A Greedy Algorithm for Convex Geometries

Yoshio Okamoto &

Kenji Kashiwabara

Department of Systems Science

Graduate School of Arts & Sciences

University of Tokyo

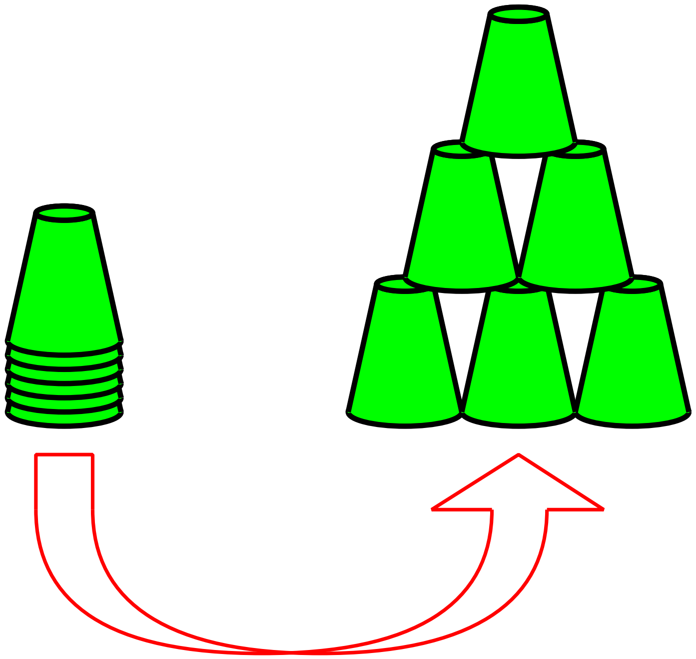
Our Result in One Sentence

We extend Krüger's work to convex geometries

contents

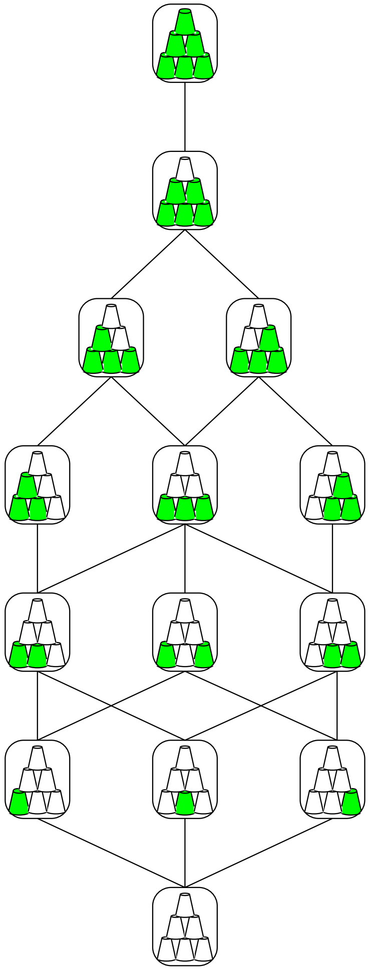
1 Convex Geometries

cupstacks



Stack the cups and unstack them as quickly as possible

the orders to unstack the cups



We can see the precedence relation among the cups

$\Downarrow$

Abstraction of precedence relation

$\Downarrow$

Convex geometries

convex geometries

E : finite set

$\mathcal{L} \subseteq 2^E$: a family of subsets of E s.t.

- $\emptyset \in \mathcal{L}, E \in \mathcal{L}$;
- $X \in \mathcal{L} \setminus \{E\} \Rightarrow \exists x \in E \setminus X$ s.t. $X \cup \{x\} \in \mathcal{L}$;
- $X, Y \in \mathcal{L} \Rightarrow X \cap Y \in \mathcal{L}$;

$\mathcal{L}$: convex geometry on E

$X \in \mathcal{L}$: convex set

E : the ground set of $\mathcal{L}$

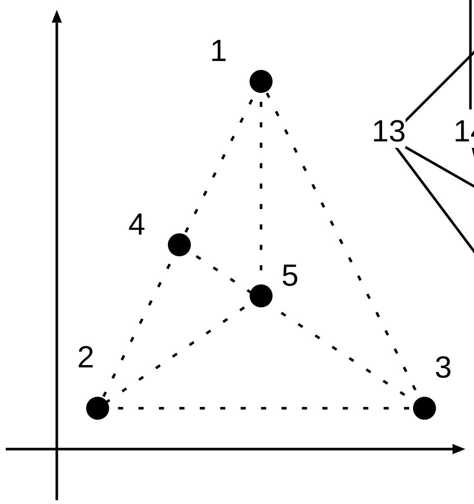
example of convex geometries (1)

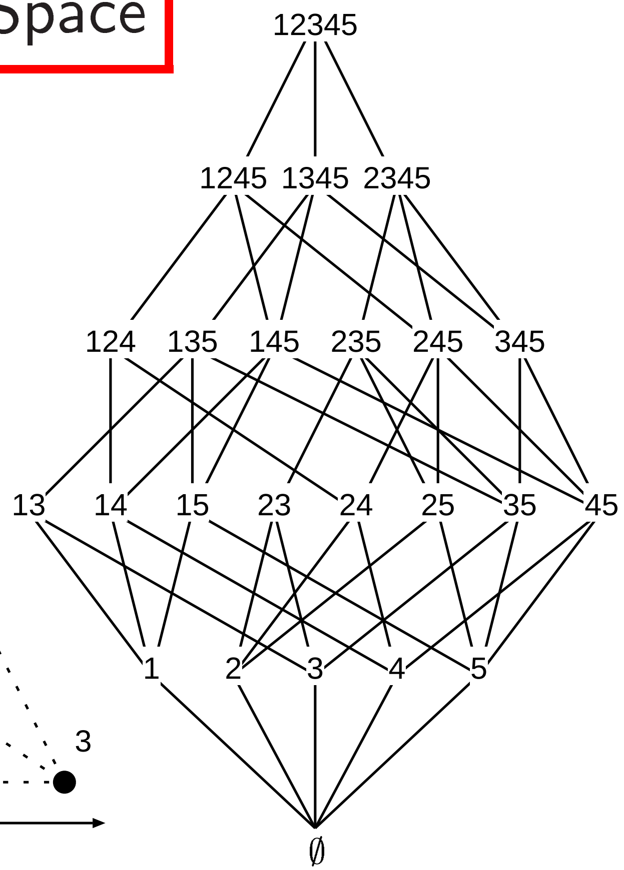
Convex Shellings on the Euclidean Space

E : a finite set of points in $\mathbb{R}^n$

Delete a vertex of the convex hull

step by step



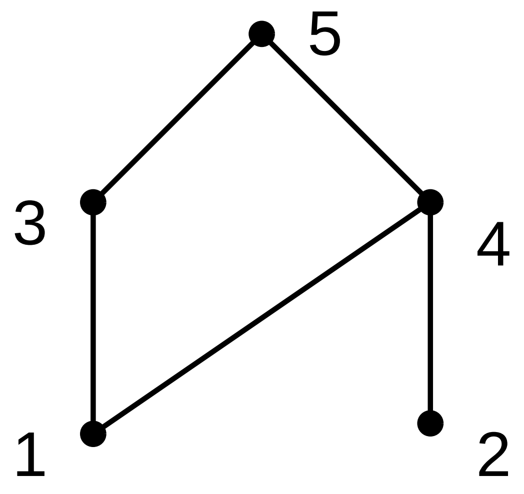


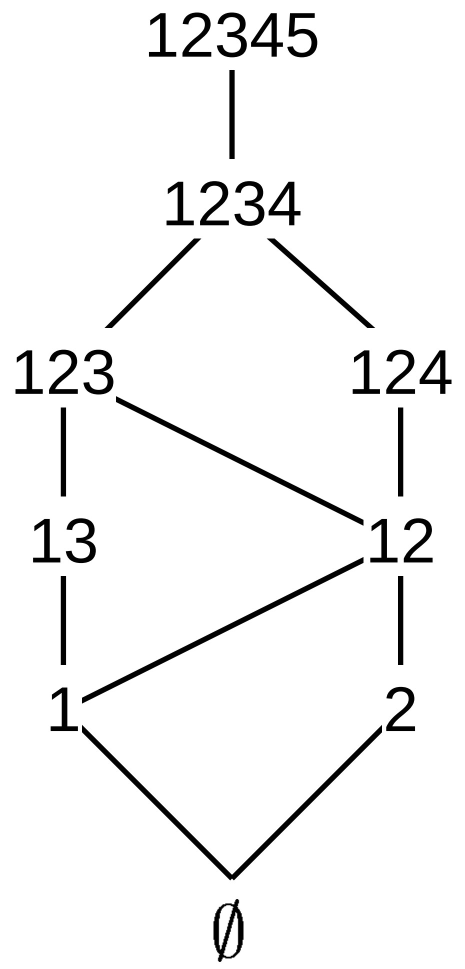
example of convex geometries (2)

Poset Shellings

E : a partially ordered set (poset)

Delete a max. element step by step



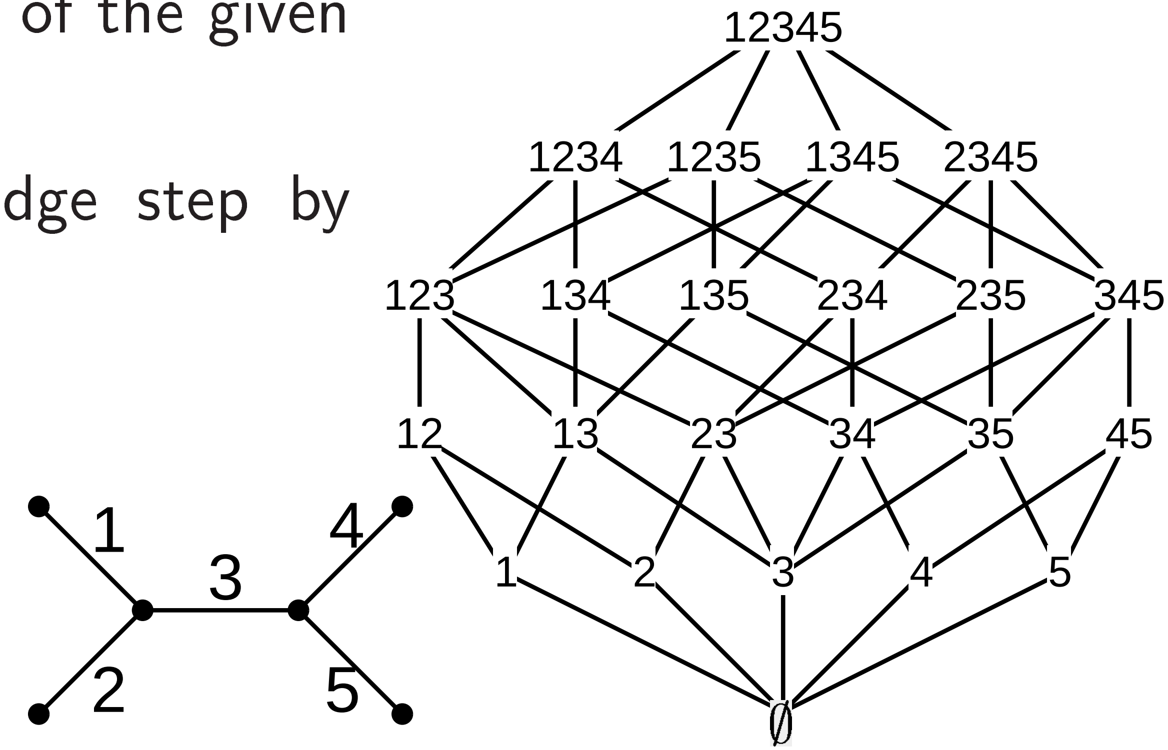


example of convex geometries (3)

Edge Shellings of Trees

E : the edge set of the given tree

Delete an end-edge step by step



some classes of convex geometries

- node shellings of trees
- double poset shellings
- point searches of rooted directed graphs
- point searches of rooted undirected graphs
- simplicial schemes of chordal graphs
- transitive schemes of transitive graphs

$\vdots$

$\Rightarrow$ Various combinatorial objects yield convex geometries

extreme sets

$\mathcal{L}$: convex geometry on E

$X \in \mathcal{L}$

$\boxed{\text{ex}(X) = \{a \in X : X \setminus \{a\} \in \mathcal{L}\}}$: the extreme set of X

$\text{ex}(\mathcal{L}) = \{\text{ex}(X) : X \in \mathcal{L}\}$

$\Rightarrow$ $\text{ex} : \mathcal{L} \to \text{ex}(\mathcal{L})$ is bijective.

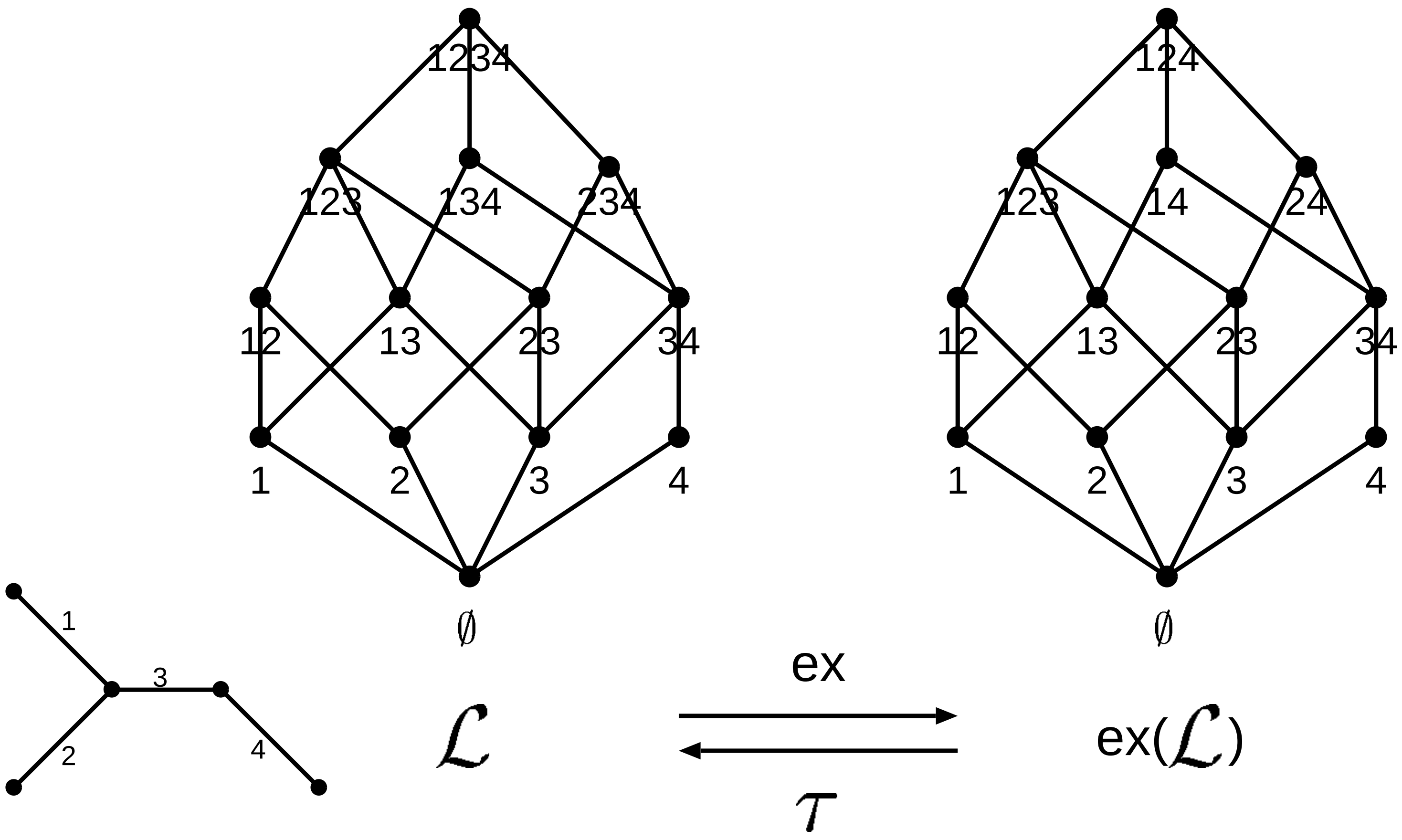
τ : the inverse of $\text{ex} : \mathcal{L} \to \text{ex}(\mathcal{L})$

$\preceq$: order on $\text{ex}(\mathcal{L})$

$\text{ex}(X) \preceq \text{ex}(Y) \Leftrightarrow X \subseteq Y$

$\Rightarrow$ $\text{ex}(\mathcal{L})$ is a lattice isomorphic to $\mathcal{L}$.

example : extreme sets



★ex($\mathcal{L}$) is a lattice isomorphic to $\mathcal{L}$.

2 Review of Krüger's Work

set functions & associated polyhedra

E : finite set

$\mathcal{L}$: convex geometry on E

f : ex$(\mathcal{L}) \to \mathbb{R}$ with $f(\emptyset) = 0$

$$\mathbb{P}(f) = \left\{ x \in \mathbb{R}^E : \sum_{e \in X} x(e) \leq f(X) \quad \forall X \in \mathrm{ex}(\mathcal{L}) \right\}$$

$\mathbb{P}(f)$: polyhedron associated with f

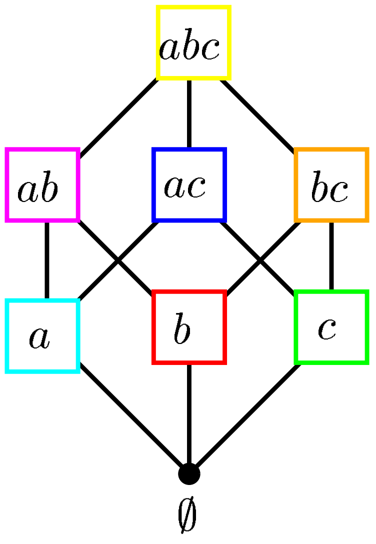
maximization on ex$(\mathcal{L})$

[Problem (P)]

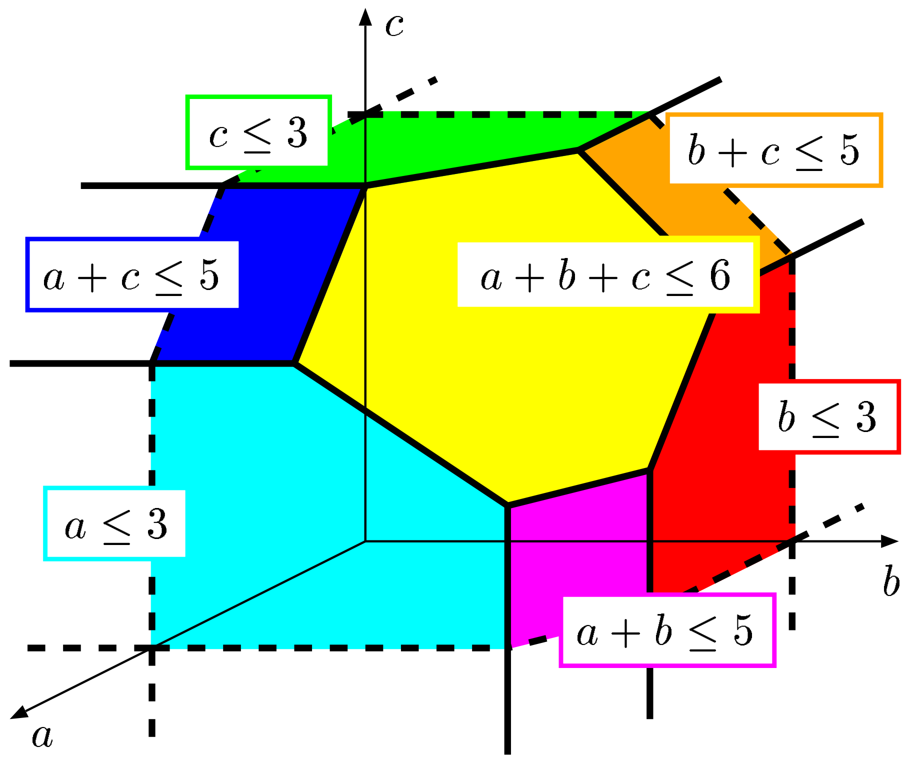
For a non-negative vector $c \in \mathbb{R}_+^E$

$$\boxed{\begin{array}{ll} \text{Max} & \displaystyle\sum_{e \in E} c(e)x(e) \\ \text{s.t.} & x \in \mathbb{P}(f) \end{array}}$$

example of associated polyhedra (1)



2^E



Submodular Polyhedron

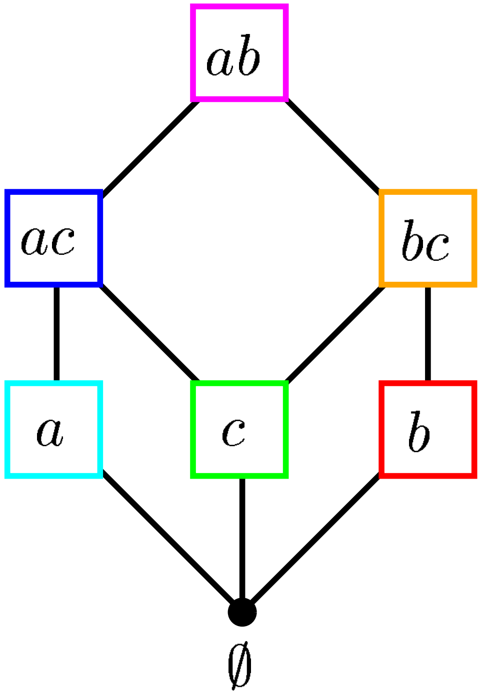
Faigle-Kern's dual greedy algorithm (G)

To solve the dual problem of (P) (Faigle-Kern, 1996)

$i \leftarrow 1$;
<u>while</u> $c \neq 0$ <u>do</u> <u>begin</u>
    $A_i \leftarrow \mathrm{ex}(\{e : c(e) > 0\})$;
    $\lambda_i \leftarrow \min\{c(e) : e \in A_i\}$; $(\lambda > 0)$
    $c \leftarrow c - \lambda_i \chi^{A_i}$; ($\chi^A$: char. vector of A)
    $i \leftarrow i + 1$;
<u>end of while</u>;
<u>compute</u> $\sum \lambda_i f(A_i)$

---

how (G) works (1)



$$
\begin{aligned}
c &= \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \\
&= 3\times \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \\
&= 3\times \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + 2\times \boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \\
&= 3\times \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + 2\times \boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \\
&\quad + 2\times \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}
\end{aligned}
$$

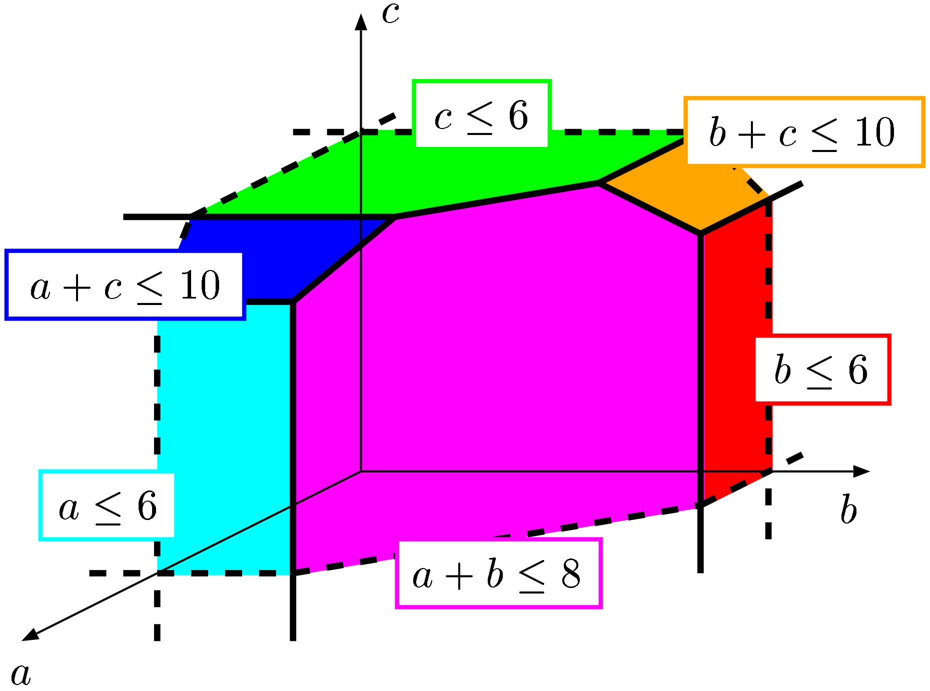
Now we have decomposed the input c.

Then we compute the output value.

how (G) works (2)

Output

$$= 3\times f(\{a,b\}) + 2\times f(\{b,c\}) + 2\times f(\{c\}) = 3\times 8 + 2\times 10 + 2\times 6 = 56$$



In this example, $\boxed{\text{output} = \text{optimal value}}$.

But, this is not always the case.

property of the dual greedy algorithm (G)

$\mathcal{L}$: convex geometry

f : ex($\mathcal{L}$) $\to \mathbb{R}$

(G) works $\Leftrightarrow$

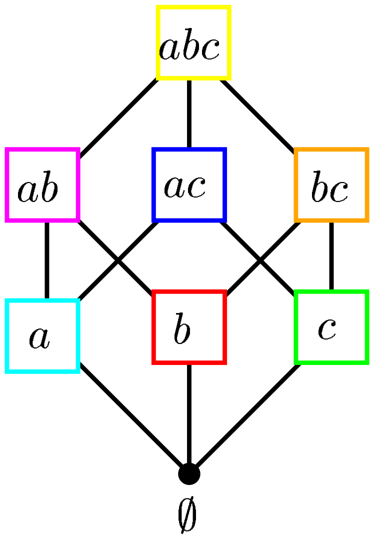
$\forall$ maximal chain $\mathcal{C} : \emptyset \prec X_1 \prec \ldots \prec X_n$ of ex($\mathcal{L}$)

the unique solution of

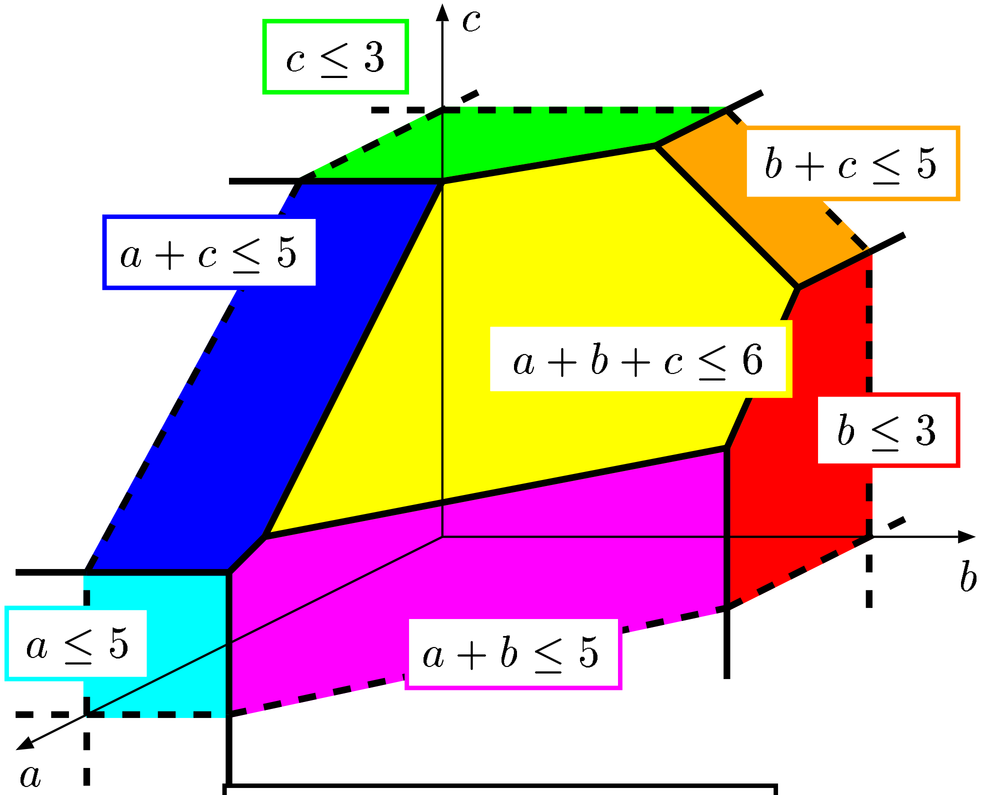
$$\sum_{e \in X_i} x(e) = f(X_i) \quad i = 1, \ldots, n$$

is a vertex of $\mathbb{P}(f)$

example of associated polyhedra (2)



2^E



Not Submodular Polyhedron

history of this problem (1)

1983	Lovász
	$\mathcal{L} = 2^E$
	f is submodular $\Leftrightarrow$ (G) works

f is submodular $\Leftrightarrow$

$$\boxed{f(X) + f(Y) \geq f(X \vee Y) + f(X \wedge Y)} \qquad \forall X, Y \in 2^E$$

history of this problem (2)

2000	Krüger
	$\mathcal{L}$: poset shelling
	f is b-submodular $\Leftrightarrow$ (G) works

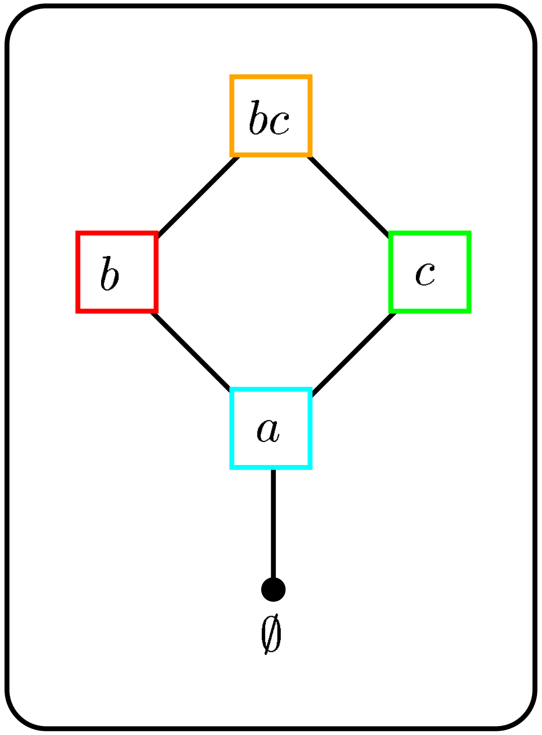
f is b-submodular $\Leftrightarrow$

$$\boxed{f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y)} \quad \forall X, Y \in \mathrm{ex}(\mathcal{L})$$

where $X \sqcap Y = (X \wedge Y) \cap (X \cup Y)$

(the reduced meet of X & Y)

reduced meet



meet

$$\{b\} \wedge \{c\} = \{a\}$$

reduced meet

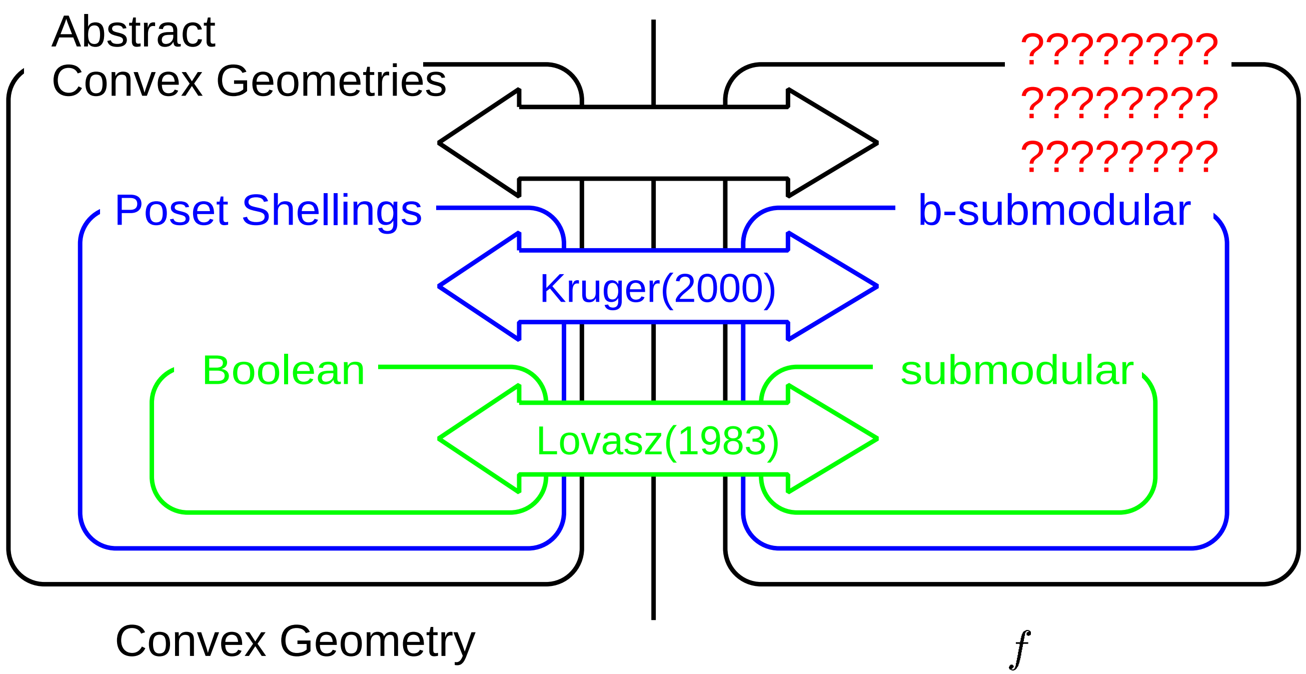
$$\begin{aligned}\{b\} \sqcap \{c\} & \\ &= (\{b\} \wedge \{c\}) \cap (\{b\} \cup \{c\}) \\ &= \{a\} \cap \{b, c\} \\ &= \emptyset\end{aligned}$$

- $|X| + |Y| \quad \leq \quad |X \vee Y| + |X \wedge Y|$
- $|X| + |Y| \quad = \quad |X \vee Y| + |X \sqcap Y|$

3 Our Main Result

motivation...

How can we extend Krüger's result to any convex geometry ?



main result

[Theorem]

2000 Kashiwabara & Okamoto

$\mathcal{L}$: any convex geometry

f is c-submodular $\Leftrightarrow$ (G) works

★ Thus, we extend the class of problems which are handled by the same framework as Krüger.

c-submodular functions

$f : \mathrm{ex}(\mathcal{L}) \to \mathbb{R}$

f is c-submodular $\Leftrightarrow$

$$\boxed{f(X) + f(Y) \geq f(X \vee Y) + f(X \sqcap Y) + f(X \diamond Y)}$$

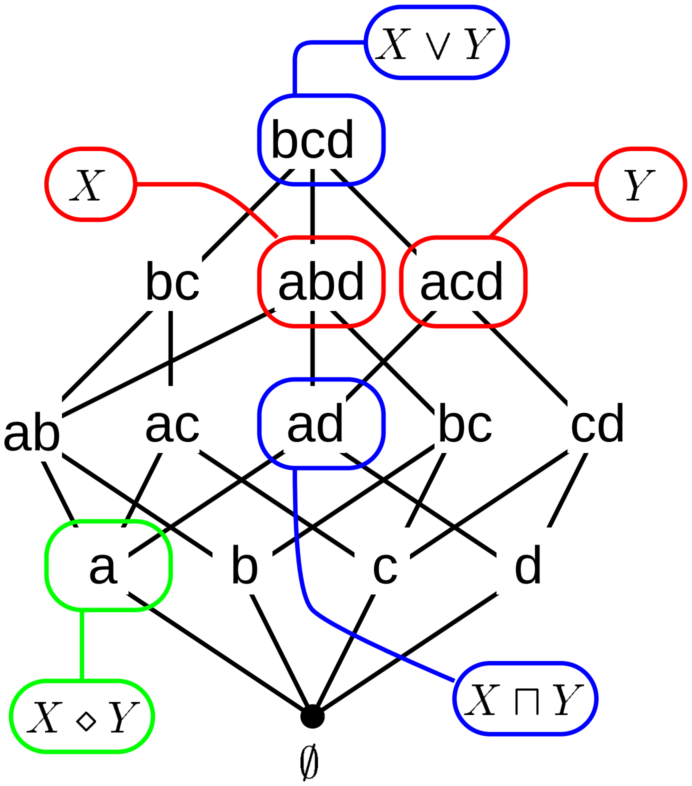
$\forall X, Y \in \mathrm{ex}(\mathcal{L})$ s.t. $\chi^X + \chi^Y = \chi^{X \vee Y} + \chi^{X \sqcap Y} + \chi^{X \diamond Y}$

where

$X \sqcap Y = (X \wedge Y) \cap (X \cup Y)$: the reduced meet of X & Y

$X \diamond Y = (X \cap Y) \setminus (X \vee Y)$: the residue of X & Y

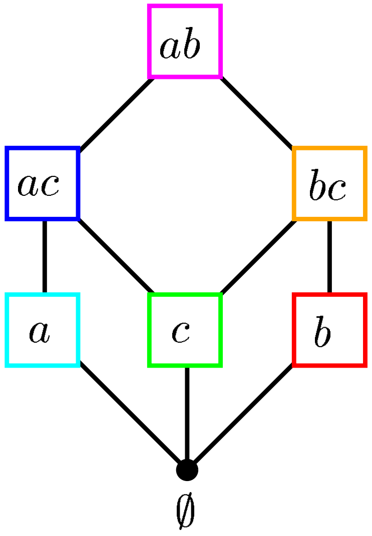
residue



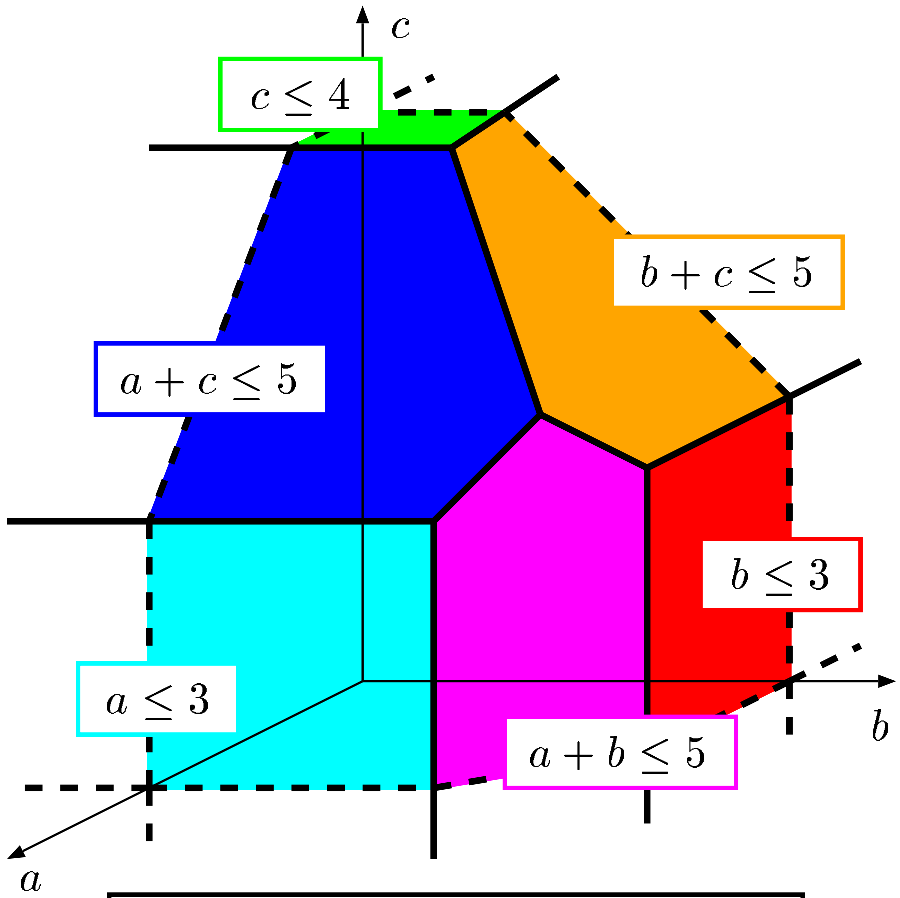
residue

$$
\begin{aligned}
\{a,b,d\} \diamond \{a,c,d\} &= (\{a,b,d\} \cap \{a,c,d\}) \\
&\qquad \setminus (\{a,b,d\} \vee \{a,c,d\}) \\
&= \{a,d\} \setminus \{b,c,d\} \\
&= \{a\}
\end{aligned}
$$

example of associated polyhedra (3)

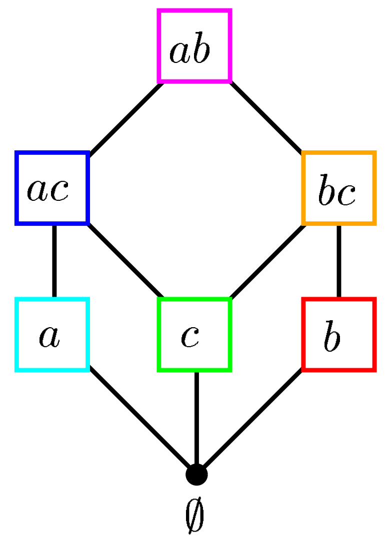


ex($\mathcal{L}$)

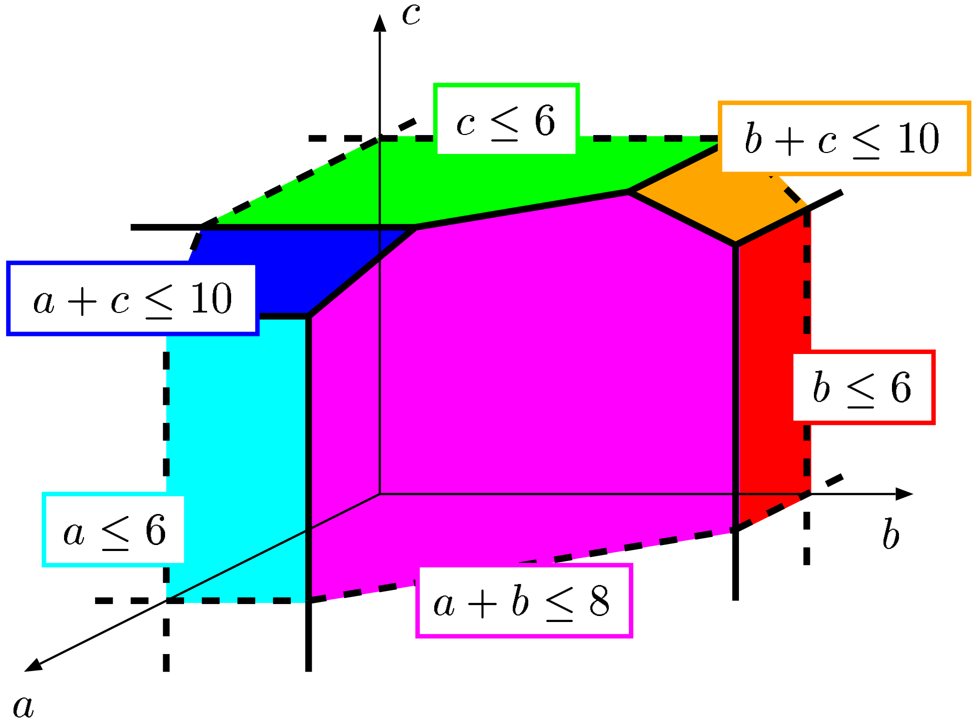


not c but b-submodular polyhedron

example of associated polyhedra (4)

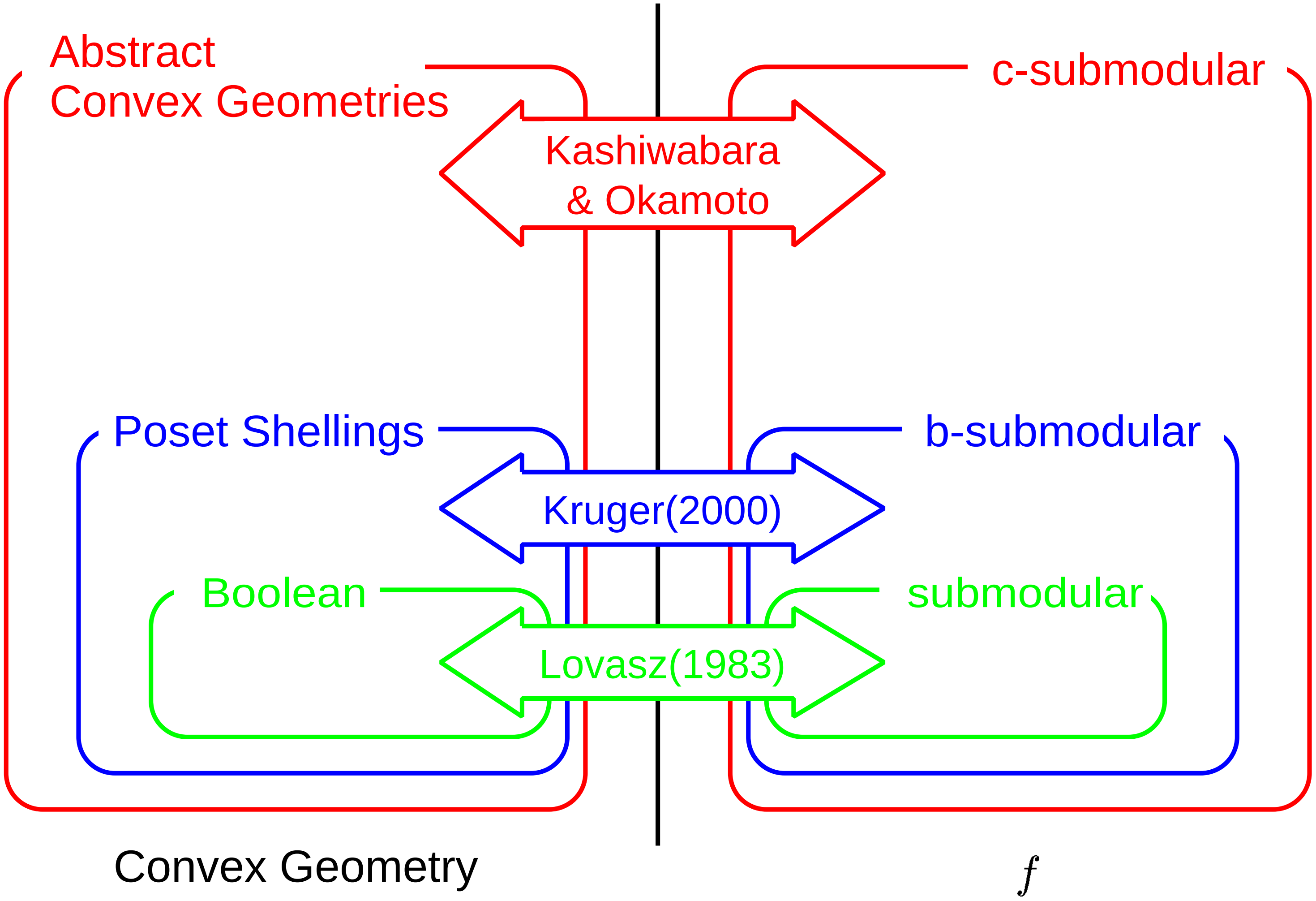


ex($\mathcal{L}$)



c-submodular polyhedron

relationship of the 3 works



4 Other Results

c-submodular functions & poset shellings

[Proposition]

$\mathcal{L}$: a convex geometry

the class of c-submodular functions

$=$ the class of b-submodular functions

$\Leftrightarrow$ $\mathcal{L}$ is a poset shelling

---

totally dual integrality

[Proposition]

$\mathcal{L}$: any convex geometry

f : ex$(\mathcal{L}) \to \mathbb{R}$

f is c-submodular $\Rightarrow \mathbb{P}(f)$ is totally dual integral

In particular,

f is integer-valued $\Rightarrow$ any vertex of $\mathbb{P}(f)$ is integral.

characterization of poset shellings

[Corollary]

$\mathcal{L}$: any convex geometry

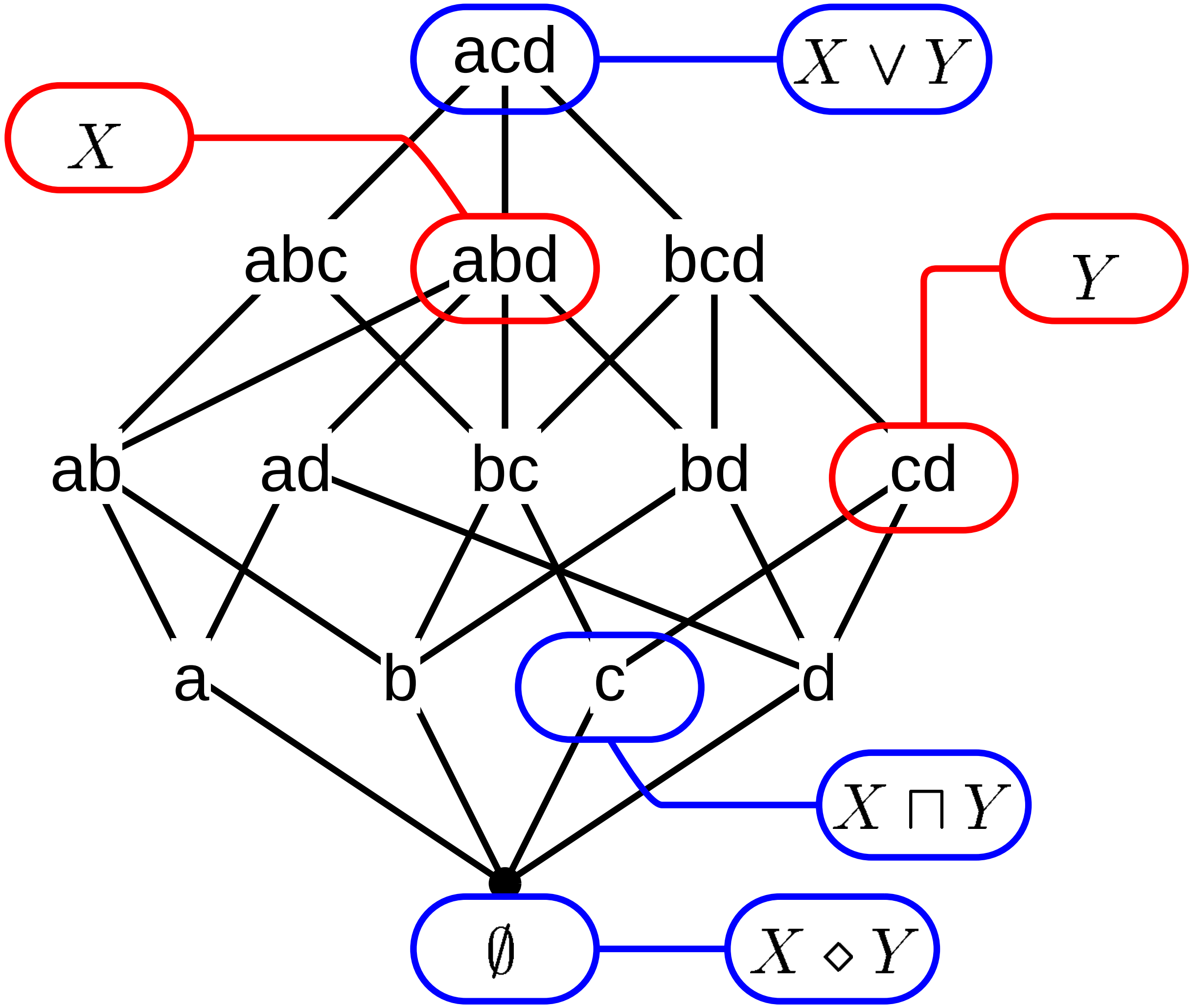
$\mathcal{L}$ is a poset shelling

$\Leftrightarrow$ (G) works for any b-submodular f

---

5 For Q & A

why we need that...



locally submodular

$f : 2^E \to \mathbb{R}$

Following two conditions are equivalent:

1. [submodularity]
   $$f(X) + f(Y) \geq f(X \vee Y) + f(X \wedge Y) \quad \forall X, Y \in 2^E$$
2. [local submodularity]
   $$f(X) + f(Y) \geq f(X \vee Y) + f(X \wedge Y) \quad \forall X, Y \in 2^E$$
   s.t.
   $$|(X \vee Y) \setminus X| = |(X \vee Y) \setminus Y| = 1,$$
   $$|X \setminus (X \wedge Y)| = |Y \setminus (X \wedge Y)| = 1$$