A tight lower bound for convexly independent subsets of the Minkowski sums of planar point sets

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Convexly independent subset

$P \subseteq \mathbb{R}^2$ a finite point set

**Definition: Convexly independent subset**

A set $S \subseteq P$ is called **convexly independent** if every point in $S$ is an extreme point of the convex hull of $S$.
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Buchin, Fulek, Kiyomi, Okamoto, Tanigawa, & Tóth
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Largest convexly independent subset

**Notation**

For a finite point set $P \subseteq \mathbb{R}^2$

$$M(P) = \max\{|S| : S \subseteq P \text{ convexly independent}\}$$
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An extremal problem

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For a natural number $n$

$$M(n) = \max\{M(P) : P \subseteq \mathbb{R}^2, |P| = n\}$$

Question

Determine $M(n)$
An extremal problem

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For a finite point set $P \subseteq \mathbb{R}^2$

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Question and Answer
Determine $M(n)$

—Well, it’s easy: $M(n) = n$
More interesting extremal problem

Let $P \oplus Q$ be the Minkowski sum of $P$ and $Q$, as defined in the next slide...

Notation

For two finite point sets $P, Q \subseteq \mathbb{R}^2$

$$M(P, Q) = \max\{|S| : S \subseteq P \oplus Q \text{ convexly independent}\};$$

For two natural numbers $m, n$

$$M(m, n) = \max\{M(P, Q) : P, Q \subseteq \mathbb{R}^2, |P| = m, |Q| = n\}$$

Question

Determine $M(m, n)$
$P, Q \subseteq \mathbb{R}^2$ finite point sets

**Definition: Minkowski sum**

The **Minkowski sum** of $P$ and $Q$ is

$$P \oplus Q = \{ p + q : p \in P, q \in Q \}$$

**Remark**

$|P \oplus Q| \leq |P| + |Q|$, and it's possible that $|P \oplus Q| < |P| + |Q|$.
Minkowski sums

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**Remark**

$|P \oplus Q| \leq |P||Q|$, and it's possible that $|P \oplus Q| < |P||Q|$.
Minkowski sums

\( P, Q \subseteq \mathbb{R}^2 \) finite point sets

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**Remark**

\[ |P \oplus Q| \leq |P| \cdot |Q|, \text{ and it's possible that } |P \oplus Q| < |P| \cdot |Q| \]
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More interesting extremal problem

**Notation**

For two finite point sets $P, Q \subseteq \mathbb{R}^2$

$$M(P, Q) = M(P \oplus Q) = \max\{|S| : S \subseteq P \oplus Q \text{ convexly independent}\};$$

For two natural numbers $m, n$

$$M(m, n) = \max\{M(P, Q) : P, Q \subseteq \mathbb{R}^2, |P| = m, |Q| = n\}$$

**Question**

Determine $M(m, n)$

For example, is it true that $M(m, n) = mn$?
Example

\[ |P \oplus Q| = 18 \]
Example

\[ |P \oplus Q| = 18, \text{ while } M(P, Q) = M(P \oplus Q) = 12 \]
Theorem (Eisenbrand, Pach, Rothvoß, Sopher ’08)

\[ M(m, n) = O\left(\frac{m^2}{3} \frac{n^2}{3} + m + n\right) \]

They only knew a linear lower bound:

\[ M(m, n) = \Omega(m + n) \]
Our result

Theorem (Eisenbrand, Pach, Rothvoß, Sopher ’08)

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\[ M(m, n) = \Omega(m^{2/3} n^{2/3} + m + n) \]

Our result was independently found by Bílka, but only when \( m = n \)
Our result (independently found by Bílka)

\[ M(m, n) = \Omega\left(m^{2/3} n^{2/3} + m + n\right) \]
Our result (independently found by Bílka)

\[ M(m, n) = \Omega\left(\frac{m^2}{3} \cdot\frac{n^2}{3} + m + n\right) \]

Contents

- Basic idea
- Fine tuning
Our result (independently found by Bílka)

\[ M(m, n) = \Omega \left( m^{2/3} n^{2/3} + m + n \right) \]

Contents

- Basic idea
- Fine tuning

1. Look at a lower-bound example for the point-line incidence problem
2. Construct two point sets from such an example
3. Simulate the point-line incidences as a large convexly independent subset of the two point sets
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1. **Look at a lower-bound example for the point-line incidence problem**

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Point-line incidences

$p$ a point, $\ell$ a line

Definition: Point-line incidence

$p$ is **incident** to $\ell$ if $p \in \ell$

\[
I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}|
\]

incident

not incident
$p$ a point, $\ell$ a line

**Definition: Point-line incidence**

$p$ is *incident* to $\ell$ if $p \in \ell$

$P$ a set of points, $L$ a set of lines

**Notation**

$$I(P, L) = \left| \{(p, \ell) \in P \times L : p \in \ell \} \right|$$
Point-line incidences

$p$ a point, $\ell$ a line

**Definition: Point-line incidence**

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**Notation**

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I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}|
\]

\[
I(P, L) = 8
\]
Point-line incidences: Lower bound

Notation

\[ I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}| \]
\[ I(m, n) = \max \{I(P, L) : |P| = m, |L| = n\} \]

Theorem (Erdős ’46)

\[ I(m, n) = \Omega(m^{2/3} n^{2/3} + m + n) \]

Remark: This is tight (due to Szeméredi and Trotter ’83)
Our result (independently found by Bílka)

\[ M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n) \]

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1. Look at a lower-bound example for the point-line incidence problem
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Crucial idea

Take $P$ and $L$ such that $I(P, L) = \Omega(m^{2/3}n^{2/3} + m + n)$

Expected consequence

$M(P, Q) \geq |S| = \Omega(m^{2/3}n^{2/3} + m + n)$
How to construct $Q (1/3)$: Sort the lines by their slopes

$\ell_i =$ the $i$th line in the sorted list of the lines in $L$
How to construct $Q (2/3)$: Align the lines to form a curve

Consider a polygonal chain $C$ consisting of $n$ line segments s.t. the $i$th segment $s_i$ has the same slope as $\ell_i$
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- $C$ is a convex chain (∵ the lines are sorted by their slopes)
How to construct $Q$ (2/3): Align the lines to form a curve

Consider a polygonal chain $C$ consisting of $n$ line segments s.t. the $i$th segment $s_i$ has the same slope as $\ell_i$.

- $C$ is a convex chain (∵ the lines are sorted by their slopes)
- Set the length of each segment sufficiently long
How to construct $Q$ (3/3): Place a point for each line

Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$

This is possible since each segment is long enough

In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$
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- In particular, \(|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|\)
Our result (independently found by Bílka)

\[ M(m, n) = \Omega\left(m^{2/3} n^{2/3} + m + n\right) \]

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- Basic idea
- Fine tuning

1. Look at a lower-bound example for the point-line incidence problem
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3. Simulate the point-line incidences as a large convexly independent subset of the two point sets
$(P \oplus Q) \cap C$ is our candidate for a large convexly independent subset.

\[ I(P, L) = \sum_{i=1}^{n} |P \cap \ell_i| = \sum_{i=1}^{n} |(P_i \oplus \{q_i\}) \cap C| = |(P \oplus Q) \cap C| \]
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\[ \therefore |(P \oplus Q) \cap C| = I(P, L) = \Omega(m^{2/3} n^{2/3} + m + n) \]
\((P \oplus Q) \cap C\) is our candidate for a large convexly independent subset.

\[
I(P, L) = \sum_{i=1}^{n} |P \cap \ell_i| = \sum_{i=1}^{n} |(P_i \oplus \{q_i\}) \cap C| = |(P \oplus Q) \cap C| \\
\therefore |(P \oplus Q) \cap C| = I(P, L) = \Omega(m^{2/3} n^{2/3} + m + n)
\]

We take \((P \oplus Q) \cap C\) as our large subset.
**Our result (independently found by Bílka)**

\[ M(m, n) = \Omega(m^{2/3} n^{2/3} + m + n) \]

**Contents**

- Basic idea
- **Fine tuning**
An issue to resolve

The set \((P \oplus Q) \cap C\) is not necessarily convexly independent since

- For each \(i\), the points in \((P \oplus Q) \cap s_i\) are collinear
Applying a nonlinear transformation — to resolve the issue

Transform \( P, L \) by the following map with suff. small \( \varepsilon > 0 \):

\[
(x, y) \quad \mapsto \quad (x, y + \varepsilon x^2)
\]
Applying a nonlinear transformation — to resolve the issue

Transform $P, L$ by the following map with suff. small $\varepsilon > 0$:

$$(x, y) \mapsto (x, y + \varepsilon x^2)$$

The line $y = a_i x + b_i$ \mapsto \begin{align*}
\text{The parabola } y &= \varepsilon x^2 + a_i x + b_i
\end{align*}
Applying a nonlinear transformation — to resolve the issue

Transform $P, L$ by the following map with suff. small $\varepsilon > 0$:

$$(x, y) \quad \mapsto \quad (x + \varepsilon x^2, y + \varepsilon x^2)$$

The line $y = a_i x + b_i$ \quad $\mapsto$ \quad The parabola $y = \varepsilon x^2 + a_i x + b_i$

We repeat the same construction as before, and then $(P \oplus Q) \cap C$ is convexly independent
Convexly independent subsets of the Minkowski sums
Our result

Theorem (Eisenbrand, Pach, Rothvoß, Sopher '08)

\[ M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n) \]

They only knew a linear lower bound:

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Our result

\[ M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n) \]

Our result was independently found by Bílka, but only when \( m = n \)
Open problem (1): More summands

Instead of two point sets, what if we are given $k$ point sets?

Notation

For $k$ finite point sets $P_1, \ldots, P_k \subseteq \mathbb{R}^2$

$$M(P_1, \ldots, P_k) = \max \{|S| : S \subseteq P_1 \oplus \cdots \oplus P_k \text{ conv'ly independent}\};$$

For $k$ natural numbers $n_1, \ldots, n_k$

$$M(n_1, \ldots, n_k) = \max \{M(P_1, \ldots, P_k) : P_i \subseteq \mathbb{R}^2, |P_i| = n_i\}$$

Open problem 1

Determine $M(n_1, \ldots, n_k)$

- Our result: $M(n_1, n_2) = \Omega(n_1^{2/3} n_2^{2/3} + n_1 + n_2)$
What about algorithms?

Open problem 2

Given \( P, Q \subseteq \mathbb{R}^2 \), \(|P| = m\), \(|Q| = n\), how quickly can we find a largest convexly independent subset of \( P \oplus Q \)?

Remark

A largest convexly independent subset of a single point set \( P \) can be found in \( O(n^3) \) time where \(|P| = n\) \hspace{1em} (Chvátal, Klincsek ’80)

- Improving the \( O(n^3) \) bound is a long-standing open problem (see Edelsbrunner’s book ’87)
- A naive application of Chvátal–Klincsek’s algorithm just yields an \( O(m^3 n^3) \)-time algorithm \( \rightarrow \) Improve!
Open problem 1
Determine $M(n_1,\ldots,n_k)$

Open problem 2
Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m$, $|Q| = n$,
how quickly can we find a largest convexly independent subset of
$P \oplus Q$?
Open problem 1
Determine $M(n_1, \ldots, n_k)$

Open problem 2
Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m$, $|Q| = n$, how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Bottom line
Surprisingly, we know only little about Minkowski sums, even for planar point sets
Open problem 1
Determine $M(n_1, \ldots, n_k)$

Open problem 2
Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m$, $|Q| = n$,
how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Bottom line
Surprisingly, we know only little about Minkowski sums, even for planar point sets.

[Thank you]