## The Geodesic Diameter of Polygonal Domains

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## Matias gave talks at JCCGG '09, EWCG '10, ESA '10

## Motivation and result

## Joe Mitchell said ...

In Handbook of Discrete and Computational Geometry ('97, '04)

How efficiently can one compute a geodesic diameter for a polygonal domain?





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The Geodesic Diameter of Polygonal Domains

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How efficiently can one compute a geodesic diameter for a polygonal domain?

#### Result

Interesting geometric observations

(that constitute the main theorem)

The first polynomial-time algorithm for this problem





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## An improvement claimed, but ...

## Claimed improvement (Koivisto, Polishchuk @ arXiv, June 2010)

The geodesic diameter can be computed faster than our algorithm

- There's a serious bug in their argument
- So, our algorithm is still the fastest...

## 1 Terminology and Concepts

## 2 Theorem

3 Algorithm

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#### Polygonal domains ...that we won't define, but we define "by example"



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• n = the number of corners (= 29)

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Paths and shortest paths ...okay, we don't define it as usual



Paths and shortest paths ... okay, we don't define it as usual



This is a path between the two points

Algorithm

Paths and shortest paths ...okay, we don't define it as usual



Algorithm

Paths and shortest paths ...okay, we don't define it as usual



This is NOT a path between the two points

Algorithm

Paths and shortest paths ...okay, we don't define it as usual



Algorithm

Paths and shortest paths ...okay, we don't define it as usual



This is a path between the two points but not a shortest path

Algorithm

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This is a shortest path between the two points

## Remark

A shortest path between any two points always exists

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#### Geodesic diameter

Definition: (Geodesic) diameter of  $\mathcal{P}$ 

The diameter of  $\mathcal P$  is the max shortest path-length of two pts in  $\mathcal P$ 

## A maximizer is called a diametral pair



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A diametral pair

Geodesic diameter computation: What's known?

$$n = \#$$
 corners,  $h = \#$  holes

When h = 0 (simple polygons)

- ► O(n<sup>2</sup>)-time algo (Chazelle '82)
- ► O(n log n)-time algo (Suri '87)
- O(n)-time algo
  (Hershberger & Suri '97)



Geodesic diameter computation: What's known?

n = # corners, h = # holes







## Why are simple polygons easy?

## Crucial observation

The diameter is determined by two corners for simple polygons



... This is not necessarily the case for general polygonal domains (as we'll see)

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## Our results

#### Results

For a polygonal domain with (possibly many) holes

- Classification of the patterns of diametral pairs
  ...according to the location in a given polygonal domain
- The first polynomial-time algorithm

...based on the classification above



Precise statements will come soon

## 1 Terminology and Concepts

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## Non-uniqueness of diametral pairs

#### Notice 1



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## Non-uniqueness of shortest paths

#### Notice 2

Between two points, a shortest path is not unique in general



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# Maximal pairs

## Definition: maximal pair

A pair of points  $(p, q) \in \mathcal{P} \times \mathcal{P}$  is maximal if (p, q) is a local maximum of the distance function



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A maximal pair, and diametral as well

# Maximal pairs

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## Observation (easy but important)

Every diametral pair is maximal

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### Observation (easy but important)

Every diametral pair is maximal

#### Consequence

List all maximal pairs, then you find a diametral pair

# Main theorem

- ${\mathcal P}$  a polygonal domain
- V the set of corners, E the set of edges, I the interior of  ${\cal P}$

#### Theorem

## Main theorem: Case (V-V)

#### Theorem

$$p \in V$$
,  $q \in V \Rightarrow \pi(p,q) \geq 1$ 



## Main theorem: Case (V-E)

#### Theorem

$$p \in V$$
,  $q \in E \Rightarrow \pi(p,q) \ge 2$ 



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## Main theorem: Case (V-I)

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## Main theorem: Case (E-E)

#### Theorem

$$p \in E, q \in E \Rightarrow \pi(p,q) \ge 3$$



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# Main theorem: Case (I-I)

### Theorem

$$p \in I, q \in I \Rightarrow \pi(p,q) \geq 5$$



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In this example, 
$$\pi(p,q) = 9$$

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In this example, 
$$\pi(p,q) = 6$$
### Main theorem: Case (I-I)

#### Theorem

(p,q) a maximal pair,  $\pi(p,q)$  # of shortest paths btw p,q

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#### Theorem

(p,q) a maximal pair  $\pi(p,q)$  # of shortest paths btw p,q

#### Proof

#### 7 pages long, too long to present even the idea only

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# 1 Terminology and Concepts

#### 2 Theorem

3 Algorithm

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# Algorithmic ingredients

#### Basic idea

- List all maximal pairs, with their distances
- Exhaust all six cases
  - Case (I-I) is the bottleneck

### Ingredients

- Shortest-path map
  - $O(n \log n)$  time construction
  - O(log n) time per query

- (Mitchell '96)
- Two-point shortest-path query data structure
  - $O(n^{5+10\delta+\epsilon})$  time construction
  - $O(n^{1-\delta} \log n)$  time per query

(Chiang and Mitchell '99)

# We won't define them ...

# Algorithmic consequence

#### Theorem

The geodesic diameter of a given polygonal domain  $\mathcal{P}$  with *n* corners and *h* holes can be computed in

- $O(n^{7.33})$  time or
- $O(n^7(\log n + h))$  time

 $O(n^7(\log n + h))$  is better when  $h = o(n^{0.33})$ 

- Running times are disappointing
  - ▶ Exact: O(n<sup>7.33</sup>)
  - Approx:  $O(n^2 \log n)$
- Chiang-Mitchell two-pt shortest-path data str is disappointing
  - $O(n^{5+10\delta+\epsilon})$  time construction
  - $O(n^{1-\delta} \log n)$  time per query
- How many maximal pairs can there be in worst case?
  - Upper bound:  $O(n^7)$  (from Main Theorem)
  - Lower bound:  $\Omega(n^2)$
- How about the geodesic center of a polygonal domain?



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# Approximation

# Polynomial-time approximation scheme (by Ahn, priv. comm.)

- Idea: Overlay the fine grid, and compute all pairwise distances
- Running time:  $O((\frac{n}{\epsilon^2} + \frac{n^2}{\epsilon}) \log n)$



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Geodesic diameter computation: What's known? (2) ...in a somewhat different scenario

#### For 3-dimensional convex polytopes with n vertices

► O(n<sup>14</sup> log n)-time algo

(O'Rourke & Schevon '89)

O(n<sup>8</sup> log n)-time algo

(Agarwal, Aronov, O'Rourke & Schevon '97)

•  $O(n^7 \log n)$ -time algo

(Cook IV & Wenk '09)

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► O(n<sup>7</sup> log n)-time algo

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#### Crucial observation

(O'Rourke & Schevon '89)

The diameter is determined by two non-vertex points p, q

 $\Longrightarrow \exists$  at least 5 shortest paths between p,q