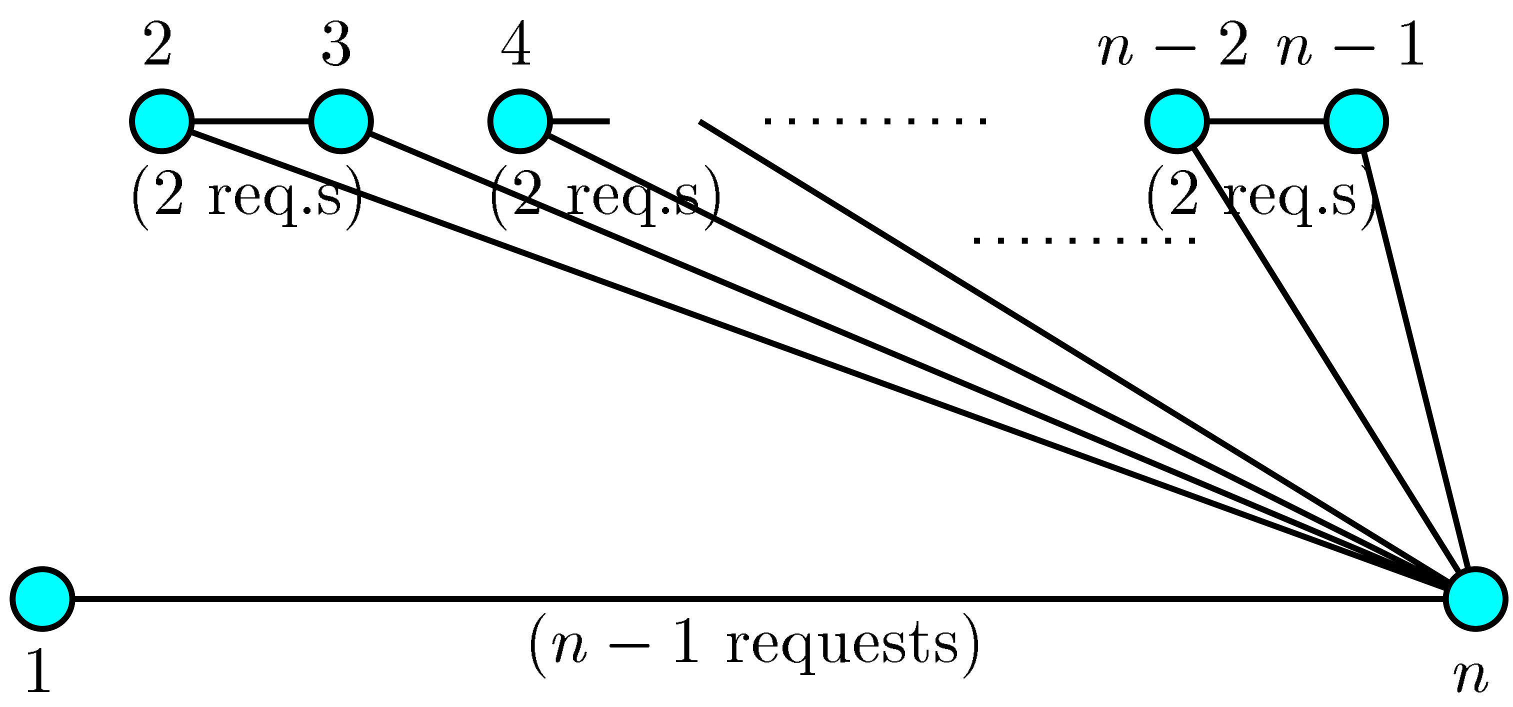
Example

There is an instance of MEDP in complete graphs which shows that SGA cannot be better than a 2-approximation algorithm.

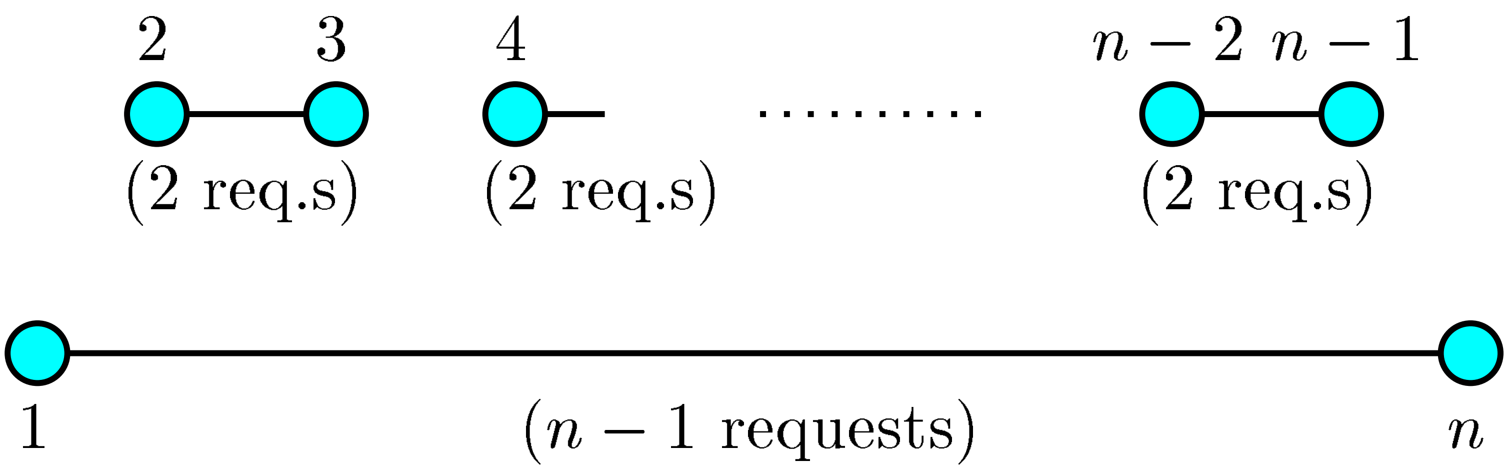


◀◀ An instance for a lower bound ▶▶

SGA may accept only $n - 1$ requests.



◀◀ An instance for a lower bound ▶▶



The optimal solution accepts all the $2n-3$ requests.
SGA may accept only $n-1$ requests.

$$\Longrightarrow \frac{\mathrm{Opt}}{\mathrm{Approx}} = \frac{2n-3}{n-1} = 2 - \frac{1}{n-1} \longrightarrow 2 \quad (n \to \infty).$$

◀◀ Conclusion ▶▶

We show that

$\boxed{\text{SGA is a } \rho\text{-approximation algorithm for MEDP in complete graphs, where } 2 \leq \rho \leq 9.}$

Future works

I believe that the 'real' bound should be 2.

- To improve the bounds (using the primal-dual method?)
- To derive inapproximability results
- To consider UFP in complete graphs

[The end of the talk]