Forbidden minors for point-search antimatroids and line-search antimatroids of rooted graphs

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Let $\mathcal{F}$ be a family of subsets of a finite set E. $\mathcal{F}$ is an *antimatroid* on E if $\mathcal{F}$ satisfies

(1) $\emptyset \in \mathcal{F}, E \in \mathcal{F}$,

(2) $X \in \mathcal{F} \setminus \{\emptyset\}$ implies $X \setminus \{e\} \in \mathcal{F}$ for some $e \in X$,

(3) $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$.

A member of an antimatroid is called a *feasible set.* Antimatroids are known as combinatorial abstraction of convexity, and various constructions and procedures yields antimatroids. Let us see some examples of them. Suppose that E is a finite set of points on a Euclidean space. If we gather $X \subseteq E$ such that the convex hull of $E \setminus X$ does not contain any elements of X, then it forms an antimatroid called a *convex shelling.* The second example is obtained from a partially ordered set. Let $(E, \leq)$ be a partially ordered set. An order ideal of $(E, \leq)$ is a subset $X \subseteq E$ such that $y \in X$ and $x \leq y$ imply $x \in X$. Then the family of order ideals of $(E, \leq)$ satisfies the axioms of antimatroids. Such an antimatroid is called a *poset shelling.* For more examples we refer [1, 2]. Remark that an antimatroid is a lattice with respect to set-inclusion.

Let $\mathcal{F}$ be an antimatroid on E, and $A, B \in \mathcal{F}$ such that $A \subseteq B$. The *minor* $\mathcal{F}[A, B]$ is defined as

(4) $\mathcal{F}[A, B] = \{X \setminus A : X \in \mathcal{F}, A \subseteq X \subseteq B\}$.

It is easy to see that a minor of an antimatroid is also an antimatroid.

In this work, we characterize three classes of antimatroids by their forbidden-minors : a point-search of a rooted directed graph, a point-search of a rooted undirected graph and a line-search of a rooted directed graph. Let $G = (V \cup \{r\}, E)$ be a directed graph with a root r. Such a graph is called a *rooted directed graph.* Then

(5) $\mathcal{F}_{\mathrm{psd}}(G) = \{X \subseteq V :$ the subgraph induced by $X \cup \{r\}$ contains a directed tree rooted at $r\}$

is an antimatroid, called a *point-search antimatroid* of a rooted directed graph G. A *line-search antimatroid* $\mathcal{F}_{\mathrm{lsd}}(G)$ of a rooted directed graph G is defined as follows:

(6) $\mathcal{F}_{\mathrm{lsd}}(G) = \{X \subseteq E :$ every edge in X can be reached from r by a directed path$\}$.

We can analogously define a point-search antimatroid of a rooted undirected graph; let $G = (V \cup \{r\}, E)$ be a rooted undirected graph, then

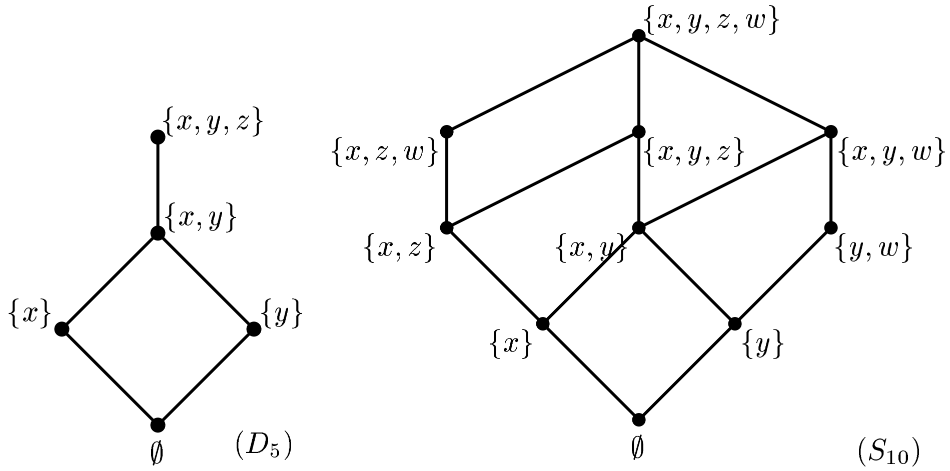
(7) $\mathcal{F}_{\mathrm{psu}}(G) = \{X \subseteq V :$ the subgraph induced by $X \cup \{r\}$ contains a directed tree rooted at $r\}$

is a point-search antimatroid of a rooted undirected graph G. We can easily check that each of these three antimatroids $\mathcal{F}_{\mathrm{psd}}(G)$, $\mathcal{F}_{\mathrm{lsd}}(G)$ and $\mathcal{F}_{\mathrm{psu}}(G)$ is closed under taking minors. Therefore, they can be characterized by their forbidden minors, i.e. minimal antimatroids which do not belong to the class.

Consider two antimatroids

(8) $D_5 = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}\}$,

(9) $S_{10} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, w\}, \{x, y, z\}, \{x, y, w\}, \{x, z, w\}, \{x, y, z, w\}\}$.

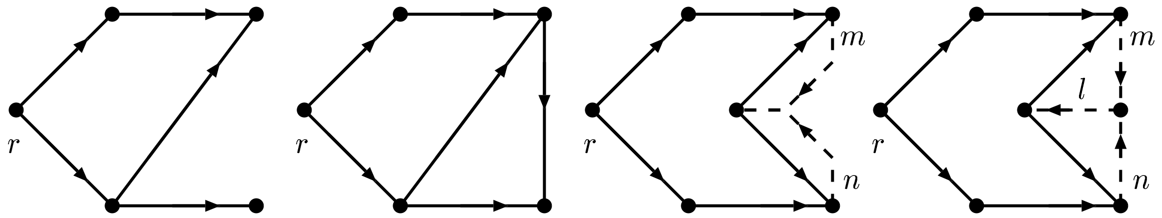


Theorem 1 ([3]). *An antimatroid is a point-search antimatroid of a rooted directed graph if and only if it has no minor isomorphic to D_5.* □

It is also easy to see that a point-search antimatroid of a rooted undirected graph and a line-search antimatroid of a rooted directed graph belong to the class of point-search antimatroids of rooted directed graphs. Hence, in order to characterize these two classes, we only have to know minimal point-search antimatroids of rooted directed graphs which do not belong to each of the classes.

Theorem 2 ([3]). *A point-search antimatroid of a rooted directed graph is also a point-search antimatroid of a rooted undirected graph if and only if it has no minor isomorphic to S_{10}.* □

Theorem 3 ([4]). *A point-search antimatroid of a rooted directed graph is also a line-search antimatroid of a rooted directed graph if and only if it has no minor isomorphic to point-search antimatroids of rooted directed graphs $A, B, C_{m,n}, D_{l,m,n}$ shown in figures below, where broken arrows indexed by l, m, n are directed paths whose length are $l, m, n \geq 1$, respectively.*



(A) (B) $(C_{m,n})$ $(D_{l,m,n})$

□

Analogously, we can define a line-search antimatroid of a rooted undirected graph, which is also closed taking minors. However, we have not known its minimal forbidden minors yet.

References

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