A tight lower bound for convexly independent subsets of the Minkowski sums of planar point sets

Kevin Buchin[1] Radoslav Fulek[2] Masashi Kiyomi[3]
Yoshio Okamoto[4] Shin-ichi Tanigawa[5] Csaba D. Tóth[6]

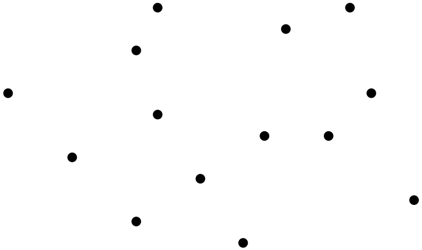
[1]TU Eindhoven [2]EPFL [3]JAIST [4]Tokyo Tech [5]Kyoto Univ [6]Univ Calgary

Japan Conference on Computational Geometry and Graphs,
Kanazawa, November 11, 2009

$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

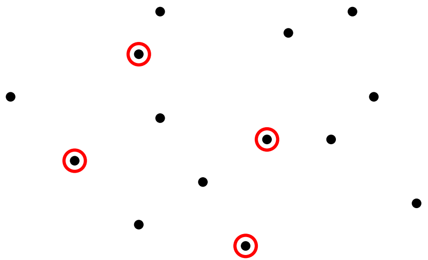
A set $S \subseteq P$ is called **convexly independent** if every point in S is an extreme point of the convex hull of S



$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

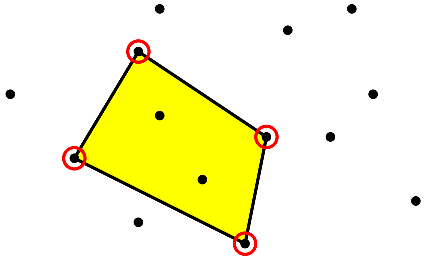
A set $S \subseteq P$ is called **convexly independent** if
every point in S is an extreme point of the convex hull of S



$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

A set $S \subseteq P$ is called **convexly independent** if
every point in S is an extreme point of the convex hull of S

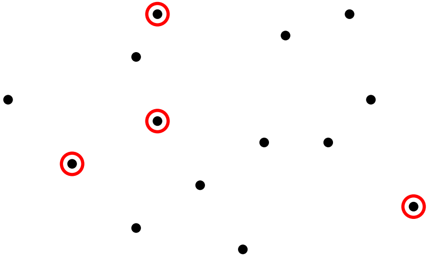


Convexly independent

$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

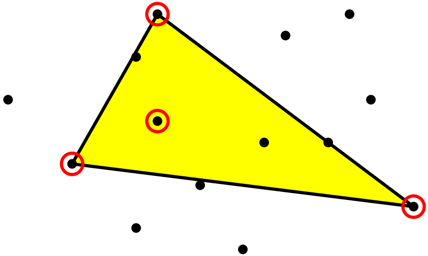
A set $S \subseteq P$ is called **convexly independent** if every point in S is an extreme point of the convex hull of S



$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

A set $S \subseteq P$ is called **convexly independent** if every point in S is an extreme point of the convex hull of S

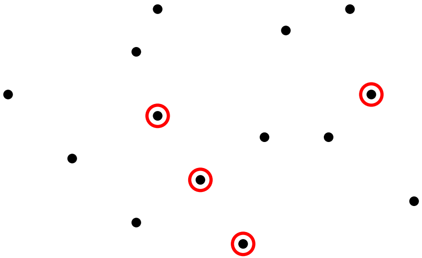


Not convexly independent

$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

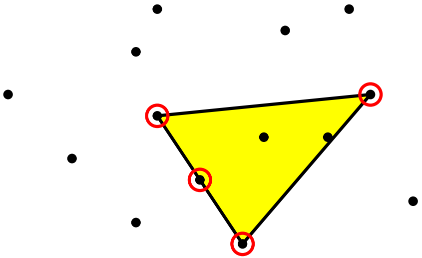
A set $S \subseteq P$ is called **convexly independent** if every point in S is an extreme point of the convex hull of S



$P \subseteq \mathbb{R}^2$ a finite point set

Definition: Convexly independent subset

A set $S \subseteq P$ is called **convexly independent** if
every point in S is an extreme point of the convex hull of S

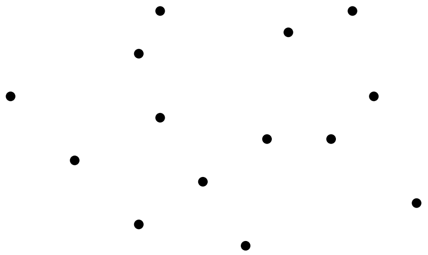


Not convexly independent

Notation

For a finite point set $P \subseteq \mathbb{R}^2$

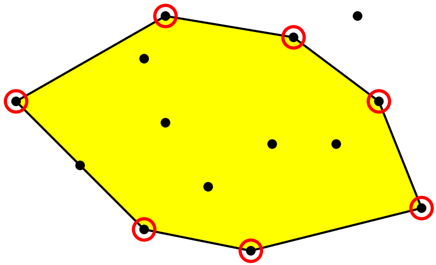
$$M(P) = \max\{|S| : S \subseteq P \text{ convexly independent}\}$$



Notation

For a finite point set $P \subseteq \mathbb{R}^2$

$$M(P) = \max\{|S| : S \subseteq P \text{ convexly independent}\}$$



$M(P) = 7$

Notation

For a finite point set $P \subseteq \mathbb{R}^2$

$$M(P) = \max\{|S| : S \subseteq P \text{ convexly independent}\};$$

For a natural number n

$$M(n) = \max\{M(P) : P \subseteq \mathbb{R}^2, |P| = n\}$$

Question

Determine $M(n)$

Notation

For a finite point set $P \subseteq \mathbb{R}^2$

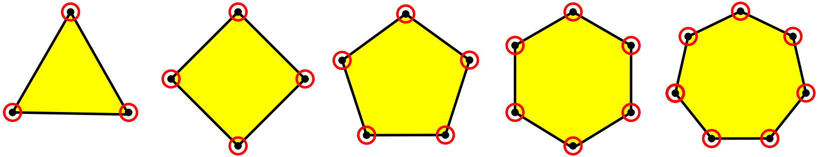
$$M(P) = \max\{|S| : S \subseteq P \text{ convexly independent}\};$$

For a natural number n

$$M(n) = \max\{M(P) : P \subseteq \mathbb{R}^2, |P| = n\}$$

Question and Answer

Determine $M(n)$ —Well, it's easy: $M(n) = n$



Let $P \oplus Q$ be the Minkowski sum of P and Q,
as defined in the next slide...

Notation

For two finite point sets $P, Q \subseteq \mathbb{R}^2$

$$M(P, Q) = \max\{|S| : S \subseteq P \oplus Q \text{ convexly independent}\};$$

For two natural numbers m, n

$$M(m, n) = \max\{M(P, Q) : P, Q \subseteq \mathbb{R}^2, |P| = m, |Q| = n\}$$

Question

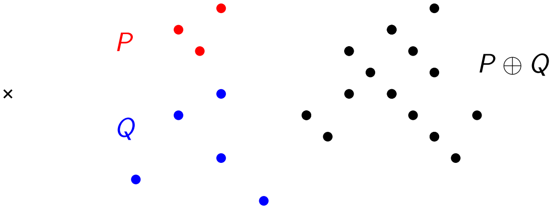
Determine $M(m, n)$

$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

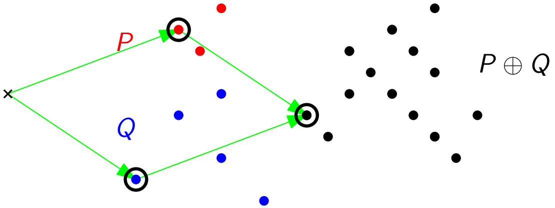


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

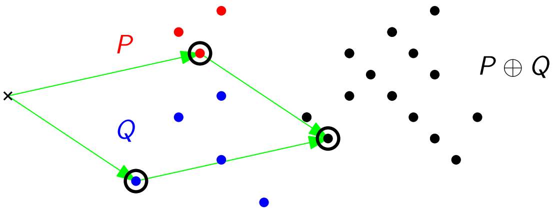


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

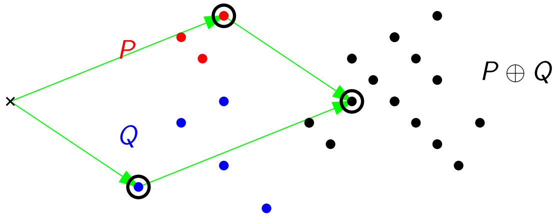


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

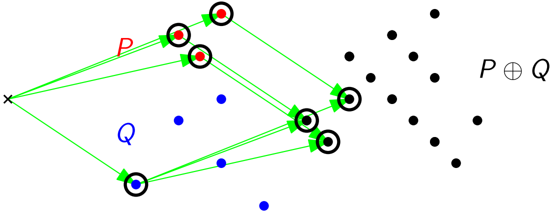


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

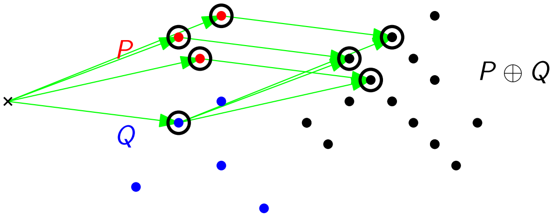


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

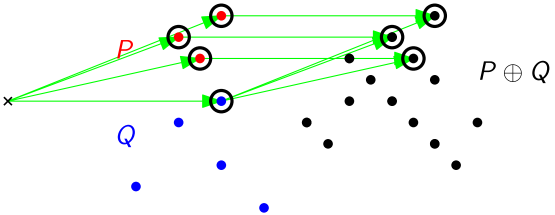


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

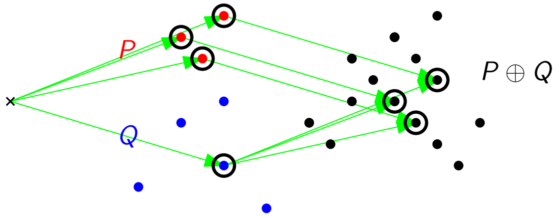


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

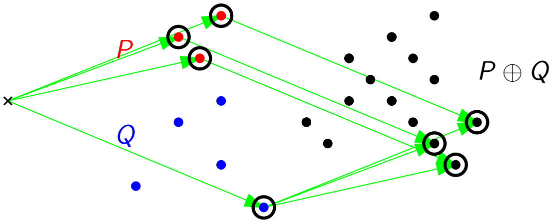


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

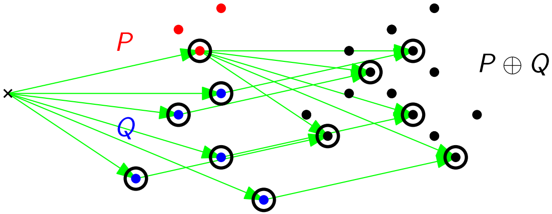


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$

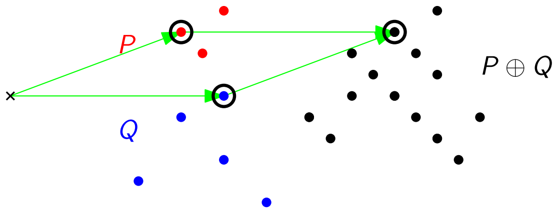


$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$



Remark

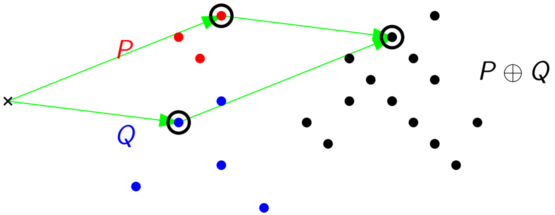
$|P \oplus Q| \leq |P||Q|$, and it's possible that $|P \oplus Q| < |P||Q|$

$P, Q \subseteq \mathbb{R}^2$ finite point sets

Definition: Minkowski sum

The **Minkowski sum** of P and Q is

$$P \oplus Q = \{p + q : p \in P, q \in Q\}$$



Remark

$|P \oplus Q| \leq |P||Q|$, and it's possible that $|P \oplus Q| < |P||Q|$

Notation

For two finite point sets $P, Q \subseteq \mathbb{R}^2$

$$\begin{aligned} M(P,Q) &= M(P \oplus Q) \\ &= \max\{|S| : S \subseteq P \oplus Q \text{ convexly independent}\}; \end{aligned}$$

For two natural numbers m, n

$$M(m,n) = \max\{M(P,Q) : P, Q \subseteq \mathbb{R}^2, |P| = m, |Q| = n\}$$

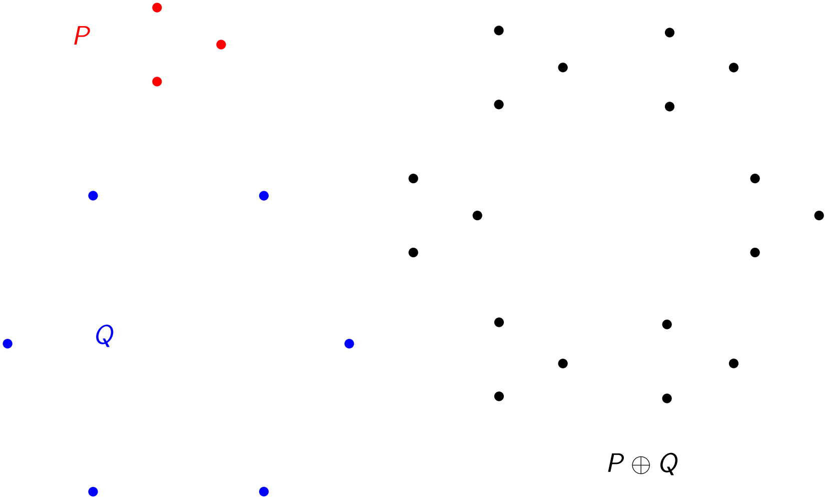
Question

Determine $M(m,n)$

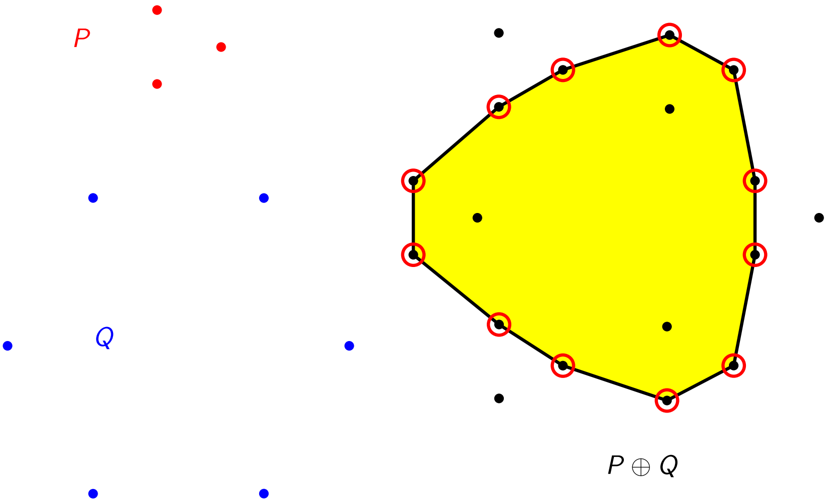
For example, is it true that $M(m,n) = mn$?

Example

$|P \oplus Q| = 18$



$|P \oplus Q| = 18$, while $M(P, Q) = M(P \oplus Q) = 12$



Theorem (Eisenbrand, Pach, Rothvoß, Sopher '08)

$$M(m, n) = \mathrm{O}(m^{2/3}n^{2/3} + m + n)$$

They only knew a linear lower bound:

$$M(m, n) = \Omega(m + n)$$

Theorem (Eisenbrand, Pach, Rothvoß, Sopher '08)

$$M(m, n) = \mathrm{O}(m^{2/3} n^{2/3} + m + n)$$

They only knew a linear lower bound:

$$M(m, n) = \Omega(m + n)$$

Our result

$$M(m, n) = \Omega(m^{2/3} n^{2/3} + m + n)$$

Our result was independently found by Bílka, but only when $m = n$

Our result (independently found by Bílka)

$$M(m, n) = \Omega(m^{2/3} n^{2/3} + m + n)$$

Contents

- Basic idea
- Fine tuning

Our result (independently found by Bílka)

$$M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

Contents

- **Basic idea**
- Fine tuning

Our result (independently found by Bílka)

$$M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

Contents

- **Basic idea**
- Fine tuning

1. Look at a lower-bound example for the point-line incidence problem
2. Construct two point sets from such an example
3. Simulate the point-line incidences as a large convexly independent subset of the two point sets

Our result (independently found by Bílka)

$$M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

Contents

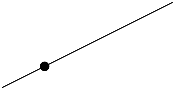
- **Basic idea**
- Fine tuning

1. **Look at a lower-bound example for the point-line incidence problem**
2. Construct two point sets from such an example
3. Simulate the point-line incidences as a large convexly independent subset of the two point sets

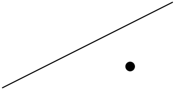
p a point, ℓ a line

Definition: Point-line incidence

p is **incident** to ℓ if $p \in \ell$



incident



not incident

p a point, ℓ a line

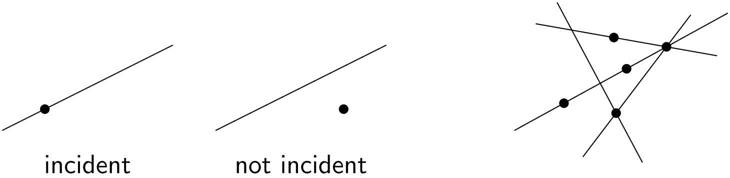
Definition: Point-line incidence

p is **incident** to ℓ if $p \in \ell$

P a set of points, L a set of lines

Notation

$$I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}|$$



Point-line incidences

p a point, ℓ a line

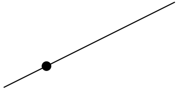
Definition: Point-line incidence

p is **incident** to ℓ if $p \in \ell$

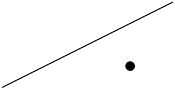
P a set of points, L a set of lines

Notation

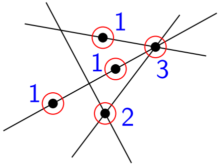
$$I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}|$$



incident



not incident



$I(P, L) = 8$

Notation

$$
\begin{aligned}
I(P, L) &= |\{(p, \ell) \in P \times L : p \in \ell\}| \\
I(m, n) &= \max\{I(P, L) : |P| = m, |L| = n\}
\end{aligned}
$$

Theorem (Erdős '46)

$$I(m, n) = \Omega(m^{2/3} n^{2/3} + m + n)$$

Remark: This is tight (due to Szeméredi and Trotter '83)

Our result (independently found by Bílka)

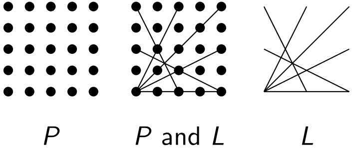
$$M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

Contents

- **Basic idea**
- Fine tuning

1. Look at a lower-bound example for the point-line incidence problem
2. **Construct two point sets from such an example**
3. Simulate the point-line incidences as a large convexly independent subset of the two point sets

Take P and L such that $I(P, L) = \Omega(m^{2/3}n^{2/3} + m + n)$



Crucial idea

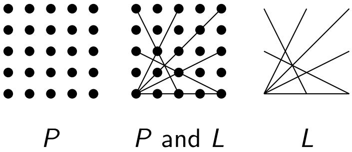
Set up a point $q_i \in Q$ for each line $\ell_i \in L$ so that $P \oplus Q$ has a convexly independent subset S satisfying

$$p \in \ell_i \iff p + q_i \in S$$

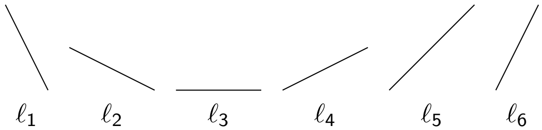
Expected consequence

$$M(P, Q) \geq |S| = \Omega(m^{2/3}n^{2/3} + m + n)$$

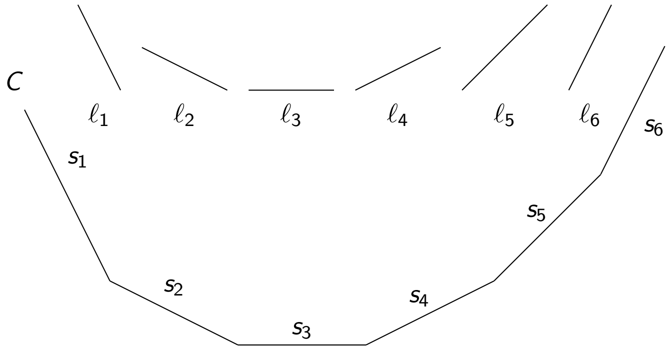
How to construct Q (1/3): Sort the lines by their slopes



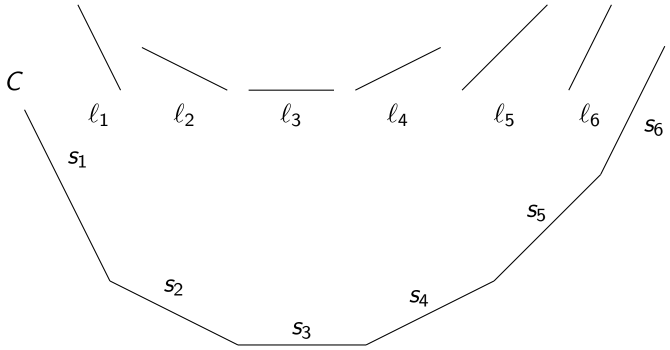
- ℓ_i = the ith line in the sorted list of the lines in L



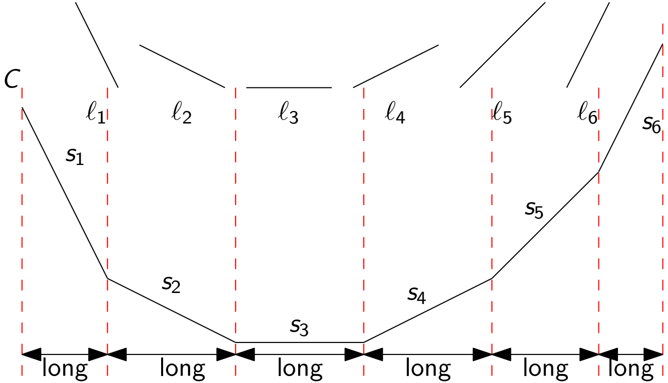
- Consider a polygonal chain C consisting of n line segments s.t. the ith segment s_i has the same slope as ℓ_i



- Consider a polygonal chain C consisting of n line segments s.t. the ith segment s_i has the same slope as ℓ_i

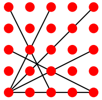


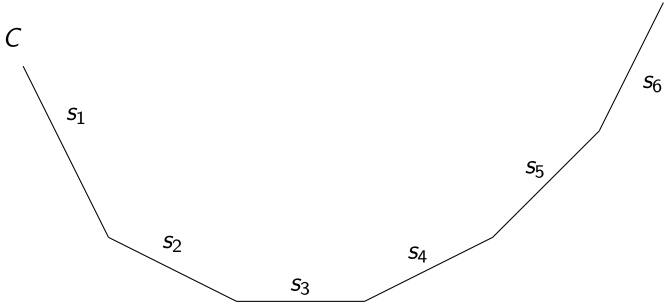
- Consider a polygonal chain C consisting of n line segments s.t. the ith segment s_i has the same slope as ℓ_i
  - C is a convex chain ($\because$ the lines are sorted by their slopes)



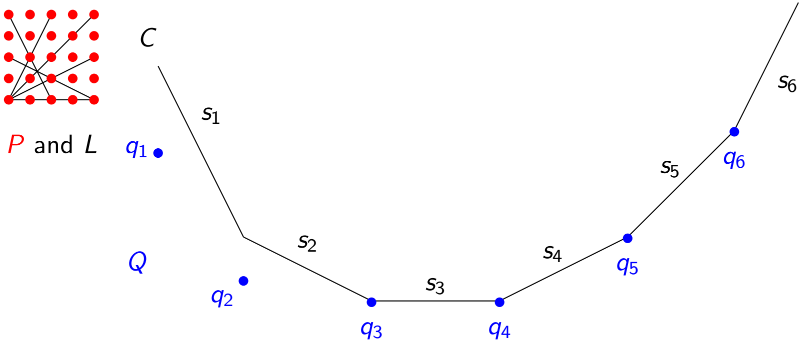
- Consider a polygonal chain C consisting of n line segments s.t. the ith segment s_i has the same slope as ℓ_i
  - C is a convex chain ($\because$ the lines are sorted by their slopes)
- Set the length of each segment sufficiently long

How to construct Q (3/3): Place a point for each line

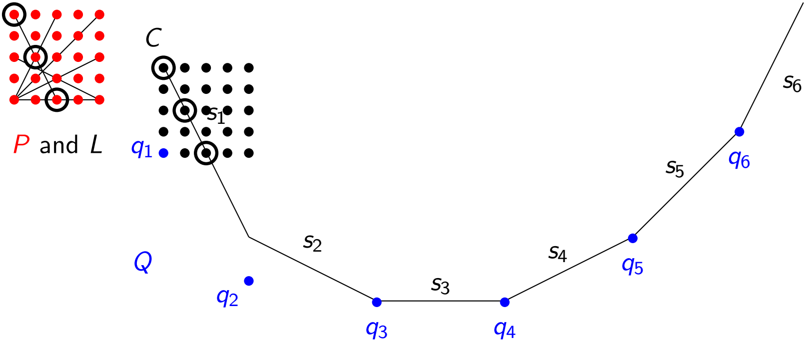




- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$

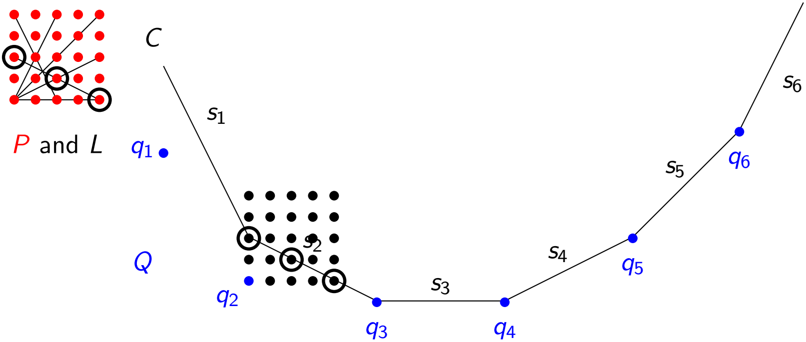


- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$



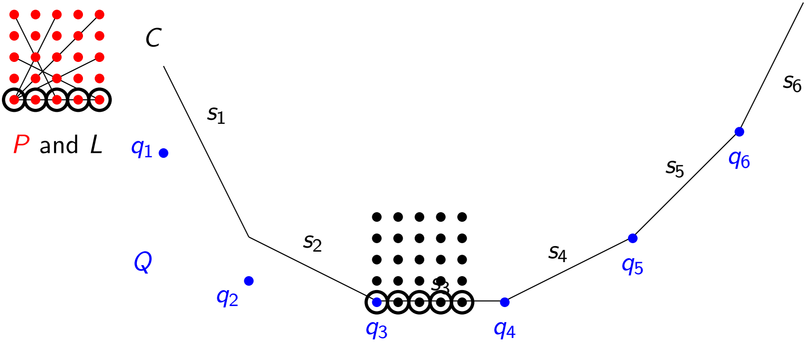
- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$

How to construct Q (3/3): Place a point for each line



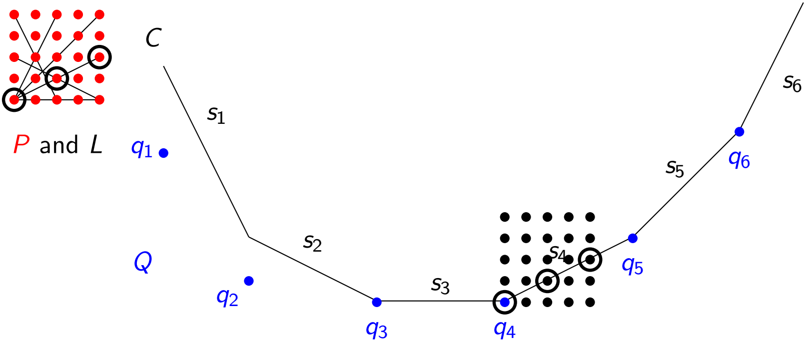
- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$

How to construct Q (3/3): Place a point for each line

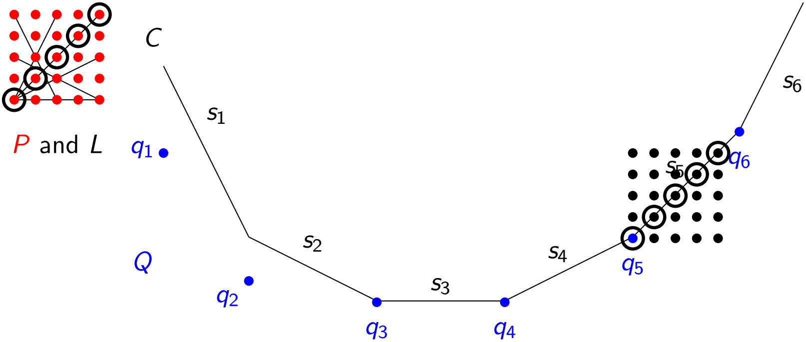


- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$

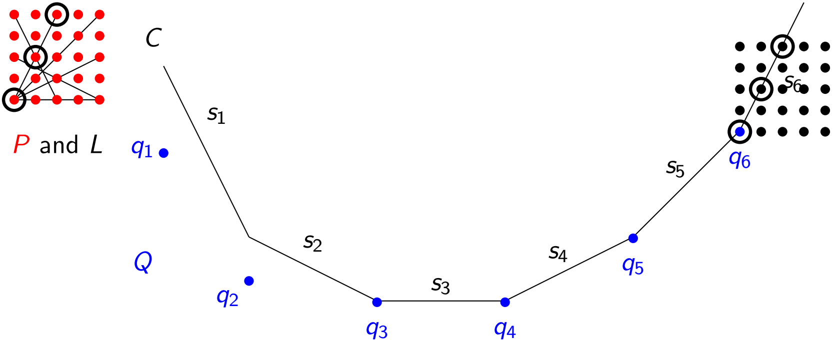
How to construct Q (3/3): Place a point for each line



- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$



- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$



- Place $Q = \{q_1, \ldots, q_n\}$ s.t. $(P \oplus \{q_i\}) \cap C = (P \cap \ell_i) \oplus \{q_i\}$
  - This is possible since each segment is long enough
- In particular, $|(P \oplus \{q_i\}) \cap C| = |(P \cap \ell_i) \oplus \{q_i\}| = |P \cap \ell_i|$

Our result (independently found by Bílka)

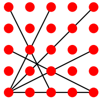
$$M(m,n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

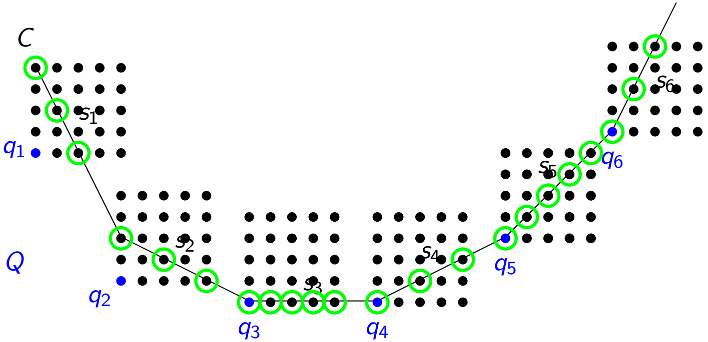
Contents

- **Basic idea**
- Fine tuning

1. Look at a lower-bound example for the point-line incidence problem
2. Construct two point sets from such an example
3. **Simulate the point-line incidences as a large convexly independent subset of the two point sets**

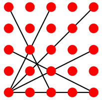
$(P \oplus Q) \cap C$ is our candidate for a large convexly independent subset

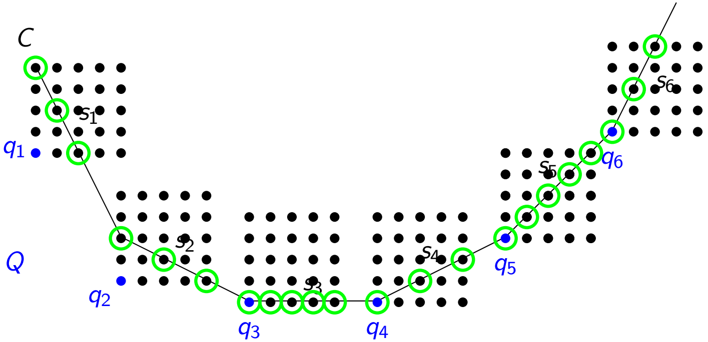




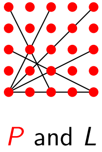
- $I(P, L) = \sum_{i=1}^{n} |P \cap \ell_i| = \sum_{i=1}^{n} |(P_i \oplus \{q_i\}) \cap C| = |(P \oplus Q) \cap C|$

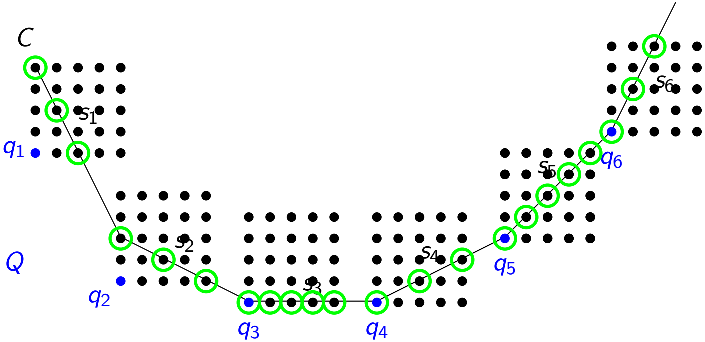
$(P \oplus Q) \cap C$ is our candidate for a large convexly independent subset





- $I(P, L) = \sum_{i=1}^{n} |P \cap \ell_i| = \sum_{i=1}^{n} |(P_i \oplus \{q_i\}) \cap C| = |(P \oplus Q) \cap C|$
- $\therefore |(P \oplus Q) \cap C| = I(P, L) = \Omega(m^{2/3} n^{2/3} + m + n)$





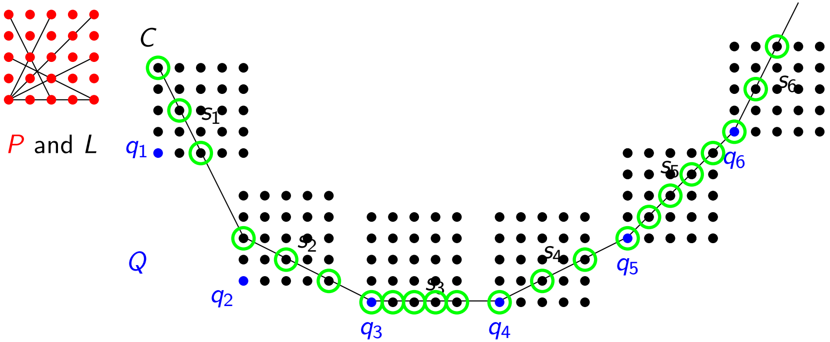
- $I(P, L) = \sum_{i=1}^{n} |P \cap \ell_i| = \sum_{i=1}^{n} |(P_i \oplus \{q_i\}) \cap C| = |(P \oplus Q) \cap C|$
- $\therefore |(P \oplus Q) \cap C| = I(P, L) = \Omega(m^{2/3} n^{2/3} + m + n)$
- **We take $(P \oplus Q) \cap C$ as our large subset**

Our result (independently found by Bílka)

$$M(m, n) = \Omega(m^{2/3}n^{2/3} + m + n)$$

Contents

- Basic idea
- **Fine tuning**



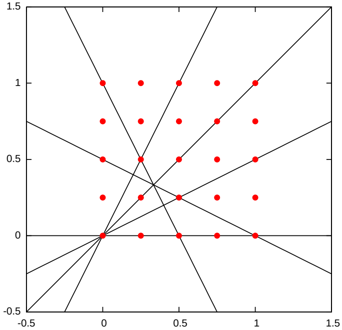
Issue

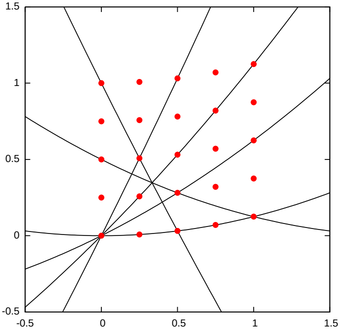
The set $(P \oplus Q) \cap C$ is not necessarily convexly independent since

- For each i, the points in $(P \oplus Q) \cap s_i$ are collinear

Transform P, L by the following map with suff. small $\varepsilon > 0$:

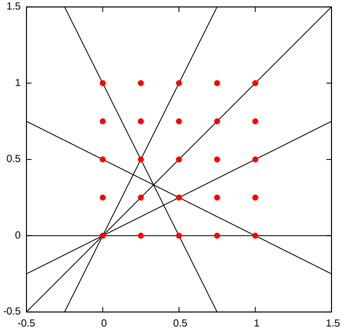
$$(x, y) \mapsto (x, y + \varepsilon x^2)$$

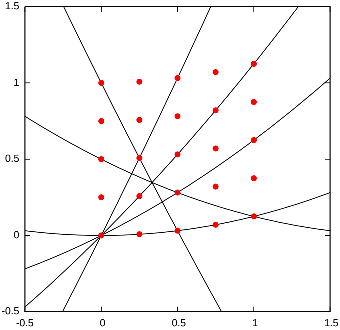




Transform P, L by the following map with suff. small $\varepsilon > 0$:

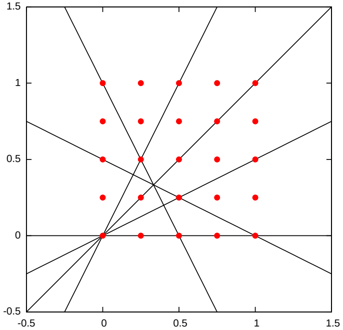
$$\begin{array}{rcl} (x, y) & \mapsto & (x, y + \varepsilon x^2) \\ \text{The line } y = a_i x + b_i & \mapsto & \text{The parabola } y = \varepsilon x^2 + a_i x + b_i \end{array}$$

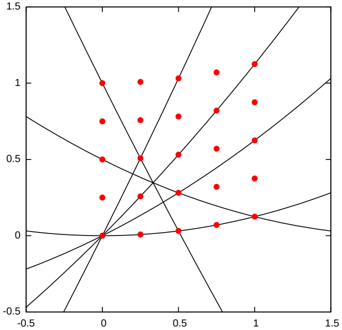




Transform P, L by the following map with suff. small $\varepsilon > 0$:

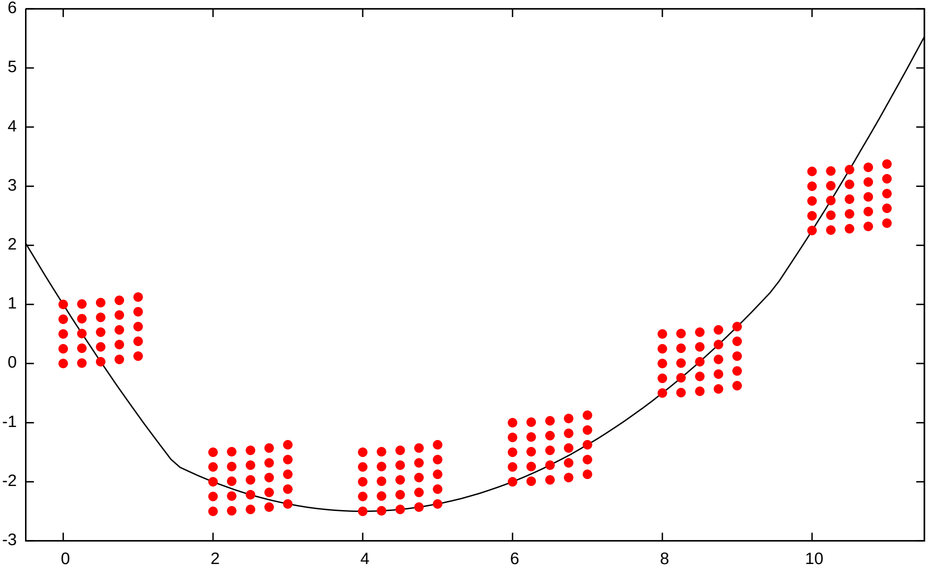
$$\begin{array}{ccc} (x,y) & \mapsto & (x, y+\varepsilon x^2) \\ \text{The line } y = a_i x + b_i & \mapsto & \text{The parabola } y = \varepsilon x^2 + a_i x + b_i \end{array}$$





We repeat the same construction as before,
and then $(P \oplus Q) \cap C$ is convexly independent $\square$

Whole picture: illustration



Theorem (Eisenbrand, Pach, Rothvoß, Sopher '08)

$$M(m, n) = \mathrm{O}(m^{2/3} n^{2/3} + m + n)$$

They only knew a linear lower bound:

$$M(m, n) = \Omega(m + n)$$

Our result

$$M(m, n) = \Omega(m^{2/3} n^{2/3} + m + n)$$

Our result was independently found by Bílka, but only when $m = n$

Instead of two point sets, what if we are given k point sets?

Notation

For k finite point sets $P_1, \ldots, P_k \subseteq \mathbb{R}^2$

$M(P_1, \ldots, P_k) = \max\{|S| : S \subseteq P_1 \oplus \cdots \oplus P_k \text{ conv'ly independent}\}$;

For k natural numbers $n_1, \ldots, n_k$

$$M(n_1, \ldots, n_k) = \max\{M(P_1, \ldots, P_k) : P_i \subseteq \mathbb{R}^2, |P_i| = n_i\}$$

Open problem 1

Determine $M(n_1, \ldots, n_k)$

- Our result: $M(n_1, n_2) = \Omega(n_1^{2/3} n_2^{2/3} + n_1 + n_2)$

What about algorithms?

Open problem 2

Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m, |Q| = n$,
how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Remark

A largest convexly independent subset of a single point set P can be found in $\mathrm{O}(n^3)$ time where $|P| = n$ (Chvátal, Klincsek '80)

- Improving the $\mathrm{O}(n^3)$ bound is a long-standing open problem (see Edelsbrunner's book '87)
- A naive application of Chvátal–Klincsek's algorithm just yields an $\mathrm{O}(m^3 n^3)$-time algorithm $\longrightarrow$ Improve!

The final slide

Open problem 1

Determine $M(n_1, \ldots, n_k)$

Open problem 2

Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m, |Q| = n$,
how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Open problem 1

Determine $M(n_1, \ldots, n_k)$

Open problem 2

Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m, |Q| = n$,
how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Bottom line

Surpringly, we know only little about Minkowski sums, even for planar point sets

Open problem 1

Determine $M(n_1, \ldots, n_k)$

Open problem 2

Given $P, Q \subseteq \mathbb{R}^2$, $|P| = m, |Q| = n$,
how quickly can we find a largest convexly independent subset of $P \oplus Q$?

Bottom line

Surpringly, we know only little about Minkowski sums, even for planar point sets

[Thank you]