Forbidden minors for point-search antimatroids and line-search antimatroids of rooted graphs

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$\boxed{\text{Our Results in One Sentence}}$

Forbidden-minor characterizations for some classes of antimatroids

Antimatroids

A combinatorial abstraction of "convexity"

(Jamison, 1982; Korte-Lovász, 1984)

E a non-empty finite set

Def. a family $\mathcal{F}$ of subsets of E satisfying:

- $\emptyset \in \mathcal{F}, E \in \mathcal{F}$
- $X \in \mathcal{F} \setminus \{\emptyset\}$ implies $X \setminus \{e\} \in \mathcal{F}$ for some $e \in X$
- $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$

is an antimatroid on E.

Notice An antimatroid $\mathcal{F}$ forms a lattice w.r.t. "$\subseteq$."

Outline

1. Some examples of antimatroids
2. Minors of antimatroids
3. Forbidden-minor characterizations
   - Point searches of directed graphs
   - Point searches of undirected graphs
   - Line searches of directed graphs

1 Some examples of antimatroids

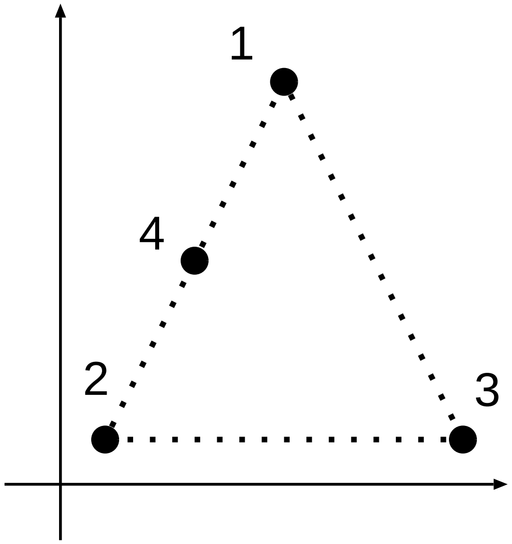
$\boxed{\text{Various combinatorial objects yield antimatroids}}$

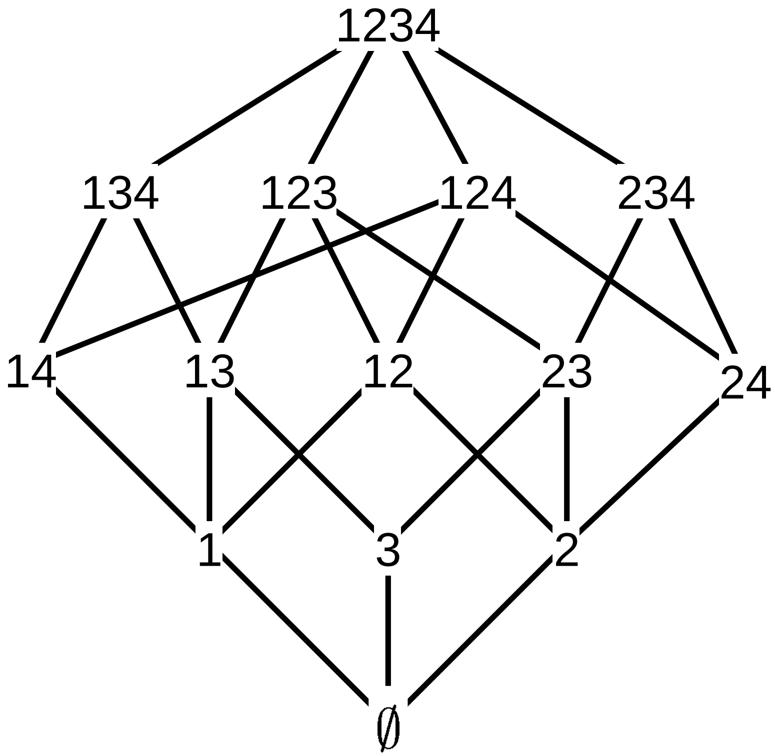
- Shelling, i.e. deleting a 'suitable' element step by step
  - convex shellings on the Euclidean space
  - poset shellings
  - edge shellings of trees
  - ....
- Search, i.e. gathering an 'adjacent' element step by step
  - point searches/line searches of graphs
  - ...

Ex. (1) Convex Shellings on the Euclidean space

E a finite set of points on $\mathbb{R}^n$

Delete a vertex of the convex hull step by step

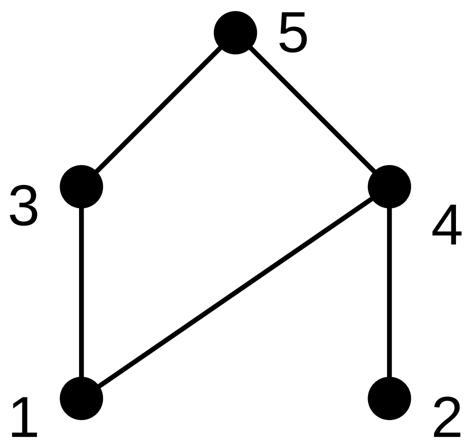


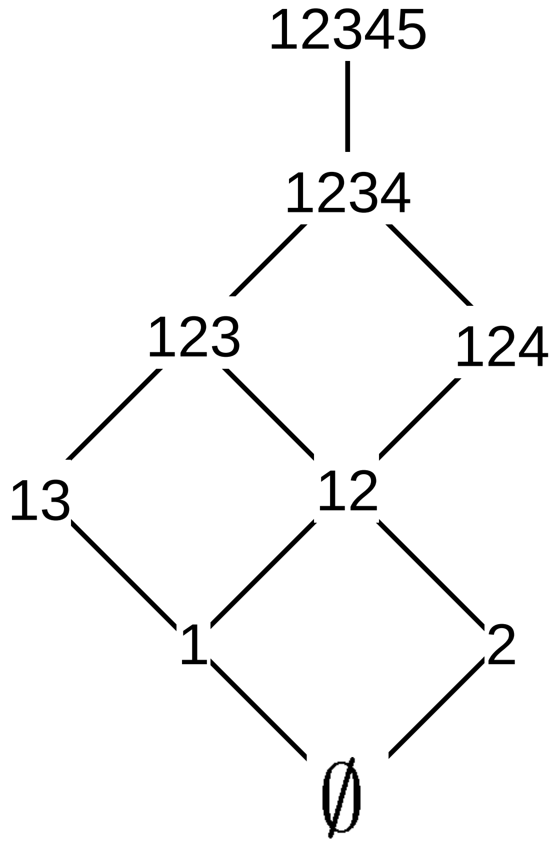


Ex. (2) Poset Shellings

E a partially ordered set (poset)

Delete a min. element step by step

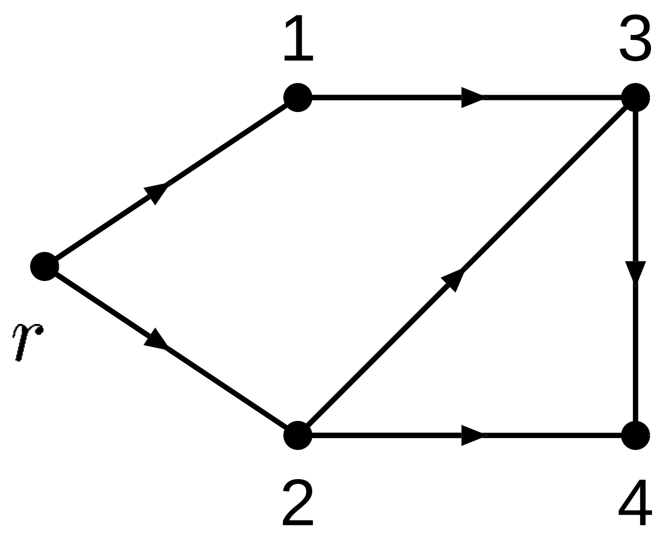


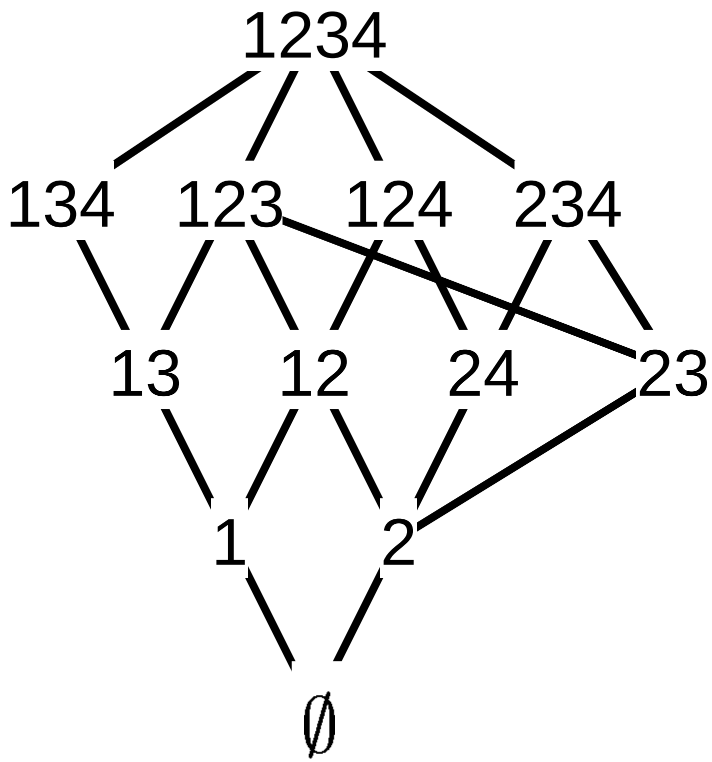


Ex. (3) Point searches of directed graphs

E the vertices of a rooted digraph

Search vertices from the root





Motivation

There are many classes of antimatroids

$$\Downarrow$$

How can we characterize each of the classes??

- classification problem
- characterization problem

Now, we concentrate on forbidden-minor characterizations.

Forbidden-minor characterization

We know some “forbidden-minor characterization” results:

- Graphs
  - Planar : $K^5, K^{3,3}$ (Kuratowski, 1930)
- Matroids
  - Binary : $U_{2,4}$ (Tutte, 1958)
  - Ternary : $U_{2,5}, U_{3,5}, F_7, F_7^*$ (Bixby, 1979; Seymour, 1979)
  - Quaternary : Seven minors (Geelen-Gerards-Kapoor, 2000)

Notice

A class can be characterized by its forbidden minors
$\Leftrightarrow$ it is closed under taking minors

2 Minors of antimatroids

Toward forbidden-minor characterizations

We want

To characterize some classes of antimatroids by their forb. minors.

Now, we have to consider..

- What is a "minor" of an antimatroid?
- What kind of classes are closed under taking their minors?
  What kind of classes can be characterized by their forb. minors?

Minors of antimatroids

$\mathcal{F}$ an antimatroids

$A, B \in \mathcal{F}$ $(A \subseteq B)$

Def.

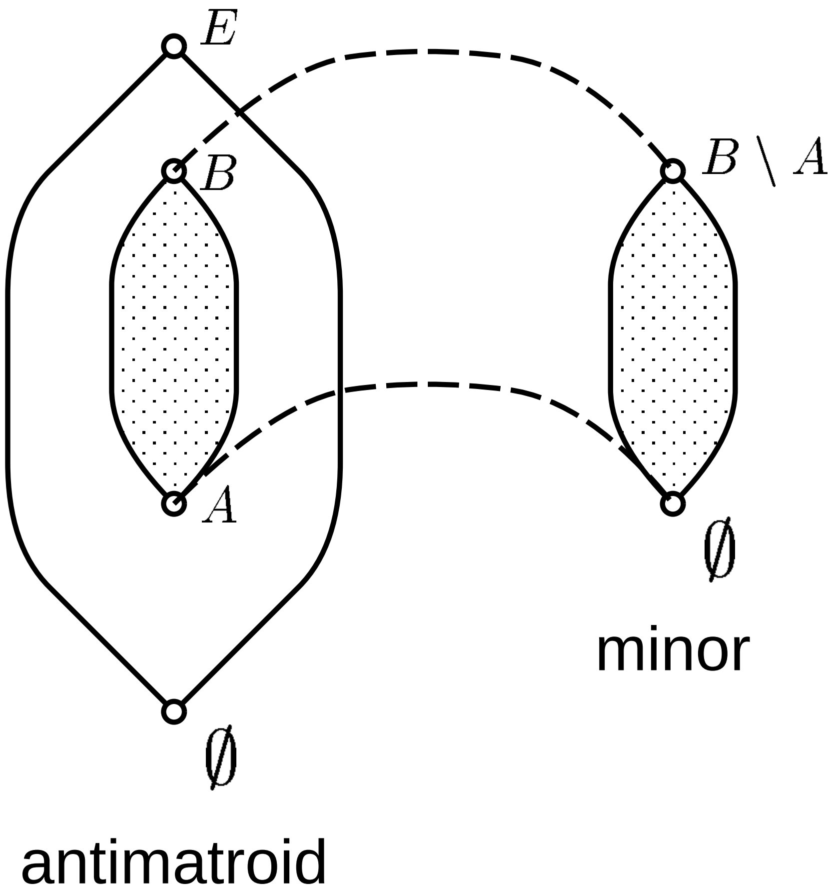
Minor of $\mathcal{F}$

$= \{X \setminus A :$

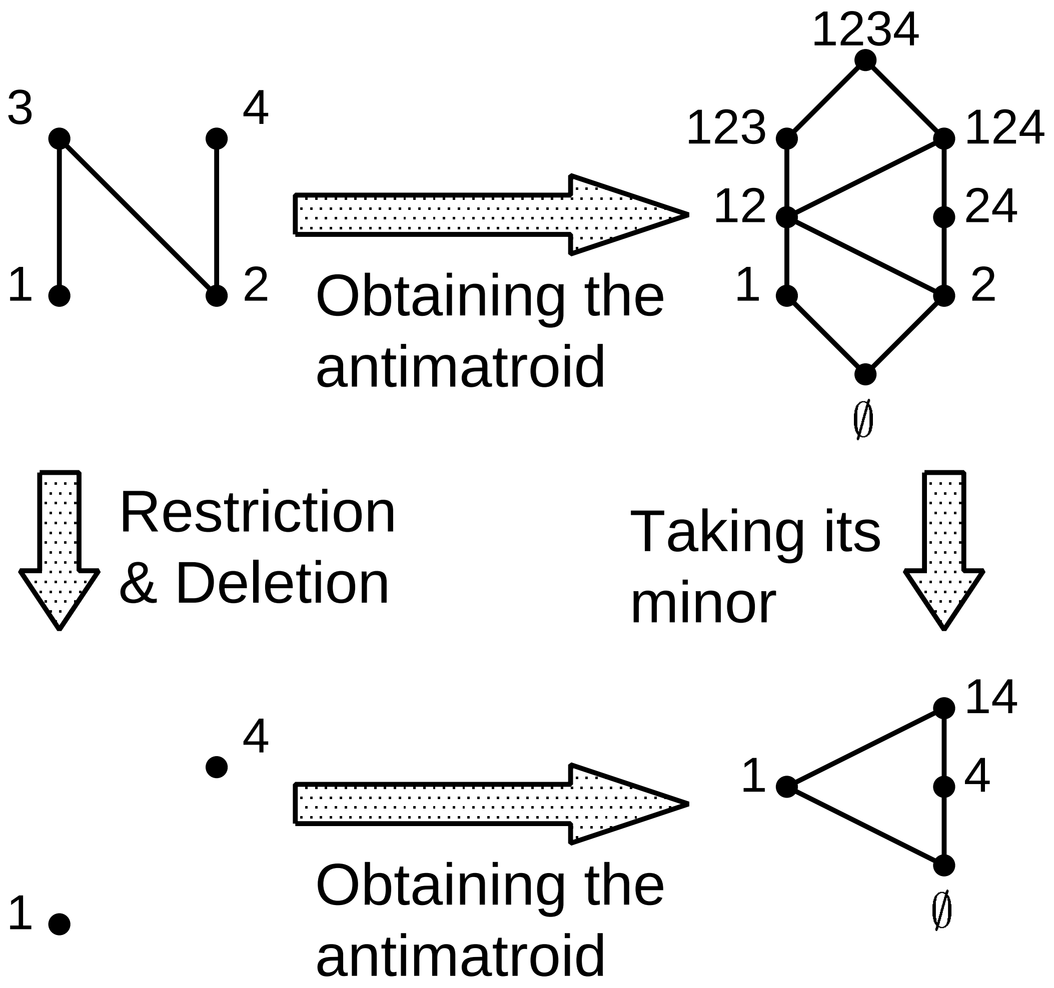
$X \in \mathcal{F}, A \subseteq X \subseteq B\}$

Notice

Minors of antimatroids are also antimatroids



Minors of poset shellings



The class of poset shellings are closed under taking minors

$$\Downarrow$$

We can characterize it by its forbidden minors

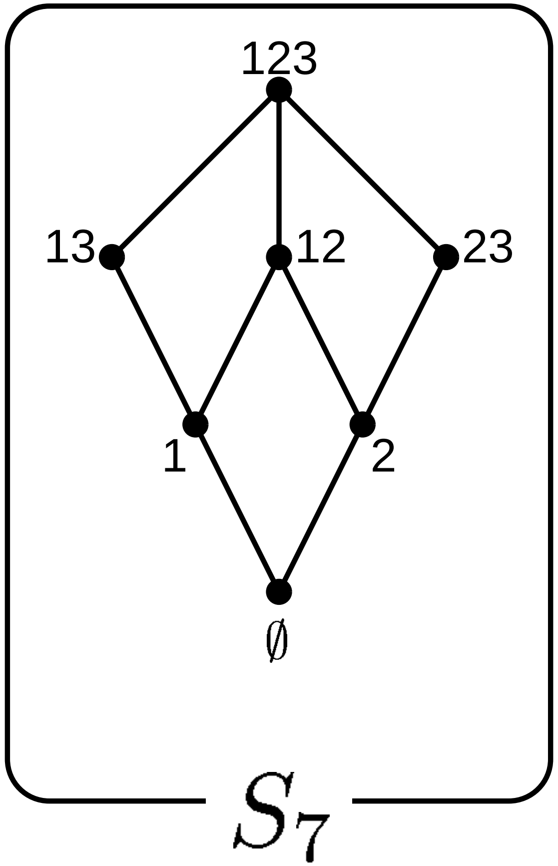
Fofbidden minors for poset shellings

$\boxed{\text{Proposition}}$ (Nakamura, 2000)

An antimatroid is a poset shelling

$$\Updownarrow$$

It has no minor isomorphic to S_7



<u>What for other classes?</u>

Classes closed under taking its minors

Closed

- poset shellings
- point searches of graphs
- line searches of graphs
- !D

Not closed

- convex shellings on $\mathbb{R}^n$
- edge shellings of trees
- !D

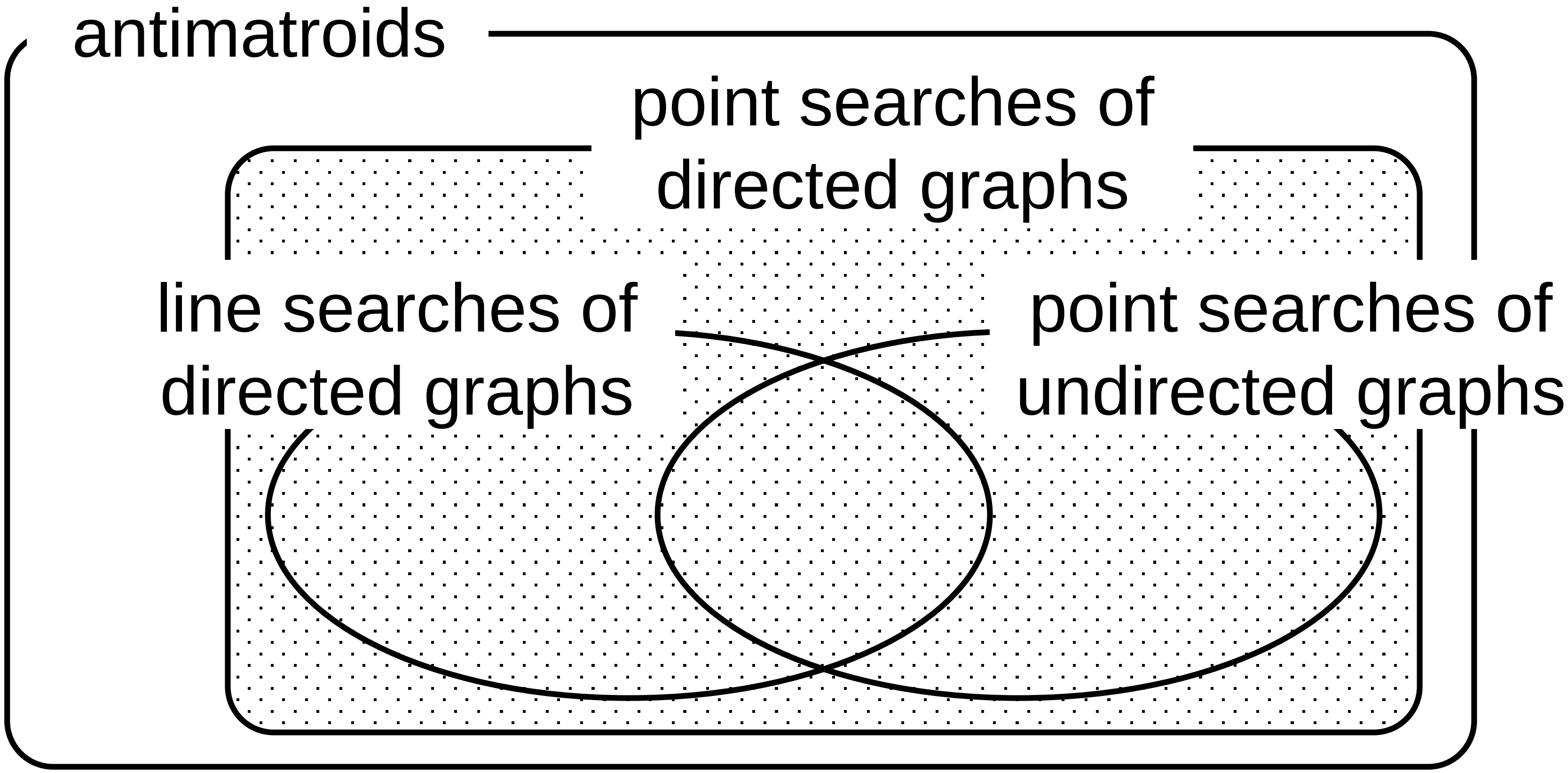
3 Forbidden-minor characterizations

Results

We give the forbidden-minor characterizations for the following classes of antimatroids:

- point searches of directed graphs
- point searches of undirected graphs
- line searches of directed graphs

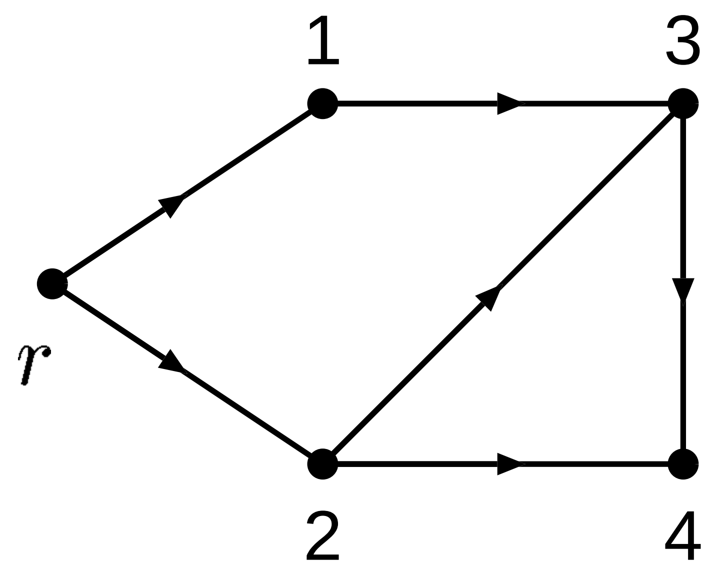
Relationship of the classes

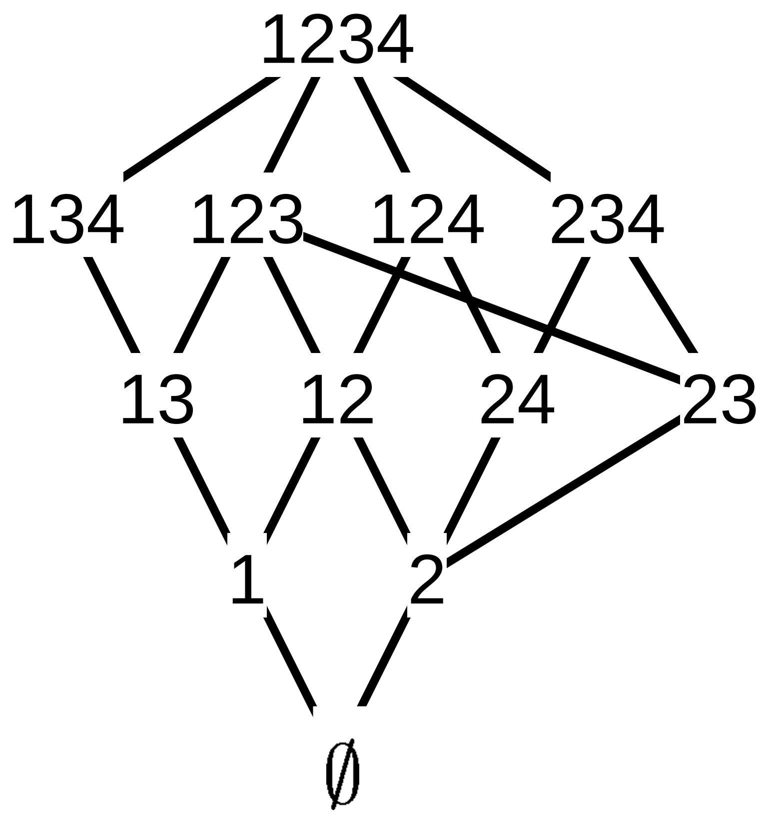


3.1 Point searches of directed graphs

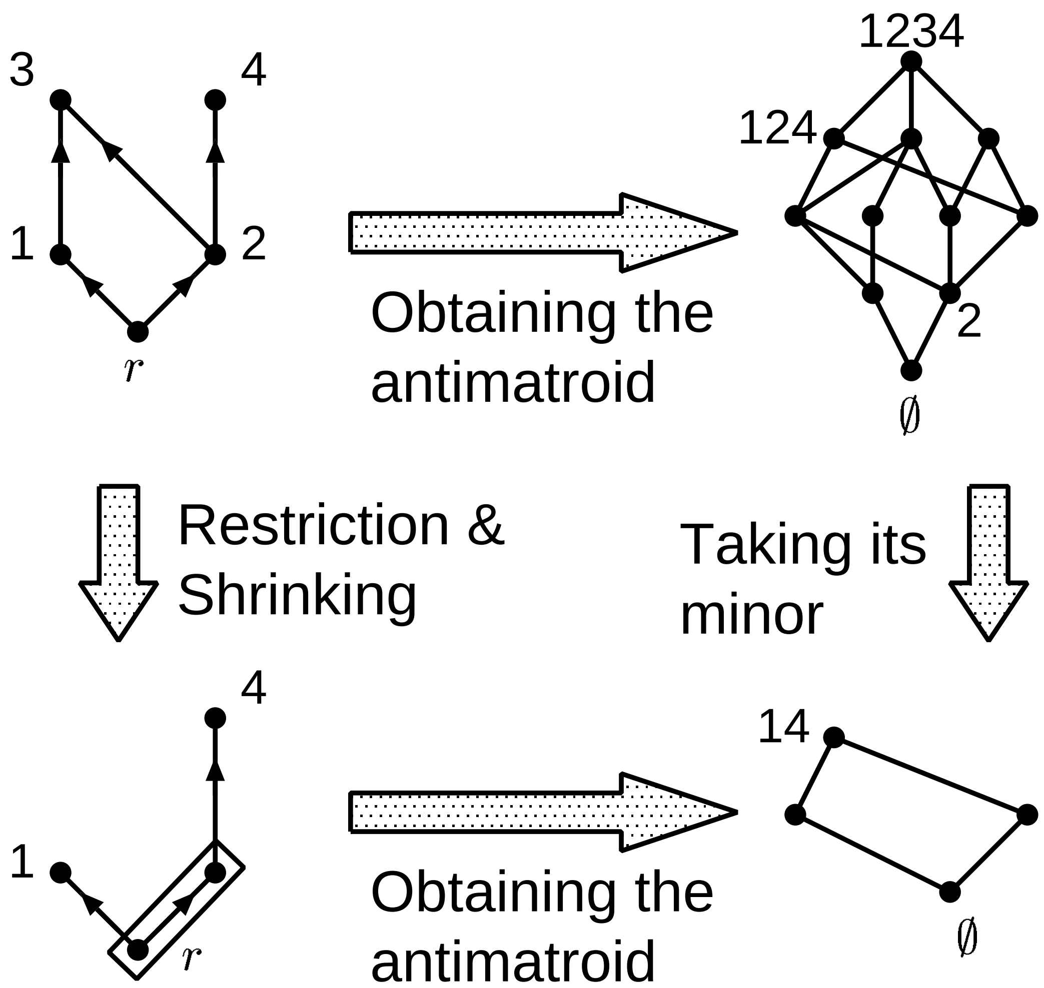
Definition

Search vertices from the root





Why is this class closed...?

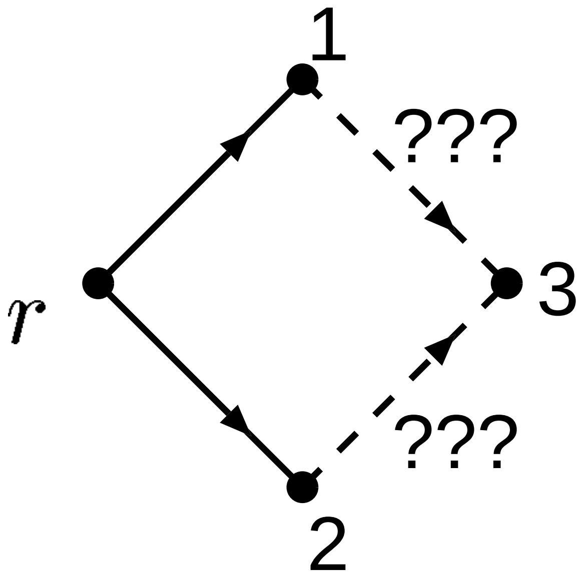


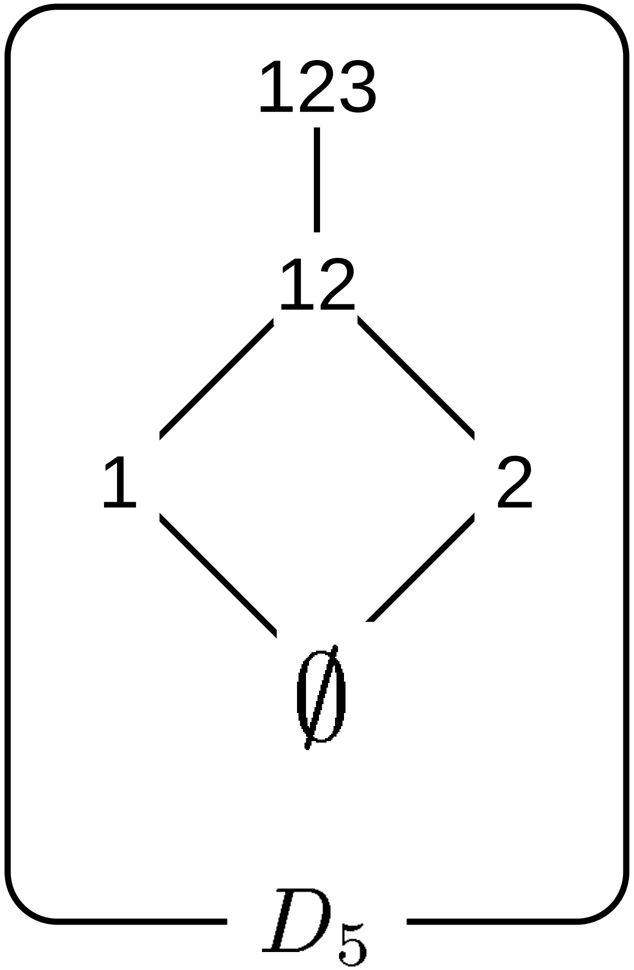
The resultant graph obtained by the leftmost procedure is called an <u>r-minor</u>.

A necessary condition

D_5 is not a point search of a directed graph.

Why?





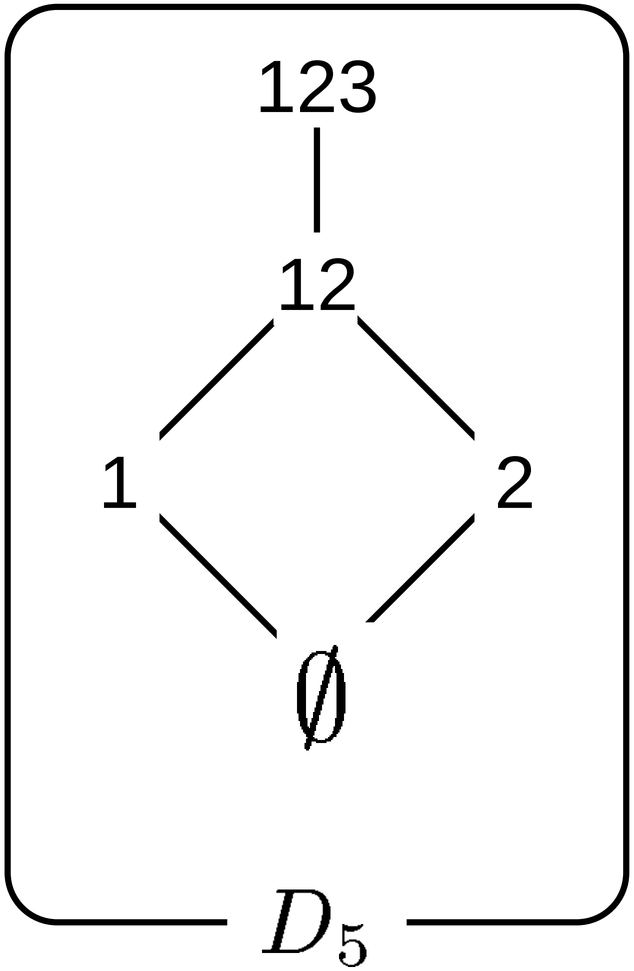
Theorem: Characterization

An antimatroid is the point search of a directed graph

$$\Updownarrow$$

It has no minor isomorphic to D_5

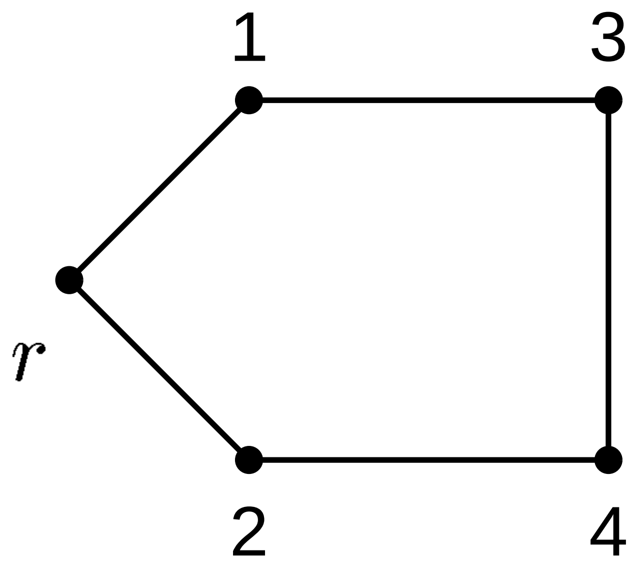
(Here, we omit the proof.)

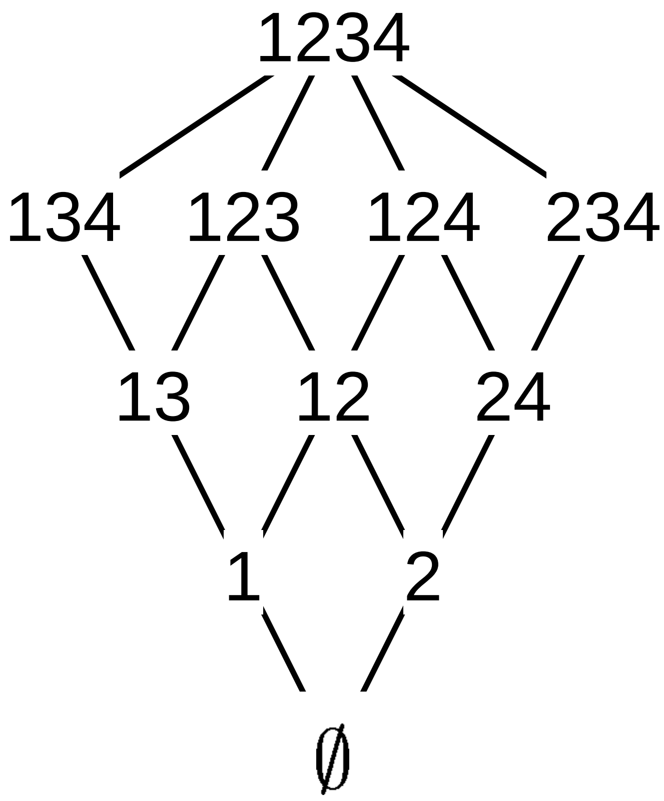


3.2 Point searches of undirected graphs

Definition

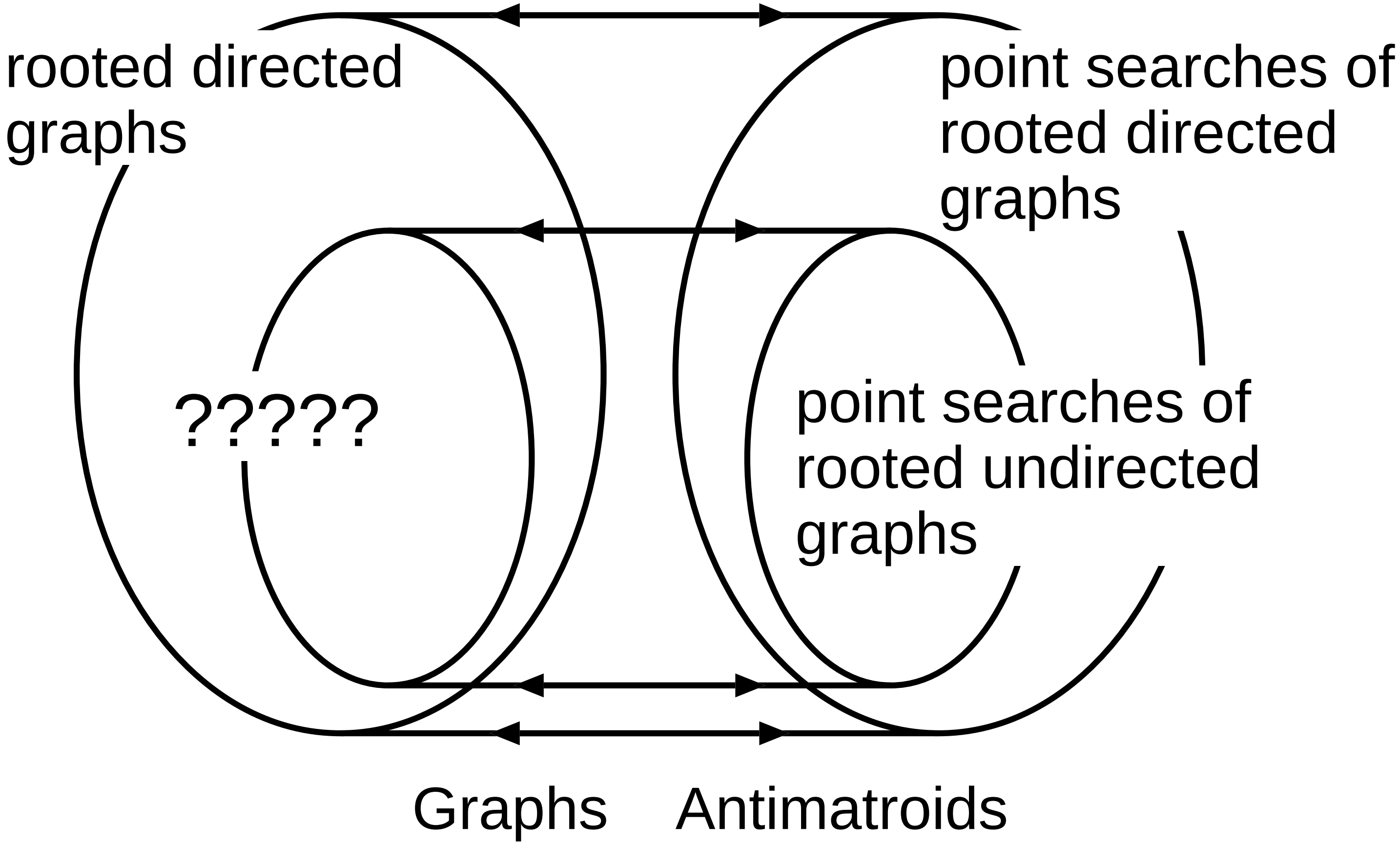
Search vertices from the root



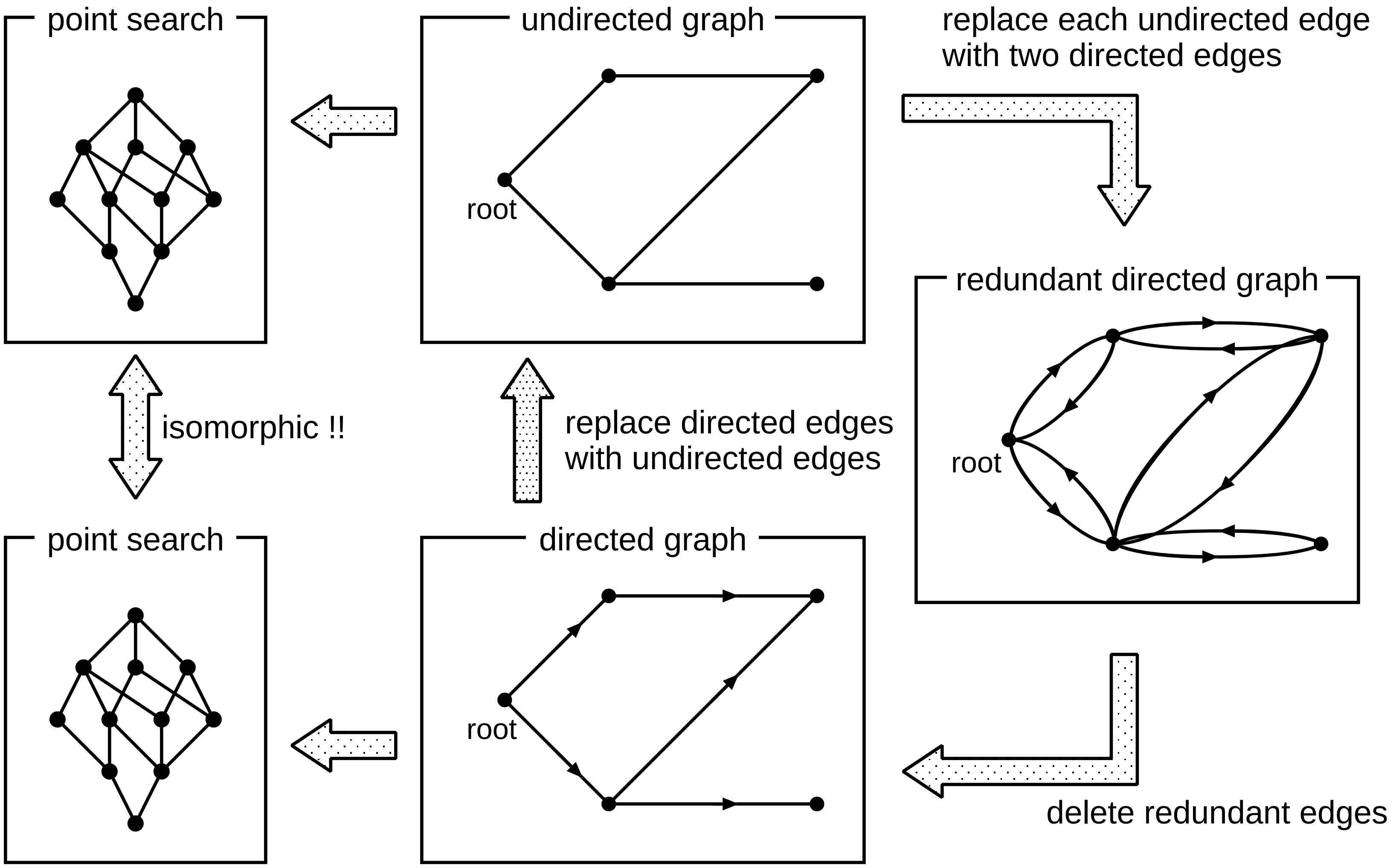


We can see that this class is closed under taking its minors.

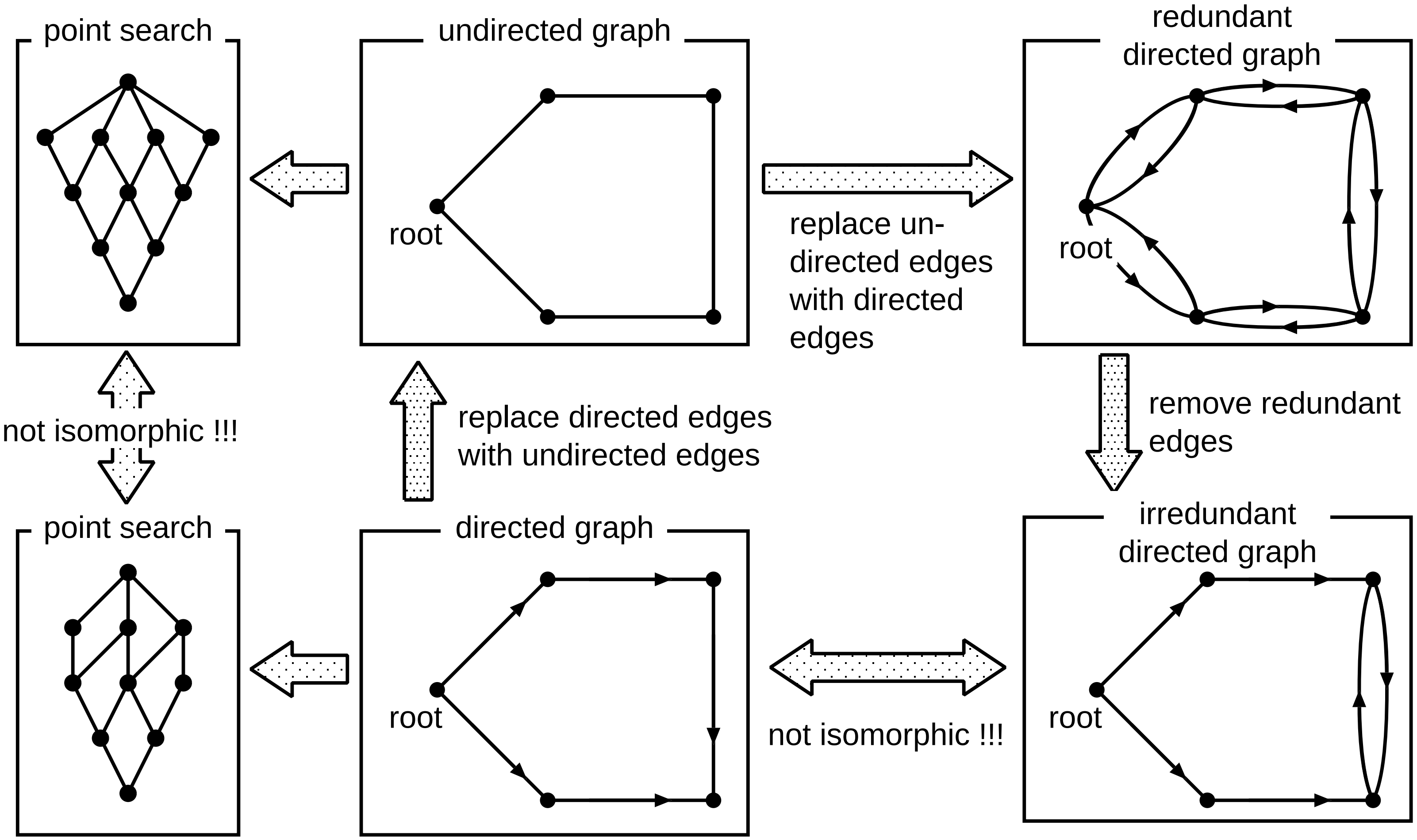
Relationship



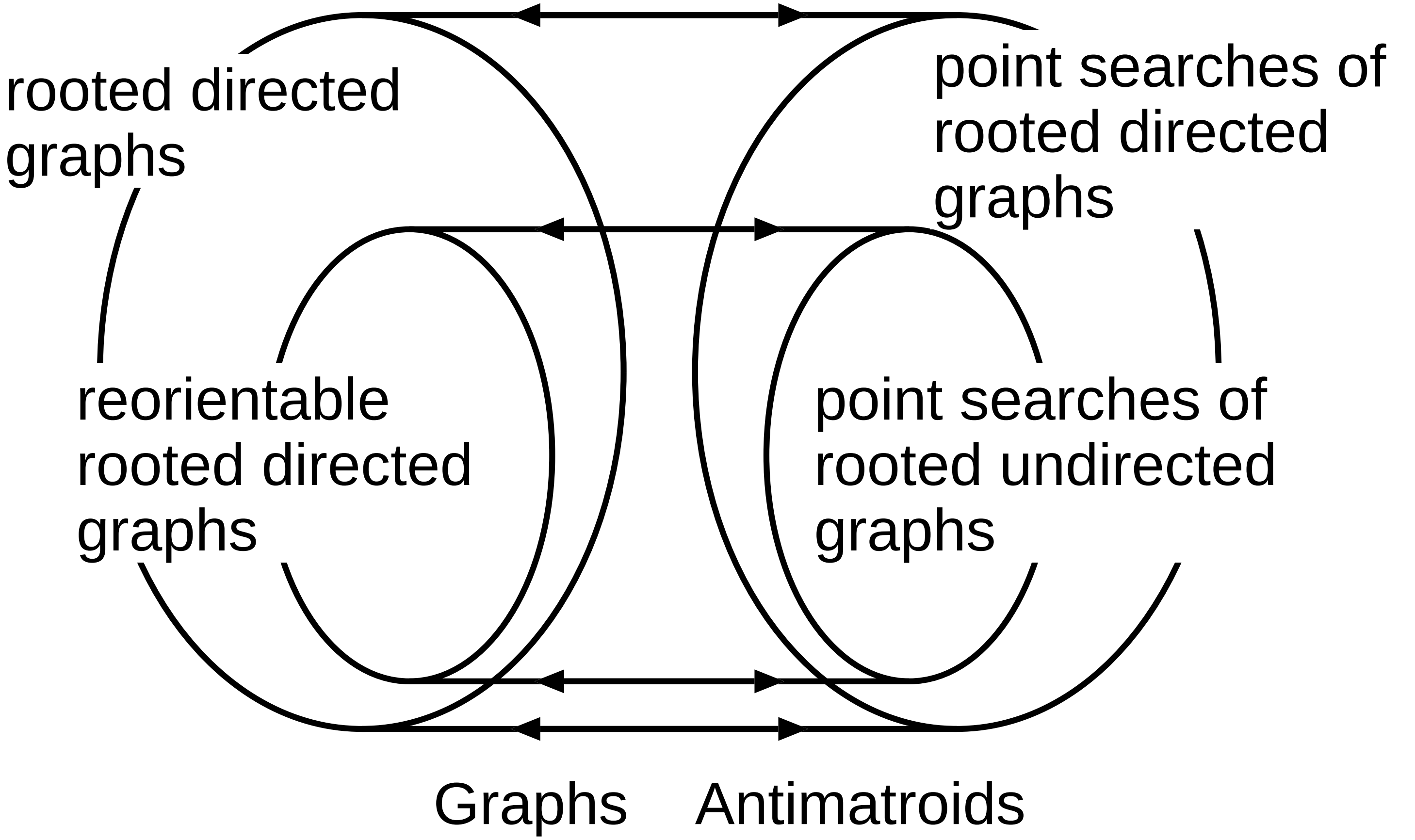
Replace : directed edge $\leftrightarrow$ undirected edge



Not reorientable graph



Relationship (again)



What we have to do

For graphs	For antimatroids
A directed graph is reorientable	The point search of a directed graph belongs to the class of point searches of undirected graphs
$\Updownarrow$	$\Updownarrow$
It has no r-minor isomorphic to $\bigcirc\bigcirc\bigcirc$	It has no minor isomorphic to $\triangle\triangle\triangle$

. . . .

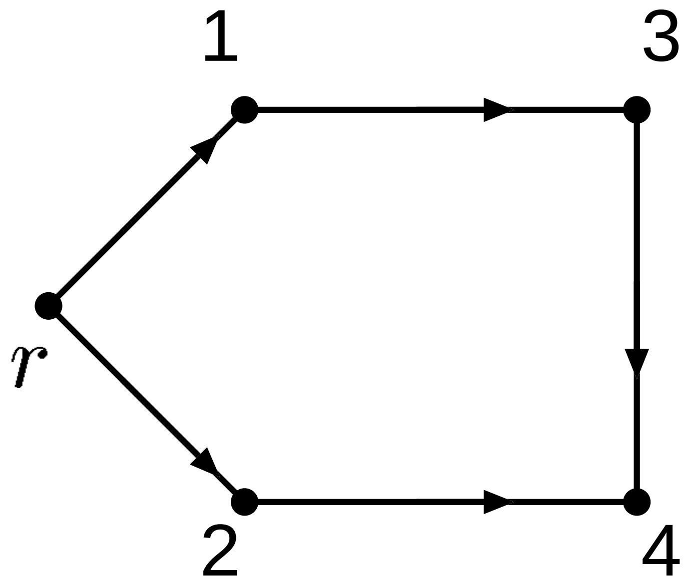
Now, we have to enumerate $\bigcirc\bigcirc\bigcirc$.

Theorem

a directed graph is reorientable

$$\Updownarrow$$

it has no r-minor isomorphic to G_5



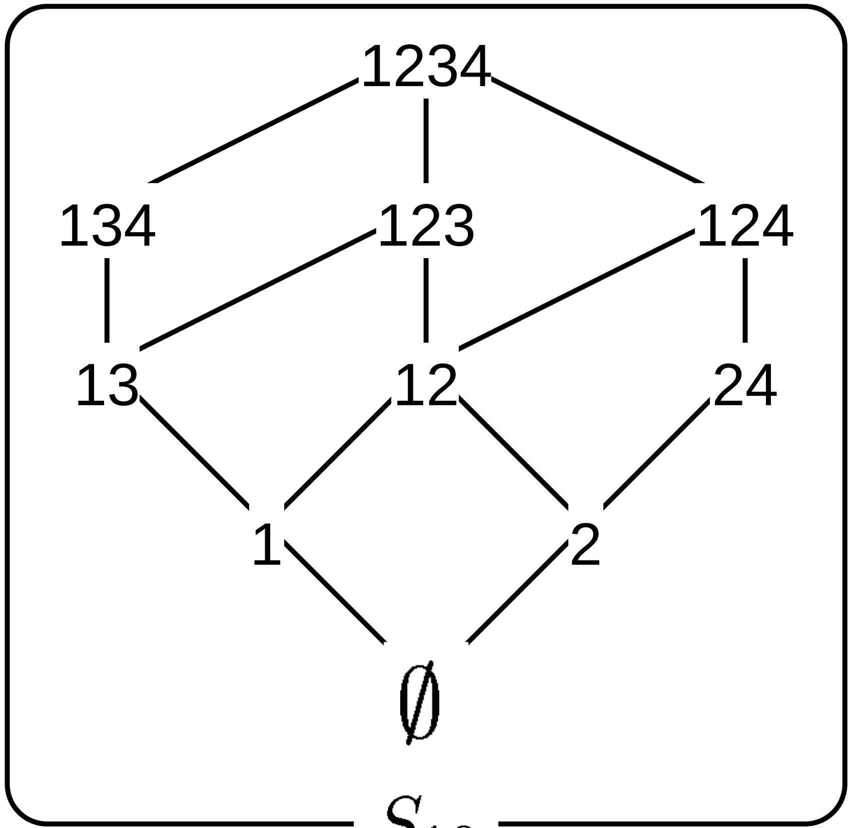
G_5

Forbidden-minor characterization

the point search of a directed graph belongs to the class of point searches of undirected graphs

$$\Updownarrow$$

it has no minor isomorphic to S_{10}

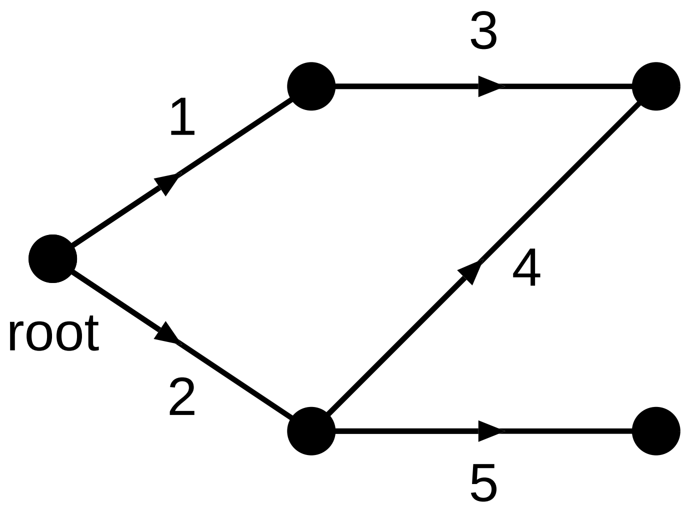


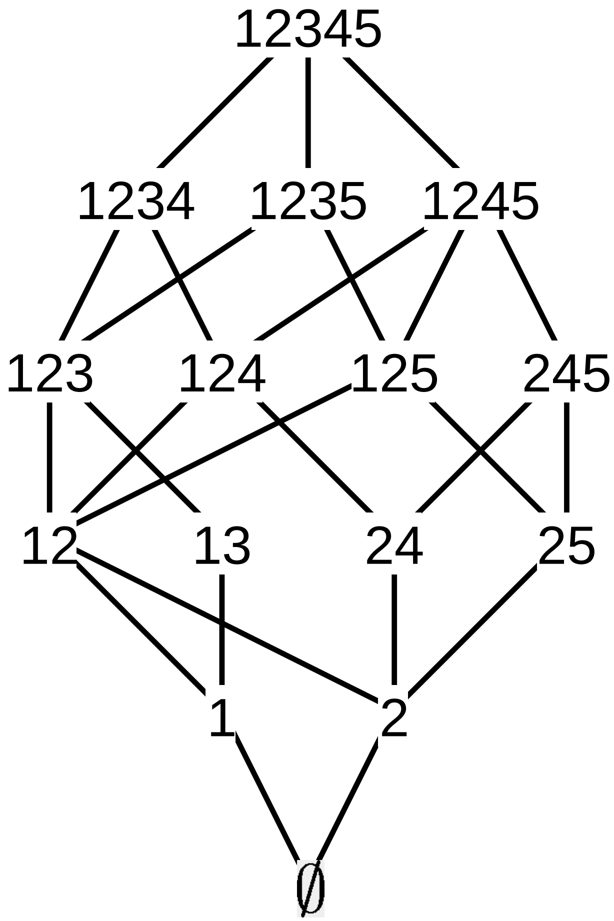
S_{10}

3.3 Line searches of directed graphs

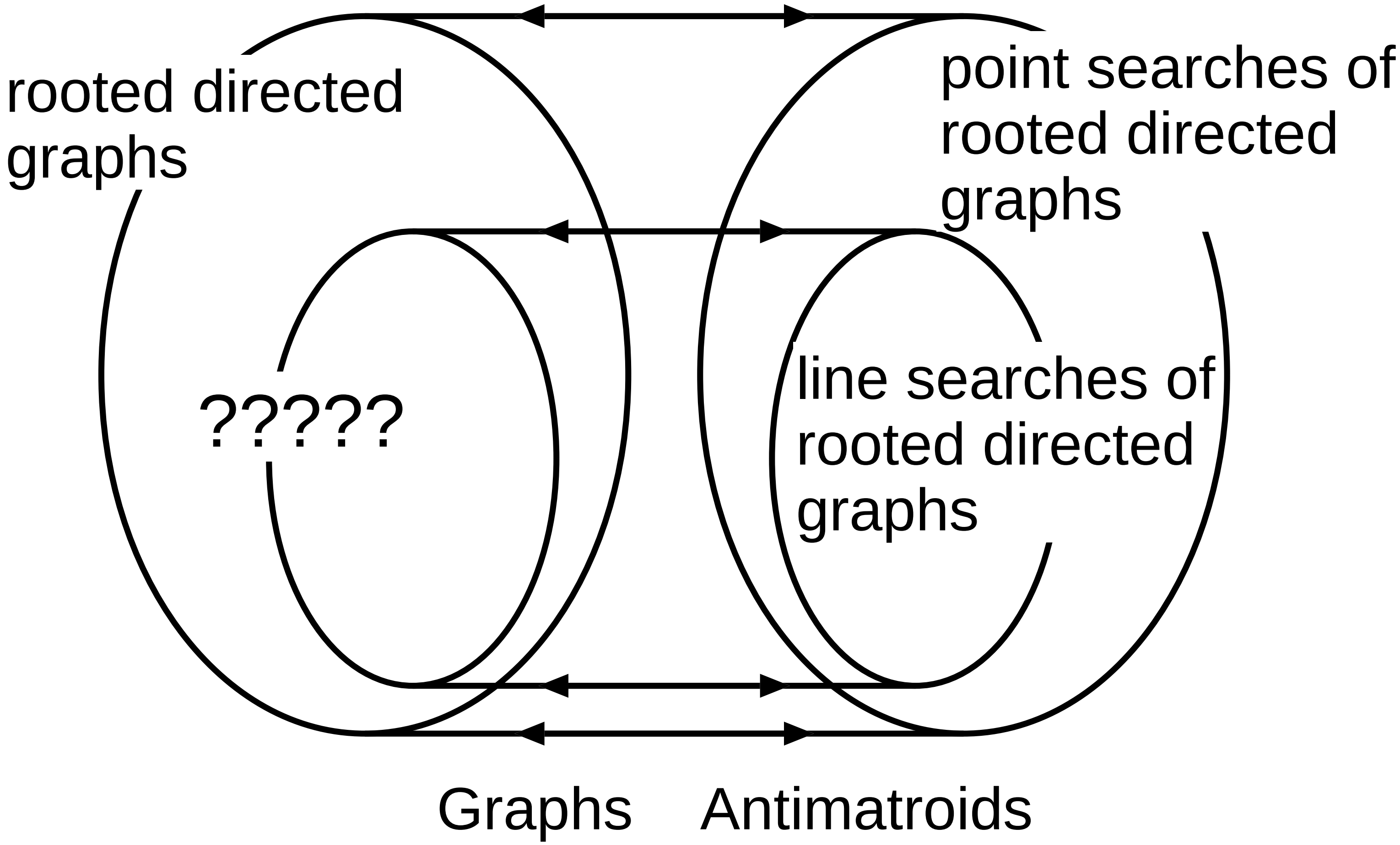
Definition

Search edges from the root

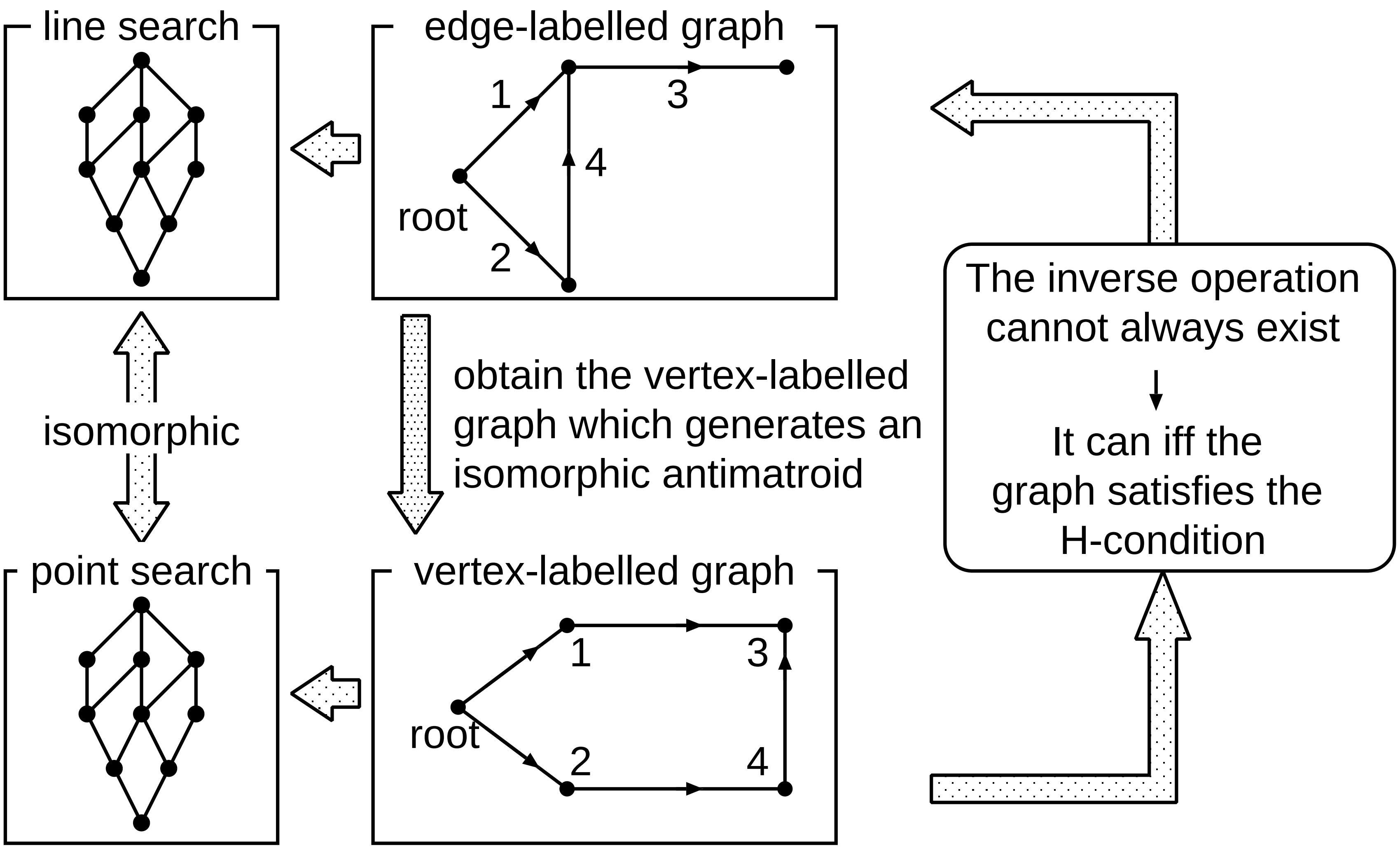




Relationship



Replace : edge-label $\leftrightarrow$ vertex-label

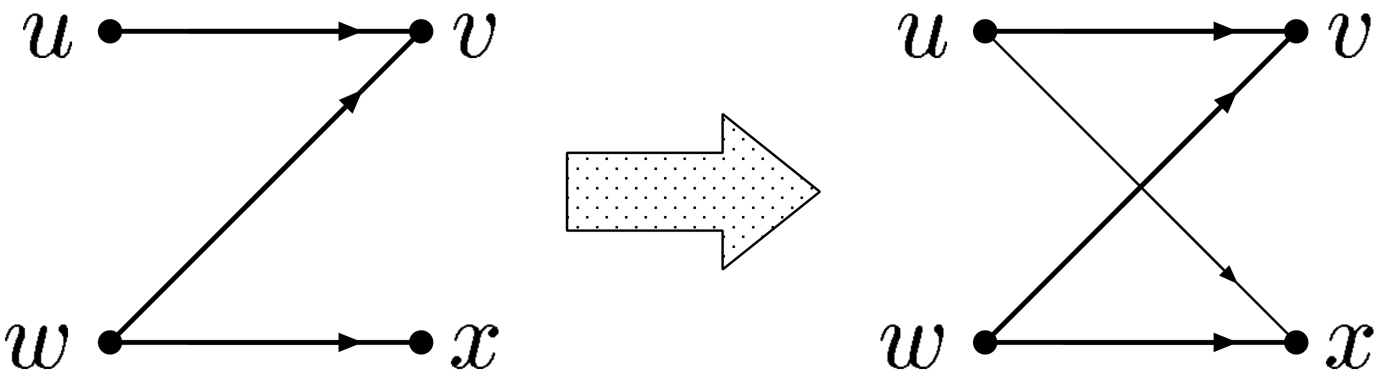


H-condition

The inverse operation exists for a vertex-labelled directed graph

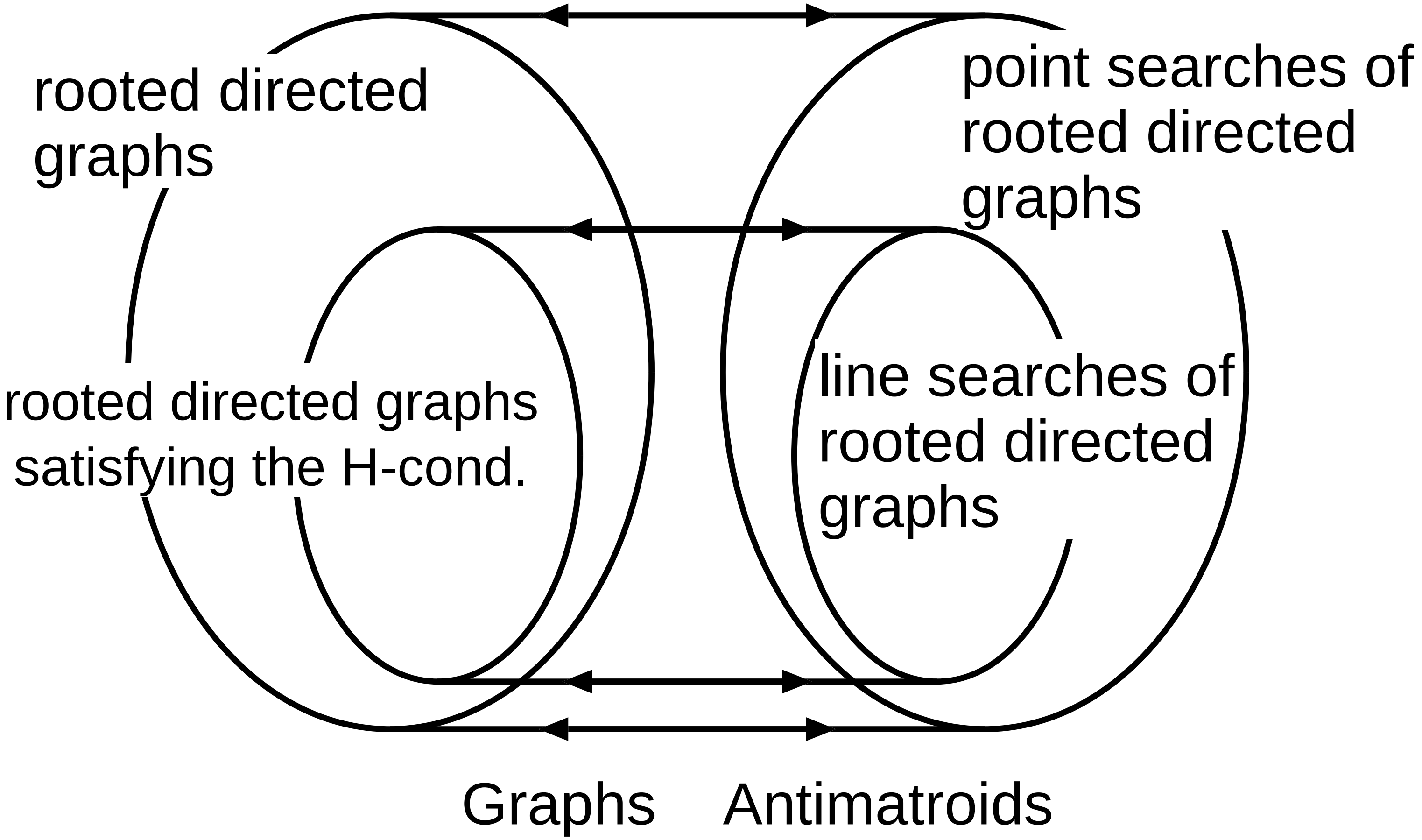
$$\Updownarrow$$

if it has u, v, w, x with edges shown in the left fig.,
then there must exist the additional edge shown in the right fig.



(Heuchenne,1964)

Relationship (again)



What we have to do

For graphs

A rooted directed graph satisfies the H-condition

$\Updownarrow$

It has no r-minor isomorphic to $\bigcirc\bigcirc\bigcirc$

For antimatroids

The point search of a directed graph belongs to the class of line searches of directed graphs

$\Updownarrow$

It has no minor isomorphic to $\triangle\triangle\triangle$

. . . .

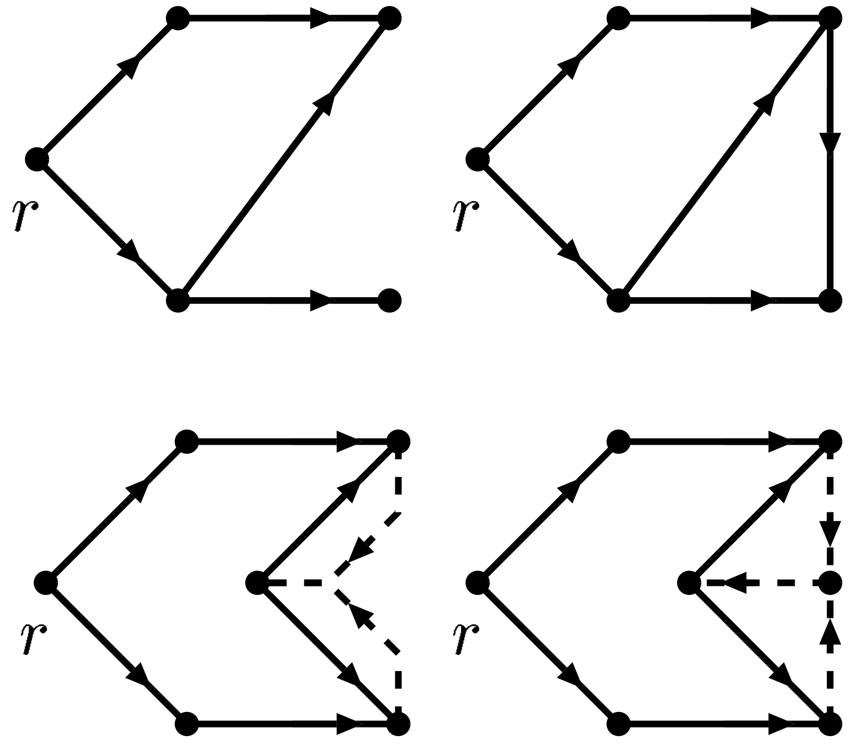
Now, we have to enumerate $\bigcirc\bigcirc\bigcirc$.

Forbidden-minor characterization

A rooted directed graph satisfies the H-condition

$$\Updownarrow$$

It has no r-minor isomorphic to the graphs shown in the Figure



(Broken arrows represent arbitrarily long paths)

Summary

We give the forbidden-minor characterizations for the following classes of antimatroids

- point searches of directed graphs
- point searches of undirected graphs
- line searches of directed graphs

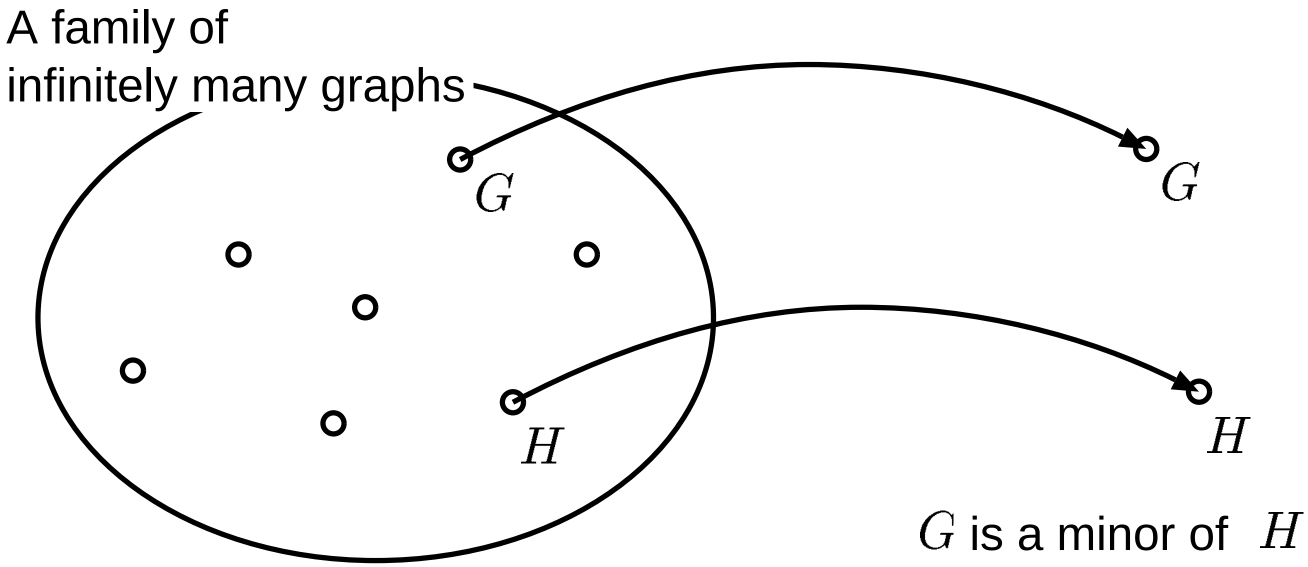
Remark

Open for line searches of undirected graphs.

Implication to the Minor Theorem

$\boxed{\text{Graph Minor Theorem}}$ (Robertson-Seymour, 1986–)

In any family of infinitely many graphs, there exists two graphs such that one is a minor of the other.



$\boxed{\text{How about antimatroids}}$ $\rightarrow$ It does not hold !!