A greedy algorithm for convex geometries

Yoshio Okamoto and **Kenji Kashiwabara**

Department of Systems Science,
Graduate School of Arts and Sciences,
The University of Tokyo
3-8-1, Komaba, Meguro,
Tokyo, 153-8902, Japan
yoshio@klee.c.u-tokyo.ac.jp

Department of Systems Science,
Graduate School of Arts and Sciences,
The University of Tokyo
3-8-1, Komaba, Meguro,
Tokyo, 153-8902, Japan
kashiwa@graco.c.u-tokyo.ac.jp

Abstract

Convex geometries are closure spaces which satisfy anti-exchange property, and they are known as the dual of antimatroids [2, 5, 7]. We consider set functions defined on the family of all extreme sets of a convex geometry. Faigle and Kern [3, 4] presented a greedy algorithm of linear programmings for poset shellings, and Krüger [6] introduced a b-submodular function on the family of all extreme sets on a poset shelling and proved that Faigle and Kern's algorithm works for a poset shelling when the given set function is b-submodular. We extend their results to all convex geometries. That is, we define a new class of submodular-type functions called c-submodular functions and prove that Faigle and Kern's algorithm works for all convex geometries precisely when the given set function is c-submodular.

1 Convex Geometries

Let E be a nonempty finite set. The family $\mathcal{L}$ of subsets of E is a *closure space* if it satisfies the following properties

(1) $\emptyset \in \mathcal{L}, E \in \mathcal{L}$,

(2) $X, Y \in \mathcal{L}$ implies $X \cap Y \in \mathcal{L}$.

We define the *closure operator* $\tau : 2^E \to 2^E$ associated with a closure space $\mathcal{L}$ as

(3) $\tau(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}$,

that is, $\tau(A)$ is the minimum closed set containing A. A closure space $\mathcal{L}$ with the closure operator τ is a *convex geometry* if the following *anti-exchange property* is satisfied: if distinct $x, y \in E$ satisfy $x, y \notin \tau(A)$ and $y \in \tau(A \cup \{x\})$ then $y \notin \tau(A \cup \{y\})$ for all $A \subseteq E$.

Proposition 1.1 ([5]). *Let $\mathcal{L}$ be a closure space on E. Then $\mathcal{L}$ is a convex geometry if and only if $\mathcal{L}$ satisfies the following property:*

(4) $X \in \mathcal{L} \setminus \{E\}$ *implies* $X \cup \{x\} \in \mathcal{L}$ *for some* $x \in E \setminus X$.

Several combinatorial objects give rise to convex geometries. Let us consider some examples. Suppose that E is a finite set of points on a Euclidean space. Then $\{X \subseteq E : X = \text{conv.hull}(X) \cap E\}$ forms a convex geometry called a *convex shelling*. The second example is obtained from a partially ordered set. Let $(E, \leq)$ be a partially ordered set. An order ideal of E is a subset $X \subseteq E$ such that $y \in X$ and $x \leq y$ imply $x \in X$. Then the family of all order ideals of E satisfies the axioms of convex geometries. Such a convex geometry is called a *poset shelling*. For further examples, see [2, 5].

We call an element a of a subset $A \subseteq E$ an *extreme point* of A if $a \notin \tau(A \setminus \{a\})$. The set of all extreme points of A is denoted by $\mathrm{ex}(A)$, which is called a *extreme set* of A. For a convex geometry $\mathcal{L}$ we denote $\mathrm{ex}(\mathcal{L}) = \{\mathrm{ex}(A) : A \in \mathcal{L}\}$.

A convex geometry $\mathcal{L}$ forms a lattice with respect to set-inclusion [2]. Then the join $\vee_{\mathcal{L}}$ and the meet $\wedge_{\mathcal{L}}$ of $\mathcal{L}$ are defined as follows: for all $A, B \in \mathcal{L}$

$$(5) \quad A \vee_{\mathcal{L}} B = \tau(A \cup B),$$

$$(6) \quad A \wedge_{\mathcal{L}} B = \tau(A \cap B) = A \cap B.$$

By setting a partial order $\preceq$ on $\mathrm{ex}(\mathcal{L})$ so that

$$(7) \quad X \preceq Y \Leftrightarrow \tau(X) \subseteq \tau(Y) \text{ for all } X, Y \in \mathrm{ex}(\mathcal{L}),$$

$\mathrm{ex}(\mathcal{L})$ is a lattice isomorphic to $\mathcal{L}$. Then we have the join $\vee_{\mathrm{ex}(\mathcal{L})}$ and the meet $\wedge_{\mathrm{ex}(\mathcal{L})}$:

$$(8) \quad A \vee_{\mathrm{ex}(\mathcal{L})} B = \mathrm{ex}(\tau(A) \vee_{\mathcal{L}} \tau(B)),$$

$$(9) \quad A \wedge_{\mathrm{ex}(\mathcal{L})} B = \mathrm{ex}(\tau(A) \wedge_{\mathcal{L}} \tau(B)).$$

Remark that for a convex geometry $\mathcal{L}$ with the closure operator τ, the restriction of τ to $\mathrm{ex}(\mathcal{L})$ is a lattice-isomorphism from $\mathrm{ex}(\mathcal{L})$ to $\mathcal{L}$, whose inverse operation is ex.

When there is no risk of confusion, we use $\vee$ instead of $\vee_{\mathrm{ex}(\mathcal{L})}$, and $\wedge$ for $\wedge_{\mathrm{ex}(\mathcal{L})}$.

We also two other operators on the family $\mathrm{ex}(\mathcal{L})$ of all extreme sets of a convex geometry $\mathcal{L}$. For $X, Y \in \mathrm{ex}(\mathcal{L})$, the *reduced meet* $X \sqcap Y$ is defined as $X \sqcap Y = (X \wedge Y) \cap (X \cup Y)$ and the *residue* $X \diamond Y$ is defined as $X \diamond Y = (X \cap Y) \setminus (X \vee Y)$. Remark that $X \sqcap Y$ and $X \diamond Y$ are also in $\mathrm{ex}(\mathcal{L})$.

Proposition 1.2. *Let $\mathcal{L}$ be a convex geometry and $\preceq$ be the partial order on $\mathrm{ex}(\mathcal{L})$ defined by (7). Then $X \diamond Y \preceq X \sqcap Y \preceq X \vee Y$ for all $X, Y \in \mathrm{ex}(\mathcal{L})$.*

2 Linear Programming on $\mathrm{ex}(\mathcal{L})$ and a greedy algorithm

Let $\mathcal{L}$ be a convex geometry on E, $c \in \mathbb{R}_+^E$ be a non-negative vector and $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ be a set function on $\mathrm{ex}(\mathcal{L})$ with $f(\emptyset) = 0$, then we consider a linear programming (P) as follows:

$$(10) \quad \text{Maximize} \quad \sum_{e \in E} c(e) x(e)$$

$$(11) \quad \text{subject to} \quad \sum_{e \in X} x(e) \leq f(X) \quad (X \in \mathrm{ex}(\mathcal{L})).$$

The problem (P) leads to the dual problem (D):

$$(12) \quad \text{Minimize} \quad \sum_{X \in \mathrm{ex}(\mathcal{L})} f(X) y(X)$$

$$(13) \quad \text{subject to} \quad \sum_{\substack{X \in \mathrm{ex}(\mathcal{L}) \\ \text{s.t. } e \in X}} y(X) = c(e) \quad (e \in E),$$

$$(14) \quad y(X) \geq 0 \quad (X \in \mathrm{ex}(\mathcal{L})).$$

`Input` : $c \in \mathbb{R}^E$

`Output` : $y \in \mathbb{R}^{\mathrm{ex}(\mathcal{L})}$ which gives the optimum value of (D)

`Initialize:`

  $y(X) \leftarrow 0 \quad (\forall X \in \mathrm{ex}(\mathcal{L}))$;

  $T \leftarrow E$;

  $w(e) \leftarrow c(e) \quad (\forall e \in E)$;

`Iterate:`

  `while` $T \neq \emptyset$ `do`

    determine $e \in \mathrm{ex}(T)$ s.t. $w(e) = \min\{w(e') : e' \in \mathrm{ex}(T)\}$;

    $y(\mathrm{ex}(T)) \leftarrow w(e)$;

    $w(x) \leftarrow w(x) - w(e) \quad (\forall x \in \mathrm{ex}(T))$;

    $T \leftarrow T \setminus \{e\}$.

  `end of while`

Algorithm 1: Greedy Algorithm(G)

Since the optimum value of (P) coincides with the optimum value of (D), we consider the dual problem (D) instead of (P).

In order to solve the dual problem (D) Faigle and Kern [3] introduced a greedy algorithm (G) shown in Algorithm 1.

Theorem 2.1 (Krüger [6], also see Ando [1]). *Let $\mathcal{L}$ be a poset shelling. Then the greedy algorithm* (G) *gives the optimum of the problem* (D) *if and only if $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is b-submodular, that is, for all $X, Y \in \mathrm{ex}(\mathcal{L})$*

$$f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y), \tag{15}$$

and $f(\emptyset) = 0$.

We extend Krüger's b-submodular function to a slightly different submodular function in our sense, called a c-submodular function. Let $\mathcal{L}$ be a convex geometry. Then a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is *c-submodular* if for all $X, Y \in \mathrm{ex}(\mathcal{L})$ satisfying the condition that $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$,

$$f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y) \tag{16}$$

and $f(\emptyset) = 0$, where χ^X is the characteristic vector of X.

Proposition 2.2. *Let $\mathcal{L}$ be a convex geometry. Then the class of c-submodular functions on $\mathrm{ex}(\mathcal{L})$ coincides with the class of b-submodular functions if and only if $\mathcal{L}$ is a poset shelling.*

Proposition 2.3. *The system of linear constraints (13) and (14) of the problem* (D) *is totally dual integral for any convex geometry $\mathcal{L}$ if $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular.*

The next is our main theorem.

Theorem 2.4. *Let $\mathcal{L}$ be a convex geometry. Then the greedy algorithm* (G) *gives the optimum of the problem* (D) *if and only if $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ is c-submodular.*

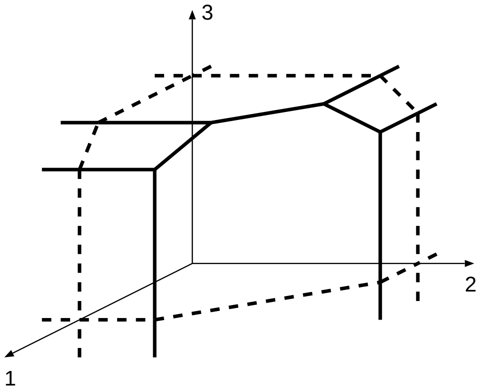

Figure 1: An example of the polyhedra described by (11).

An example of the polyhedra described by (11) is shown in Figure 1. In this example, the underlying convex geometry $\mathcal{L}$ is $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$, which means $\mathrm{ex}(\mathcal{L}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2\}\}$. And f is defined as follows: $f(\emptyset) = 0$, $f(\{1\}) = 6$, $f(\{2\}) = 6$, $f(\{3\}) = 5$, $f(\{1,3\}) = 10$, $f(\{2,3\}) = 10$, and $f(\{1,2\}) = 8$. It is easy to check that f is c-submodular.

Theorem 2.4 leads to the next corollary which gives a greedy-algorithmic characterization of a poset shelling.

Corollary 2.5. *Let $\mathcal{L}$ be a convex geometry. Then the greedy algorithm* (G) *gives the optimum of the problem* (D) *for any b-submodular function $f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$ if and only if $\mathcal{L}$ is a poset shelling.*

The importance of our consequences is due to the extension of the class of problems with the same framework as the ordinary submodular programming and the intermediation between submodular programmings and convex geometries.

Acknowledgments

We wish to give thanks to Prof. K. Ando, whose talk at the COMA Seminar based on [1] has inspired this work. We also would like to thank to Prof. M. Nakamura, Prof. T. Takabatake, Prof. T. Abe and Dr. M. Hachimori for their useful comments.

References

[1] K. Ando, The greedy algorithm for a class of lattice polyhedra and its consequences, Discussion Paper No. 830, Institute of Policy and Planning Sciences, University of Tsukuba, 1999.

[2] P. H. Edelman and R. E. Jamison, The theory of convex geometries, *Geometriae Dedicata* **19**, 1985, 247–270.

[3] U. Faigle and W. Kern, Submodular linear programs on forests, *Mathematical Programming* **72**, 1996, 195–206.

[4] U. Faigle and W. Kern, On the core of ordered submodular cost games, *Mathematical Programming* **87**, 2000, 467–481.

[5] B. Korte, L. Lovász, R. Schrader, *Greedoids*, Springer-Verlag, Berlin Heidelberg, 1991.

[6] U. Krüger, Structural aspects of ordered polymatroids, *Discrete Applied Mathematics* **99**, 2000, 125–148.

[7] J. L. Pfaltz, Closure lattices, *Discrete Mathematics* **154**, 1996, 217–236.