The Geodesic Diameter of Polygonal Domains

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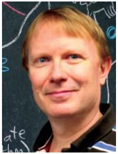
Matias gave talks at JCCGG '09, EWCG '10, ESA '10

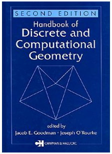
Motivation and result

Joe Mitchell said...

In Handbook of Discrete and Computational Geometry ('97, '04)

> **How efficiently can one compute a geodesic diameter for a polygonal domain?**





Motivation and result

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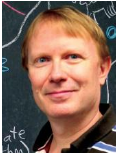
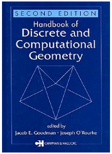
How efficiently can one compute a geodesic diameter for a polygonal domain?

Result

- Interesting geometric observations (that constitute the main theorem)
- The first polynomial-time algorithm for this problem





An improvement claimed, but ...

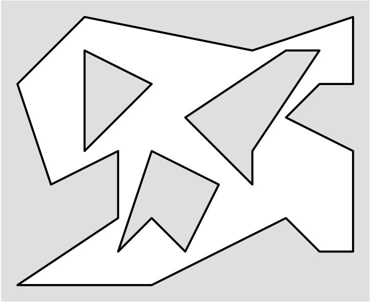
Claimed improvement (Koivisto, Polishchuk @ arXiv, June 2010)

The geodesic diameter can be computed faster than our algorithm

- There's a serious bug in their argument
- So, our algorithm is still the fastest...

Polygonal domains

...that we won't define, but we define "by example"



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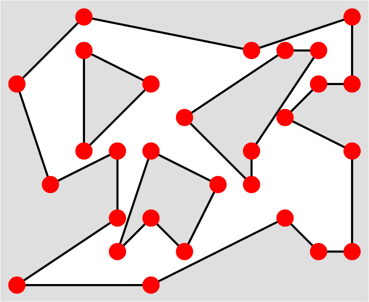

- n = the number of corners (= 29)

Polygonal domains

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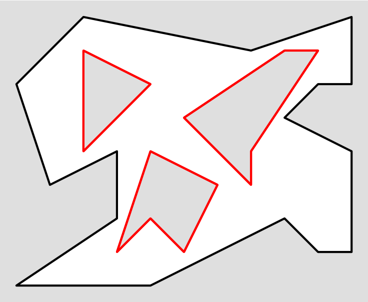

- n = the number of corners ($= 29$)
- h = the number of holes ($= 3$)

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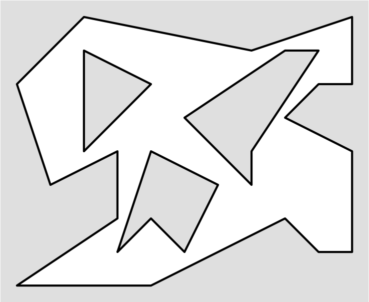
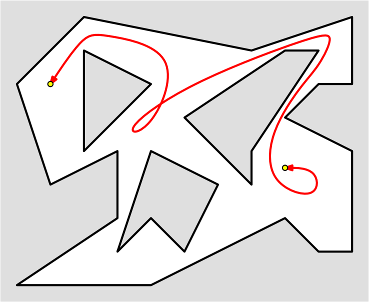


- $n =$ the number of corners ($= 29$)
- $h =$ the number of holes ($= 3$)

Paths and shortest paths

...okay, we don't define it as usual



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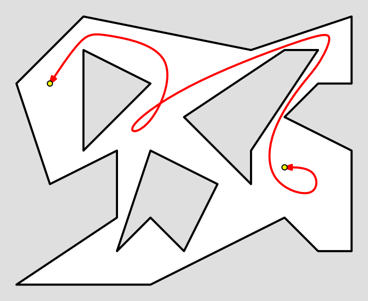

This is a path between the two points

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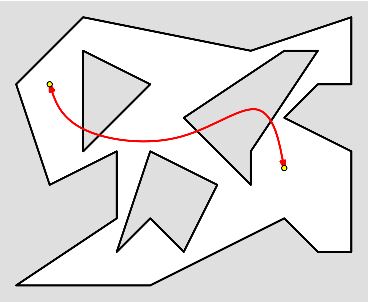

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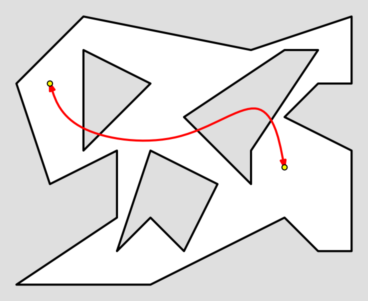

This is NOT a path between the two points

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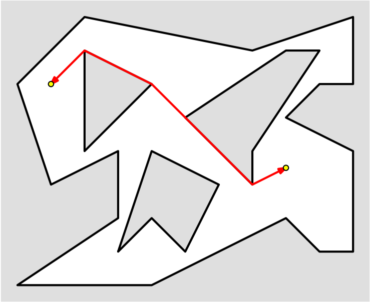

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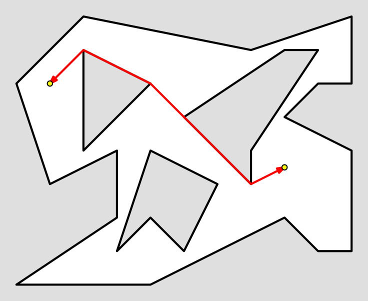

This is a path between the two points
but not a shortest path

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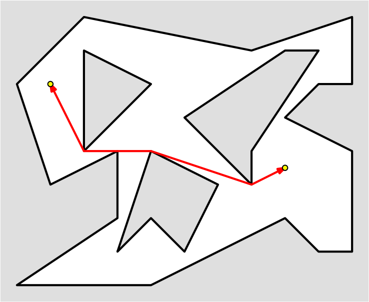

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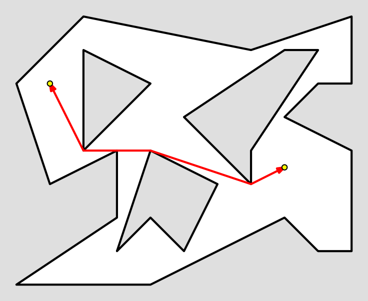

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Paths and shortest paths

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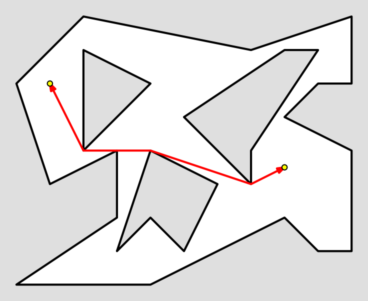


This is a shortest path between the two points

Remark

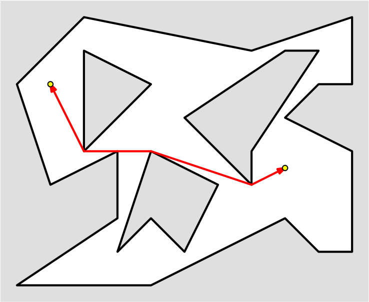
A shortest path between any two points always exists

Geodesic diameter

Definition: (Geodesic) diameter of $\mathcal{P}$

The diameter of $\mathcal{P}$ is the max shortest path-length of two pts in $\mathcal{P}$

A maximizer is called a **diametral pair**



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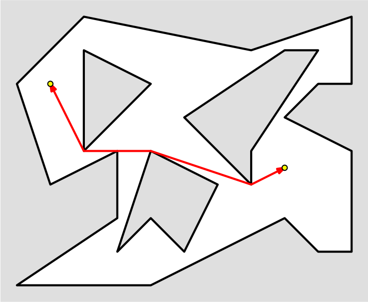

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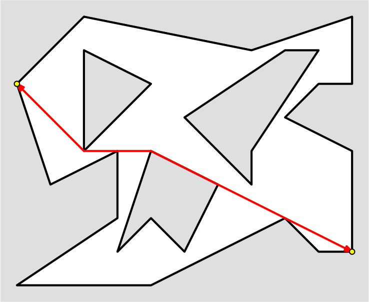

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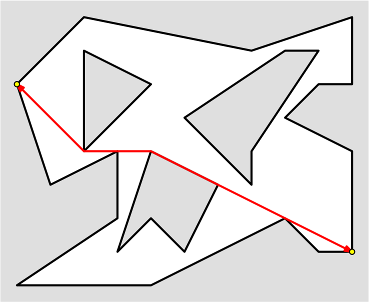

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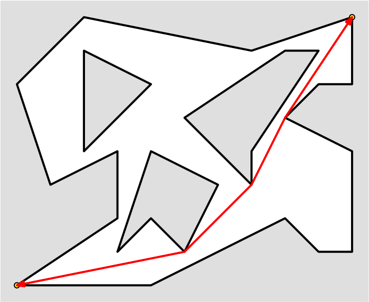

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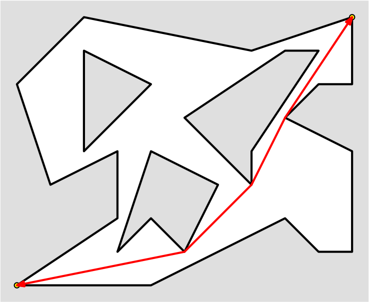


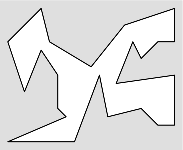
A diametral pair

Geodesic diameter computation: What's known?

$n = \#$ corners, $h = \#$ holes

When $h = 0$ (simple polygons)

- $\mathrm{O}(n^2)$-time algo (Chazelle '82)
- $\mathrm{O}(n \log n)$-time algo (Suri '87)
- $\mathrm{O}(n)$-time algo (Hershberger & Suri '97)



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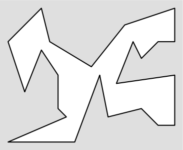
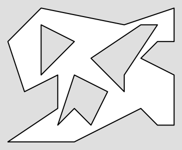
When $h > 0$

- No existing work
- Open problem (Mitchell '97)

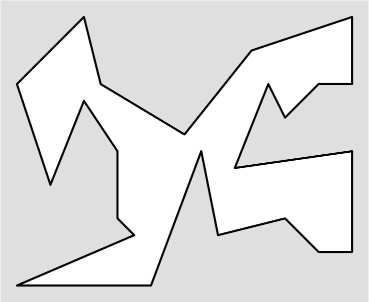




Why are simple polygons easy?

Crucial observation

The diameter is determined by two corners for simple polygons



... This is not necessarily the case for general polygonal domains (as we'll see)

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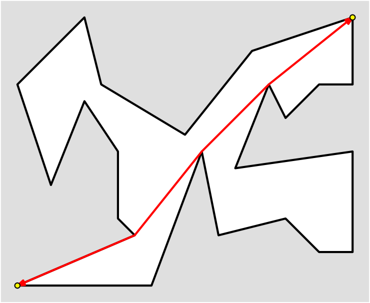

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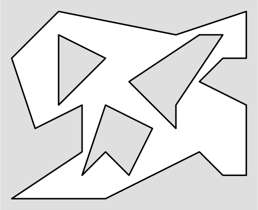
Our results

Results

For a polygonal domain with (possibly many) holes

- Classification of the patterns of diametral pairs
  ...according to the location in a given polygonal domain
- The first polynomial-time algorithm
  ...based on the classification above

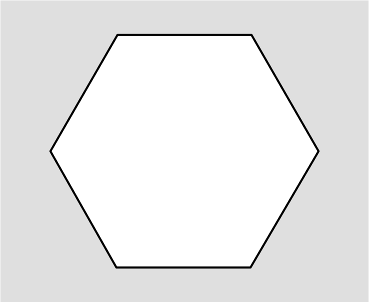


Precise statements will come soon

Non-uniqueness of diametral pairs

Notice 1

A diametral pair is not unique, in general, even for simple polygons



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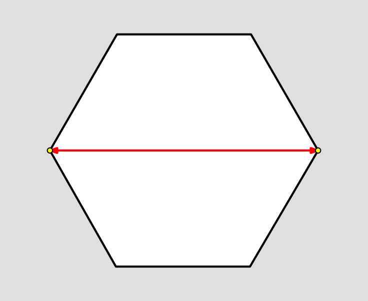

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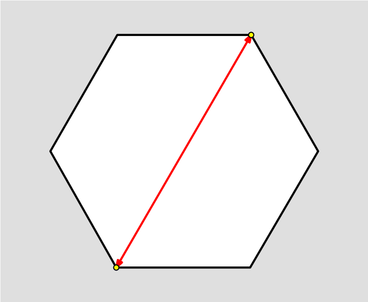

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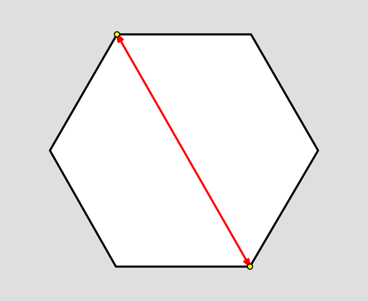
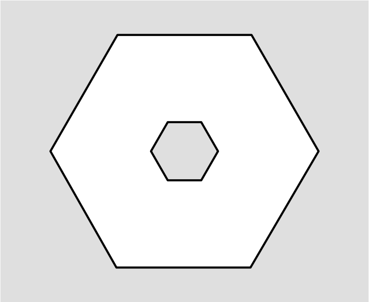


Non-uniqueness of shortest paths

Notice 2

Between two points, a shortest path is not unique in general



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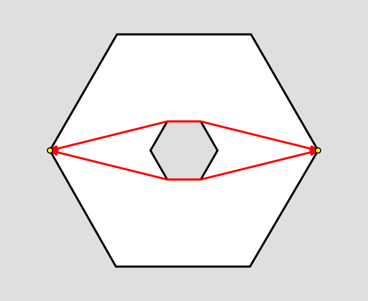
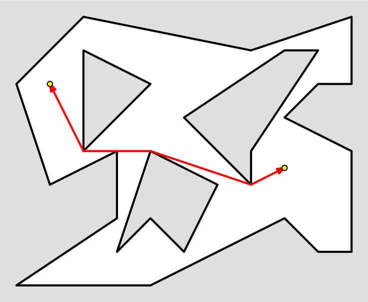


Maximal pairs

Definition: maximal pair

A pair of points $(p, q) \in \mathcal{P} \times \mathcal{P}$ is **maximal** if (p, q) is a local maximum of the distance function



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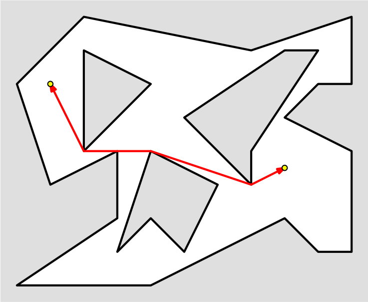

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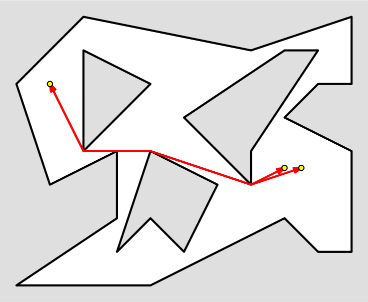

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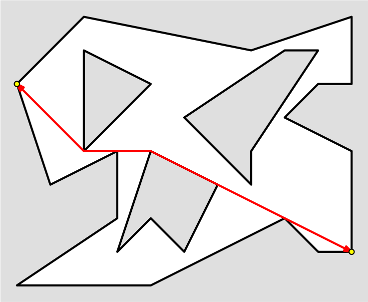

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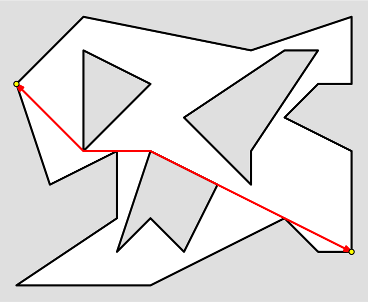

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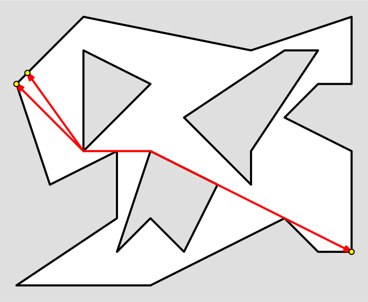

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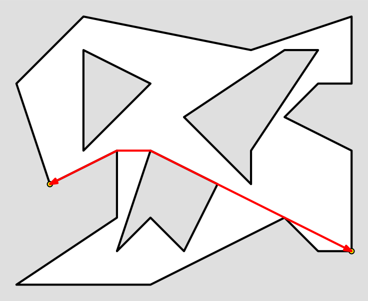

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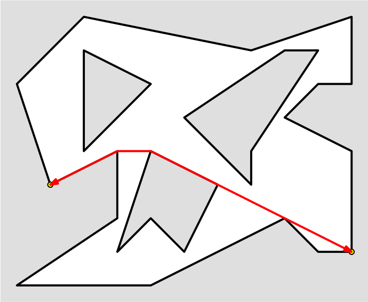

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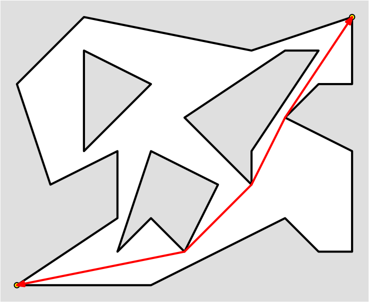

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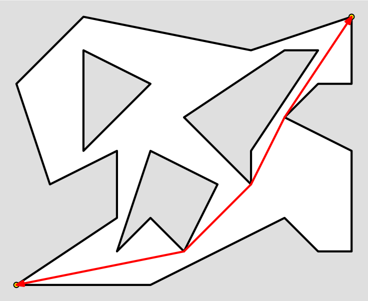

A maximal pair, and diametral as well

Maximal pairs

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Observation (easy but important)

Every diametral pair is maximal

Maximal pairs

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Observation (easy but important)

Every diametral pair is maximal

Consequence

List all maximal pairs, then you find a diametral pair

Main theorem

$\mathcal{P}$ a polygonal domain

V the set of corners, E the set of edges, I the interior of $\mathcal{P}$

Theorem

(p, q) a maximal pair

$\pi(p, q)$ # of shortest paths btw p, q

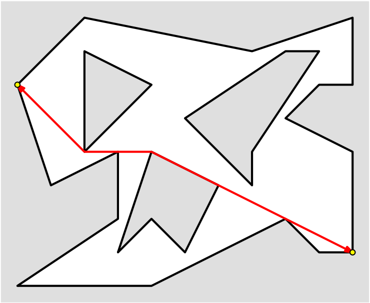
$$\begin{array}{lllll}
\textbf{(V-V)} & p \in V, & q \in V & \Rightarrow & \pi(p,q) \geq 1 \\
\textbf{(V-E)} & p \in V, & q \in E & \Rightarrow & \pi(p,q) \geq 2 \\
\textbf{(V-I)} & p \in V, & q \in I & \Rightarrow & \pi(p,q) \geq 3 \\
\textbf{(E-E)} & p \in E, & q \in E & \Rightarrow & \pi(p,q) \geq 3 \\
\textbf{(E-I)} & p \in E, & q \in I & \Rightarrow & \pi(p,q) \geq 4 \\
\textbf{(I-I)} & p \in I, & q \in I & \Rightarrow & \pi(p,q) \geq 5
\end{array}$$

Main theorem: Case (V–V)

Theorem

(p, q) a maximal pair, $\pi(p, q)$ # of shortest paths btw p, q

$$p \in V,\ q \in V \Rightarrow \pi(p, q) \geq 1$$



Main theorem: Case (V-E)

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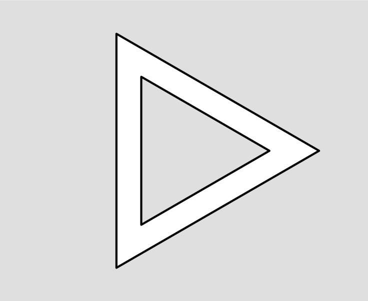

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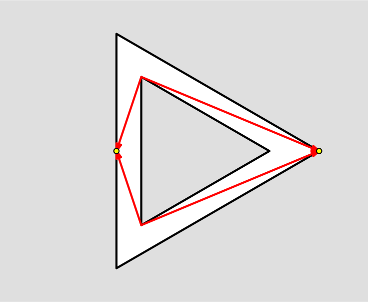
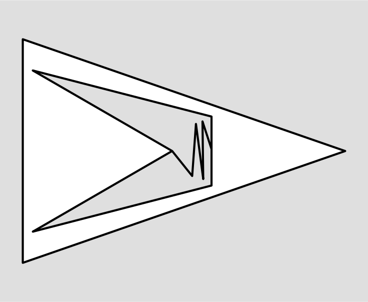


Main theorem: Case (V-I)

Theorem

(p, q) a maximal pair, $\pi(p, q)$ # of shortest paths btw p, q

$$p \in V,\ q \in I \Rightarrow \pi(p, q) \geq 3$$



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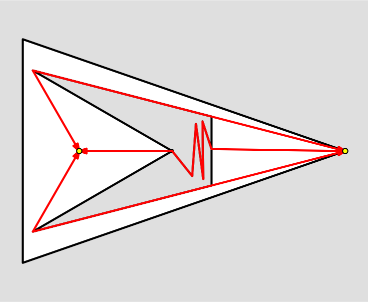

Main theorem

$\mathcal{P}$ a polygonal domain

V the set of corners, E the set of edges, I the interior of $\mathcal{P}$

Theorem

(p, q) a maximal pair

$\pi(p, q)$ # of shortest paths btw p, q

$$\begin{array}{lllll}
\textbf{(V-V)} & p \in V, & q \in V & \Rightarrow & \pi(p,q) \geq 1 \\
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\textbf{(E-I)} & p \in E, & q \in I & \Rightarrow & \pi(p,q) \geq 4 \\
\textbf{(I-I)} & p \in I, & q \in I & \Rightarrow & \pi(p,q) \geq 5
\end{array}$$

Main theorem: Case (E–E)

Theorem

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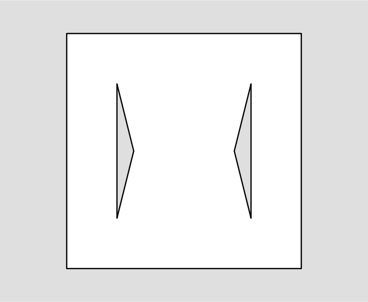

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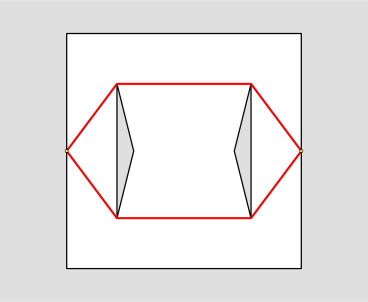

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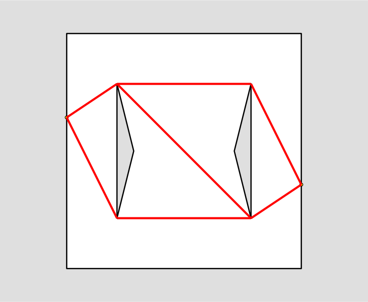

Main theorem: Case (E-I)

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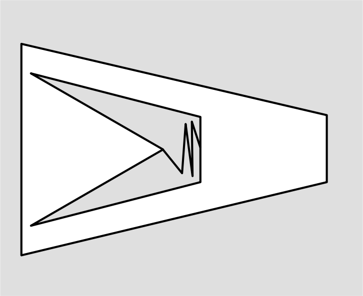

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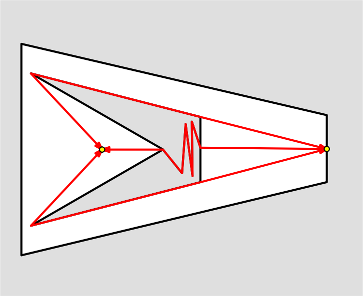

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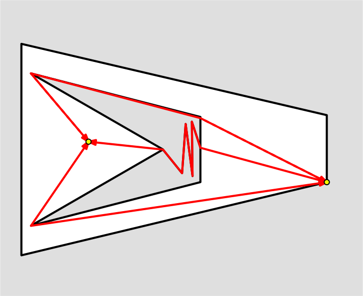

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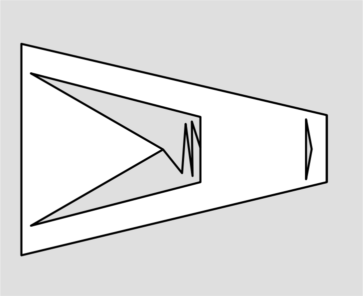

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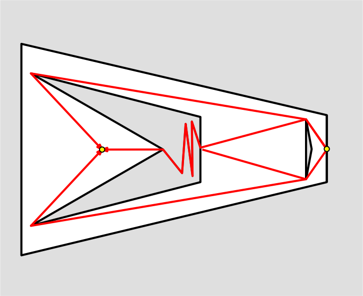

Main theorem: Case (I–I)

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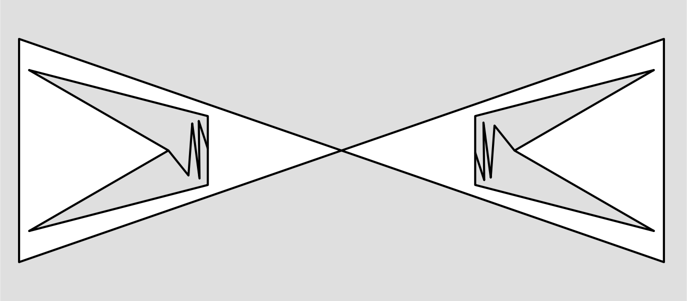

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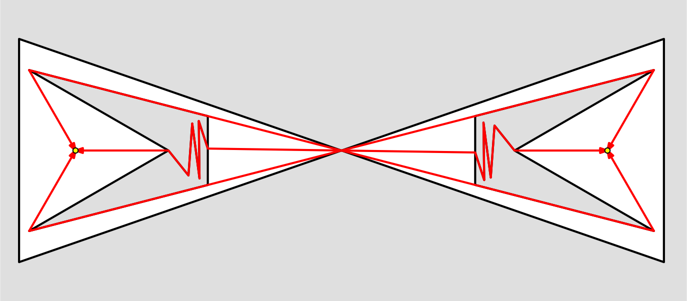

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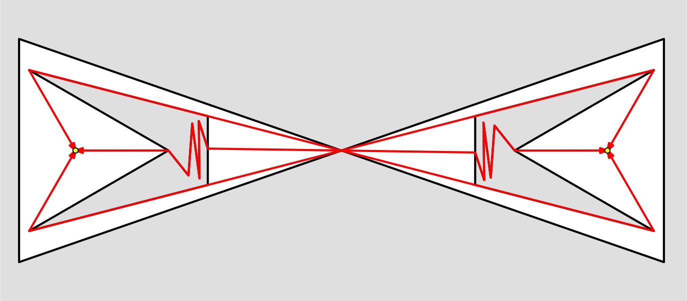

In this example, $\pi(p, q) = 9$

Main theorem: Case (I–I)

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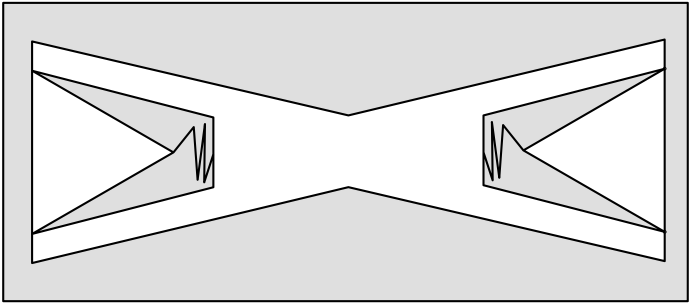

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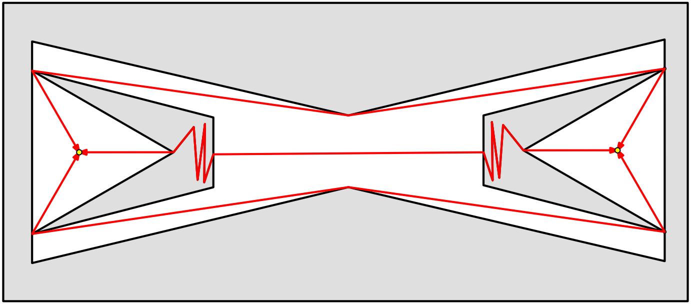

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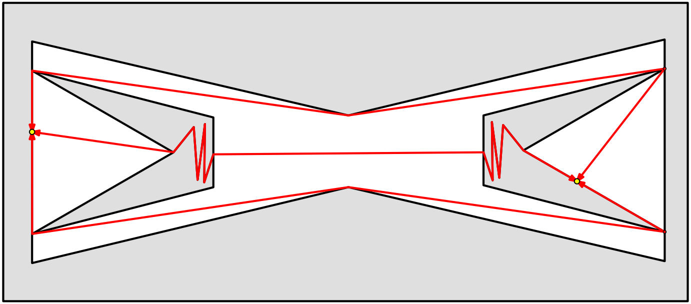

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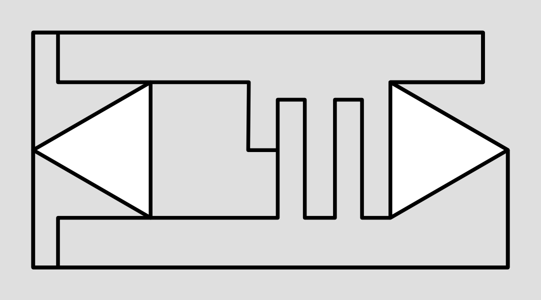

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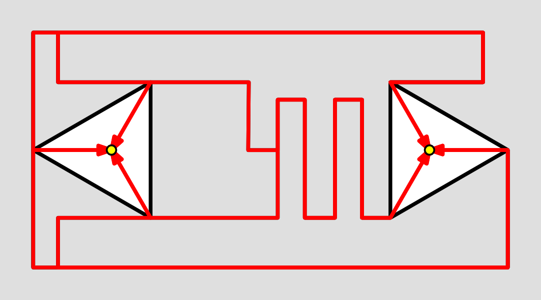

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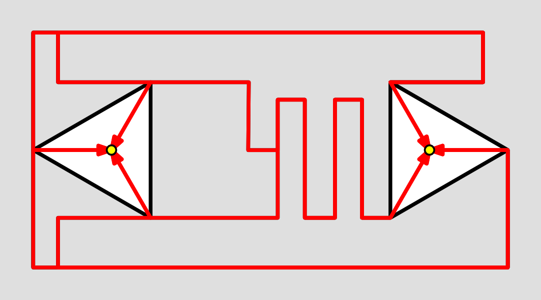

In this example, $\pi(p, q) = 6$

Main theorem: Case (I–I)

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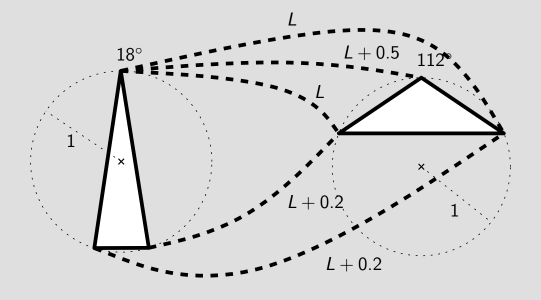

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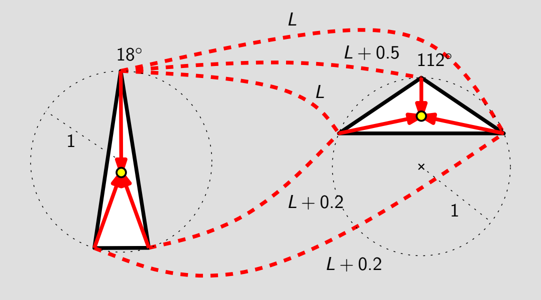

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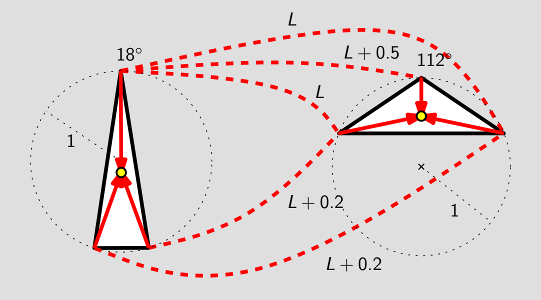

In this example, $\pi(p, q) = 5$

Main theorem

$\mathcal{P}$ a polygonal domain

V the set of corners, E the set of edges, I the interior of $\mathcal{P}$

Theorem

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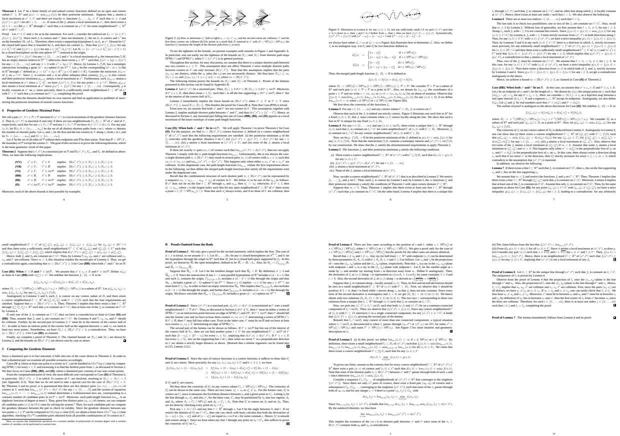
Proof

7 pages long, too long to present even the idea only



Algorithmic ingredients

Basic idea

- List all maximal pairs, with their distances
- Exhaust all six cases
  - Case (I–I) is the bottleneck

Ingredients

- Shortest-path map
  - $O(n \log n)$ time construction
  - $O(\log n)$ time per query (Mitchell '96)
- Two-point shortest-path query data structure
  - $O(n^{5+10\delta+\epsilon})$ time construction
  - $O(n^{1-\delta} \log n)$ time per query (Chiang and Mitchell '99)

We won't define them...

Algorithmic consequence

Theorem

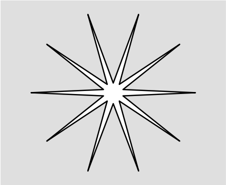
The geodesic diameter of a given polygonal domain $\mathcal{P}$ with n corners and h holes can be computed in

- $\mathrm{O}(n^{7.33})$ time or
- $\mathrm{O}(n^7(\log n + h))$ time

$\mathrm{O}(n^7(\log n + h))$ is better when $h = \mathrm{o}(n^{0.33})$

Concluding remarks

- Running times are disappointing
  - Exact: $O(n^{7.33})$
  - Approx: $O(n^2 \log n)$
- Chiang-Mitchell two-pt shortest-path data str is disappointing
  - $O(n^{5+10\delta+\epsilon})$ time construction
  - $O(n^{1-\delta} \log n)$ time per query
- How many maximal pairs can there be in worst case?
  - Upper bound: $O(n^7)$ (from Main Theorem)
  - Lower bound: $\Omega(n^2)$
- How about the geodesic center of a polygonal domain?



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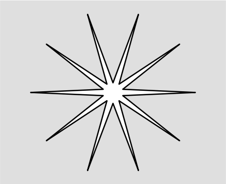

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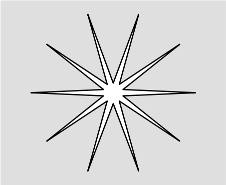

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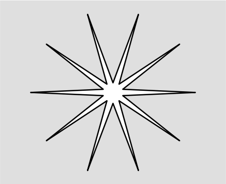

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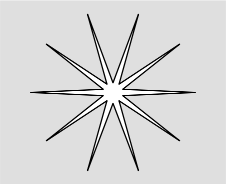

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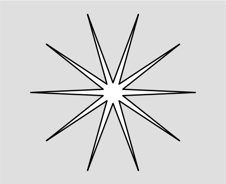
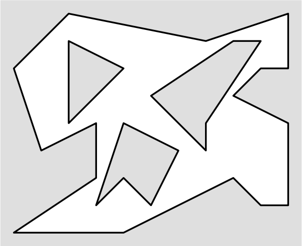


Thank you

Approximation

Polynomial-time approximation scheme (by Ahn, priv. comm.)

- Idea: Overlay the fine grid, and compute all pairwise distances
- Running time: $O((\frac{n}{\epsilon^2} + \frac{n^2}{\epsilon}) \log n)$



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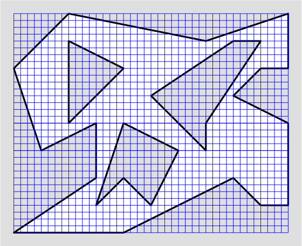

Geodesic diameter computation: What's known? (2)
...in a somewhat different scenario

For 3-dimensional convex polytopes with n vertices

- $\mathrm{O}(n^{14} \log n)$-time algo
  (O'Rourke & Schevon '89)
- $\mathrm{O}(n^{8} \log n)$-time algo
  (Agarwal, Aronov, O'Rourke & Schevon '97)
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Crucial observation (O'Rourke & Schevon '89)

The diameter is determined by two non-vertex points p, q
$\Longrightarrow \exists$ at least 5 shortest paths between p, q