Local topology of the free complex of a two-dimensional generalized convex shelling*

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Abstract

A generalized convex shelling was introduced by Kashiwabara, Nakamura & Okamoto for their representation theorem of convex geometries. Motivated by the work by Edelman & Reiner, we study local topology of the free complex of a two-dimensional separable generalized convex shelling. As a result, we prove a deletion of an element from such a complex is homotopy equivalent to a single point or two distinct points, depending on the dependency of the element to be deleted. Our result resolves an open problem by Edelman & Reiner for this case, and it can be seen as a first step toward the complete resolution from the viewpoint of a representation theorem for convex geometries by Kashiwabara, Nakamura & the author.

Keywords: abstract convex geometry, discrete geometry, topological combinatorics

1 Introduction

A convex geometry is introduced by Edelman & Jamison [4] as a combinatorial abstraction of "convexity" appearing in many objects. Recently, a representation theorem for convex geometries has been established by Kashiwabara, Nakamura & Okamoto [8], which states that every convex geometry is isomorphic to some "separable generalized convex shelling." A generalized convex shelling is defined in terms of two finite point sets in a certain dimension. Therefore, their representation theorem gives a stratification of the convex geometries by the minimum dimension in which a convex geometry can be realized as a separable generalized convex shelling. We study the topology of the free complex of a two-dimensional generalized convex shelling. As a result, we prove the following. (The necessary definitions will be given later.)

Theorem 1. *Let P and Q be nonempty finite point sets in $\mathbb{R}^2$ such that $\mathrm{conv}(P)\cap\mathrm{conv}(Q)=\emptyset$. In addition, let $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. Consider the free complex $\mathrm{Free}(\mathcal{L})$ of $\mathcal{L}$. Then the following holds.*

1. *If $\mathrm{Dep}_{\mathcal{L}}(x) \neq P$, then the deletion $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ of an element $x \in P$ is contractible (i.e., homotopy equivalent to a single point).*

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2. *If* $\mathrm{Dep}_{\mathcal{L}}(x) = P$*, then* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *is contractible or homotopy equivalent to a zero-dimensional sphere (i.e., two distinct points).*

The motivation of this work stems from Edelman & Reiner [5]. An Euler-Poincaré type formula for the number of interior points in a d-dimensional point configuration was proved by Ahrens, Gordon & McMahon [1] for $d = 2$, and proved by Edelman & Reiner [5] and Klain [10] independently for arbitrary d. The approach by Klain [10] used a more general theorem on valuation, while that by Edelman & Reiner [5] was topological. (Another proof based on oriented matroids was given by Edelman, Reiner & Welker [6].) In the paper by Edelman & Reiner [5], they studied the topology of deletions of the free complex of a convex shelling (arising from a point configuration), and also mentioned a possible generalization to a convex geometry. More precisely speaking, their open problems are as follows.

Open Problem 2 (Edelman & Reiner [5]). *Let* $\mathcal{L}$ *be a convex geometry on* E *and denote the free complex of* $\mathcal{L}$ *by* $\mathrm{Free}(\mathcal{L})$*.*

1. *Is the deletion* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *of an element* $x \in E$ *contractible if* $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$*?*
2. *Is* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *homotopy equivalent to a bouquet of spheres if* $\mathrm{Dep}_{\mathcal{L}}(x) = E$*,*

Edelman & Reiner [5] showed that this generalization can be successfully done for poset double shellings and simplicial shellings of chordal graphs. Subsequently Edelman, Reiner & Welker [6] showed that this can also successfully be done for a convex shelling of an acyclic oriented matroid. Theorem 1 states that this can also be done for a two-dimensional separable generalized convex shelling. However, our case is not just a special case. Thanks to Kashiwabara, Nakamura & Okamoto [8], every convex geometry is isomorphic to some generalized convex shelling. An explicit statement is as follows.

Proposition 3 (Kashiwabara, Nakamura & Okamoto [8]). *For every convex geometry* $\mathcal{L}$ *on a finite set* E*, there exist a natural number* d *and two point sets* $P, Q \subseteq \mathbb{R}^d$ *satisfying* $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$ *such that* $\mathcal{L}$ *is isomorphic to the generalized convex shelling of* P *with respect to* Q*.*

Therefore, our result is a step toward a resolution of Open Problem 2.

The organization of this paper is as follows. In the next section we introduce the necessary terminology about simplicial complexes and convex geometries. Section 3 sketches the proof of our theorem. We conclude the paper in Section 4 with some examples.

2 Preliminaries

In this article, we assume a moderate familiarity with graph theory.

2.1 Simplicial complexes

Let E be a finite set. An *abstract simplicial complex* on E is a nonempty family Δ of subsets of E satisfying that: if $X \in \Delta$ and $Y \subseteq X$ then $Y \in \Delta$. Often an abstract simplicial complex is simply called a simplicial complex, and in the literature they are also called independence systems and hereditary set systems. For a simplicial complex Δ on E, a subset of E is called a *face* of the simplicial complex Δ if it belongs to Δ; if not it is called a *nonface*.

For a simplicial complex Δ on E and an element $x \in E$, the *deletion* of x in Δ is defined by $\mathrm{del}_{\Delta}(x) := \{X \in \Delta : x \notin X\}$. Note that the deletion is a simplicial complex on $E \setminus \{x\}$.

When we talk about topology of a simplicial complex, we refer to a geometric realization of the simplicial complex. For details, see Matoušek's book [11].

Our topological investigation is restricted to the Euclidean case. So we just define some terms within the Euclidean space. Let X and Y be sets in $\mathbb{R}^d$. Two continuous maps $f_0, f_1 \colon X \to Y$ are *homotopic* if there exists a continuous map $F \colon X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. Two sets $X, Y \subseteq \mathbb{R}^d$ are *homotopy equivalent* if there exist two continuous maps $f \colon X \to Y$ and $g \colon Y \to X$ such that the composition $f \circ g \colon Y \to Y$ and the identity map $\mathrm{id}_Y \colon Y \to Y$ are homotopic and also the composition $g \circ f \colon X \to X$ and the identity map $\mathrm{id}_X \colon X \to X$ are homotopic.

2.2 Convex geometries

Let E be a nonempty finite set. A *convex geometry* is a family $\mathcal{L}$ of subsets of E satisfying the following three conditions:

$$\emptyset \in \mathcal{L} \text{ and } E \in \mathcal{L}, \tag{1}$$

$$\text{if } X, Y \in \mathcal{L} \text{ then } X \cap Y \in \mathcal{L}, \tag{2}$$

$$\text{if } X \in \mathcal{L} \setminus \{E\} \text{ then there exists } e \in E \setminus X \text{ such that } X \cup \{e\} \in \mathcal{L}. \tag{3}$$

For a convex geometry $\mathcal{L}$ on E, we define an operator $\tau_{\mathcal{L}} : 2^E \to 2^E$ as

$$\tau_{\mathcal{L}}(A) := \bigcap \{X \in \mathcal{L} : A \subseteq X\}.$$

The operator $\tau_{\mathcal{L}}$ is called the *closure operator* of $\mathcal{L}$. Note that $X \in \mathcal{L}$ if and only if $\tau_{\mathcal{L}}(X) = X$, by the definition. Moreover, the closure operator τ of a convex geometry $\mathcal{L}$ on E has the following important properties, which are not difficult to derive from the definition.

Extensionality: $A \subseteq \tau_{\mathcal{L}}(A)$ for all $A \subseteq E$.

Monotonicity: if $A \subseteq B \subseteq E$ then $\tau_{\mathcal{L}}(A) \subseteq \tau_{\mathcal{L}}(B)$.

For a set $A \subseteq E$, an element $e \in A$ is called an *extreme point* if $e \notin \tau_{\mathcal{L}}(A \setminus \{e\})$. We denote the set of extreme points of A by $\mathrm{ex}_{\mathcal{L}}(A)$. Namely, define the operator $\mathrm{ex}_{\mathcal{L}} \colon 2^E \to 2^E$ as

$$\mathrm{ex}_{\mathcal{L}}(A) := \{e \in A : e \text{ is an extreme point of } A\}.$$

We call $\mathrm{ex}_{\mathcal{L}}$ the *extreme point operator*. Note that the extreme point operator $\mathrm{ex}_{\mathcal{L}}$ of a convex geometry $\mathcal{L}$ on E satisfies the following properties:

Intensionality: $\mathrm{ex}_{\mathcal{L}}(A) \subseteq A$ for all $A \subseteq E$,

which is clear from the definition. Ando [2] gives a detailed treatment on closure operators and extreme point operators in a more general setting.

A set $A \subseteq E$ is called *independent* if $\mathrm{ex}_{\mathcal{L}}(A) = A$. We say that e *depends on* f if there exists an independent set A such that $f \in A$, $e \in \tau_{\mathcal{L}}(A)$ and $e \notin \tau_{\mathcal{L}}(A \setminus \{f\})$. We denote the set of all elements on which e depends by $\mathrm{Dep}_{\mathcal{L}}(e)$. A set $X \subseteq E$ is called *free* if $X \in \mathcal{L}$ and $\mathrm{ex}_{\mathcal{L}}(X) = X$. We denote the family of free sets of a convex geometry $\mathcal{L}$ by $\mathrm{Free}(\mathcal{L})$. The following lemma is well-known and relatively easy to show.

Lemma 4. *Let $\mathcal{L}$ be a convex geometry on E. Then* $\mathrm{Free}(\mathcal{L})$ *is a simplicial complex on E.*

Thus, it is natural that we call Free($\mathcal{L}$) the *free complex* of a convex geometry $\mathcal{L}$. Note that in general there might exist an element $x \in E$ such that $\{x\} \notin \mathrm{Free}(\mathcal{L})$.

Now we define a generalized convex shelling. Let P and Q be finite point sets in $\mathbb{R}^d$ (where d is a positive integer) such that $P \cap \mathrm{conv}(Q) = \emptyset$. Then the *generalized convex shelling* on P with respect to Q is a convex geometry $\mathcal{L}$ defined as follows: $\mathcal{L} = \{X \subseteq P : P \cap \mathrm{conv}(X \cup Q) = X\}$. We also call a convex geometry $\mathcal{L}$ a *d-dimensional generalized convex shelling* if there exist finite point sets P and Q in $\mathbb{R}^d$ such that $P \cap \mathrm{conv}(Q) = \emptyset$ and $\mathcal{L}$ is isomorphic to the generalized convex shelling on P with respect to Q. Note that a generalized convex shelling does not depend on Q, but only on the convex hull of Q. However, we keep the phrase "the generalized convex shelling with respect to Q," not "the generalized convex shelling with respect to the convex hull of Q," for the simplicity.

The next lemma tells us the closure operator and the extreme point operator of a generalized convex shelling.

Lemma 5. *Let $\mathcal{L}$ be a generalized convex shelling on P with respect to Q. Then, we have*

$$\begin{aligned} \tau_{\mathcal{L}}(X) &= P \cap \mathrm{conv}(X \cup Q), \\ \mathrm{ex}_{\mathcal{L}}(X) &= \{x \in X : x \textit{ is an extreme point of } \mathrm{conv}(X \cup Q)\} \end{aligned}$$

for each set $X \subseteq P$.[1] *In particular, $X \subseteq P$ is free if and only if $P \cap \mathrm{conv}(X \cup Q) = X$ and every element of X is an extreme point of* $\mathrm{conv}(X \cup Q)$.

Proof. The statement for the closure operator has already been proved by Kashiwabara, Nakamura & Okamoto [8]. Here, we prove that the extreme point operator is as claimed. The proof is based on the following chain of equivalences.

$$\begin{aligned} p \in \mathrm{ex}_{\mathcal{L}}(X) &\Leftrightarrow p \notin \tau_{\mathcal{L}}(X \setminus \{p\}) && \text{(definition of } \mathrm{ex}_{\mathcal{L}}) \\ &\Leftrightarrow p \notin P \cap \mathrm{conv}((X \setminus \{p\}) \cup Q) && \text{(the first part of this lemma)} \\ &\Leftrightarrow p \notin \mathrm{conv}((X \setminus \{p\}) \cup Q) && (p \in P) \\ &\Leftrightarrow p \notin \mathrm{conv}((X \cup Q) \setminus \{p\}) && \\ &\Leftrightarrow p \text{ is an extreme point of } \mathrm{conv}(X \cup Q) && \text{(definition of an extreme point).} \end{aligned}$$

The last part is immediate from the first two parts of this lemma and the definition of a free set. ☐

In this paper, we study the free complex of a two-dimensional separable generalized convex shelling. Since we already know that Open Problem 2 has been solved when $Q = \emptyset$ [5], we may make the following assumption, which is important in this paper.

Assumption 6. *When we talk about the generalized convex shelling on P with respect to Q in the rest of this paper, Q is always nonempty unless stated otherwise.*

Here, we define a clique complex of a graph. Let G be a graph. A *clique* of G is a vertex subset of G which induces a complete subgraph. The *clique complex* of G is the family of cliques of G. We also treat the empty set and the single vertices as cliques, so the clique complex is actually a simplicial complex. In the literature, a clique complex is also called a *flag complex.*

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[1] Here, you would notice that we are using the phrase "extreme point" in two different meanings. One for an extreme point of a convex geometry, one for an extreme point of the convex hull. But they should be clear from the context.

3 Proof of Theorem 1

3.1 Basic properties and the outline

Now we concentrate on two-dimensional separable generalized convex shellings. Let P and Q be two nonempty finite point sets in $\mathbb{R}^2$ such that $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$. Denote by $\mathcal{L}$ the generalized convex shelling on P with respect to Q. Since $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$, there exists a line which strictly separates $\mathrm{conv}(P)$ and $\mathrm{conv}(Q)$. Fix such a line, and call it ℓ. In the rest of the paper, we visualize ℓ as a vertical line, and P is put left to ℓ and Q right to ℓ.

To prove Theorem 1, we use the following fact.

Lemma 7 (Hachimori & Nakamura [7]). *A minimal nonface of the free complex* $\mathrm{Free}(\mathcal{L})$ *of a d-dimensional generalized convex shelling is of size at most d.*

It is well known that a simplicial complex whose minimal nonfaces are of size 2 is a clique complex of some graph. (Although this fact is folklore, a proof can be found in a paper by Kashiwabara, Okamoto & Uno [9], for example.) Therefore, the free complex of a two-dimensional generalized convex shelling $\mathcal{L}$ is the clique complex of some graph, and this graph is actually the one-dimensional skeleton of $\mathrm{Free}(\mathcal{L})$. Here, the *$d$-dimensional skeleton* of a simplicial complex Δ is a collection $\{X \in \Delta : |X| \leq d{+}1\}$. A one-dimensional skeleton can be regarded as a graph in the following way: The vertex set of the graph is the set of faces of size 1, and the edge set of the graph is the set of faces of size 2. Denote by $G(\mathcal{L})$ the 1-dimensional skeleton of $\mathrm{Free}(\mathcal{L})$. The following lemma tells what $G(\mathcal{L})$ is.

Lemma 8. *A point $x \in P$ is a vertex of $G(\mathcal{L})$ if and only if $P \cap \mathrm{conv}(\{x\} \cup Q) = \{x\}$ holds, i.e.,* $\mathrm{conv}(\{x\} \cup Q)$ *contains no point of P except for x. Two points $x, y \in P$ form an edge of $G(\mathcal{L})$ if and only if they are vertices of $G(\mathcal{L})$ and $P \cap \mathrm{conv}(\{x, y\} \cup Q) = \{x, y\}$ holds, i.e.,* $\mathrm{conv}(\{x, y\} \cup Q)$ *contains no point of P except for x, y.*

Proof. First of all, notice that $x \in P$ is a vertex of $G(\mathcal{L})$ if and only if $\{x\} \in \mathrm{Free}(\mathcal{L})$, and that $\{x, y\} \subseteq P$ is an edge of $G(\mathcal{L})$ if and only if $\{x, y\} \in \mathrm{Free}(\mathcal{L})$.

Assume that $x \in P$ satisfies $\{x\} \in \mathrm{Free}(\mathcal{L})$. Then, from Lemma 5, this is equivalent to saying that $P \cap \mathrm{conv}(\{x\} \cup Q) = \{x\}$ and x is an extreme point of $\mathrm{conv}(\{x\} \cup Q)$. However, x is always an extreme point of $\mathrm{conv}(\{x\} \cup Q)$ since we have the assumption that $P \cap \mathrm{conv}(Q) = \emptyset$. Thus, we have shown that $x \in P$ is a vertex of $G(\mathcal{L})$ if and only if $P \cap \mathrm{conv}(\{x\} \cup Q) = \{x\}$.

For the second part, first choose arbitrary two vertices $x, y \in V(G(\mathcal{L}))$ of $G(\mathcal{L})$. Namely, x and y satisfy the condition in the first part. Now we show that $\{x, y\}$ is an edge of $G(\mathcal{L})$ if and only if $P \cap \mathrm{conv}(\{x, y\} \cup Q) = \{x, y\}$. Assume that $\{x, y\}$ is an edge of $G(\mathcal{L})$. Again, from Lemma 5, this is equivalent to saying that $P \cap \mathrm{conv}(\{x, y\} \cup Q) = \{x, y\}$ and x and y are extreme points of $\mathrm{conv}(\{x, y\} \cup Q)$. However, the property that x and y are extreme points of $\mathrm{conv}(\{x, y\} \cup Q)$ can be derived from our assumption that x and y are vertices of $G(\mathcal{L})$. To verify this, suppose that x is not an extreme point of $\mathrm{conv}(\{x, y\} \cup Q)$. This means that $x \in \mathrm{conv}(\{y\} \cup Q)$. However, this implies that y violates the condition that $P \cap \mathrm{conv}(\{y\} \cup Q) = \{y\}$. So this is a contradiction to the first part of this lemma. Thus, we have shown the second part. □

Thanks to Lemma 8, we can regard $G(\mathcal{L})$ as a geometric graph. Namely, we can geometrically construct $G(\mathcal{L})$ in the following way. First, we remove a point $x \in P$ if and only if the condition that $P \cap \mathrm{conv}(\{x\} \cup Q) = \{x\}$ is violated. The remaining points from P are the vertices of $G(\mathcal{L})$ (by Lemma 8). Among these remaining points, we connect two points $x, y \in P$ by a line segment if and only if $P \cap \mathrm{conv}(\{x, y\} \cup Q) = \{x, y\}$ holds. This process gives the edges of $G(\mathcal{L})$. Figure 1 is an example of $G(\mathcal{L})$, where P consists of eight points $1, \ldots, 8$ and Q of two points q_1 and q_2. The right one is the resulting geometric graph $G(\mathcal{L})$. The point 2 does not remain

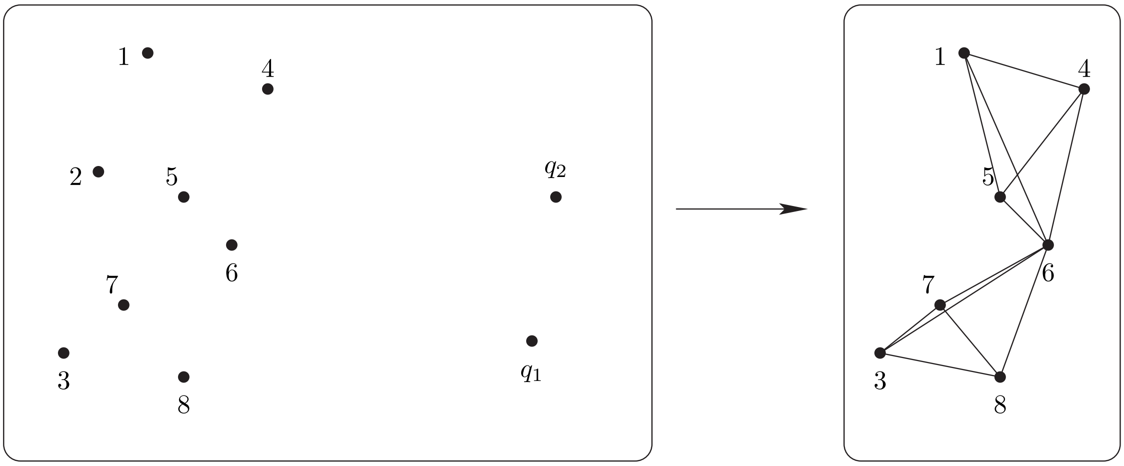

Figure 1: (Left) given sets of points. (Right) the resulting geometric graph $G(\mathcal{L})$.

in $G(\mathcal{L})$ as a vertex since $P \cap \mathrm{conv}(\{2\} \cup Q) = \{2, 5, 6\}$.

The rest of the proof is organized in the following way.

1. We prove that $G(\mathcal{L})$ is connected (Lemma 9).
2. We prove that $G(\mathcal{L})$ is chordal (Lemma 10).
3. We observe that the clique complex of a connected chordal graph is contractible (Lemma 11).
4. We show the relation of a cut-vertex of $G(\mathcal{L})$ and a dependency set (Lemmas 14 and 15).

The rest of the section is divided according to the proof scheme above.

3.2 Connectedness of the graph

First, we show the connectedness of $G(\mathcal{L})$.

Lemma 9. *In the setup above, $G(\mathcal{L})$ is connected.*

Proof. The proof is done by induction on the number of points in P. When $|P| = 1$, $G(\mathcal{L})$ consists of only one vertex. So $G(\mathcal{L})$ is connected.

Assume that $|P| > 1$. Let us choose a point v of P which is the furthest from $\mathrm{conv}(Q)$.

Let $P' = P \setminus \{v\}$ and $\mathcal{L}'$ be the generalized convex shelling on P' with respect to Q. We have two cases.

Case 1: v is not a vertex of $G(\mathcal{L})$. In this case, we claim that $G(\mathcal{L}') = G(\mathcal{L})$. First we show that the vertex sets are the same. To show that, suppose not. If $G(\mathcal{L}')$ owns a vertex u which is not a vertex of $G(\mathcal{L})$, then it must hold that $v \in \mathrm{conv}(\{u\} \cup Q)$. However, this means that v is closer to $\mathrm{conv}(Q)$ than u. This contradicts the choice of v. On the other hand, if $G(\mathcal{L})$ owns a vertex w which is not a vertex of $G(\mathcal{L}')$, then there must exist a point $x \in P' \setminus P$ such that $x \in \mathrm{conv}(\{w\} \cup Q)$. However, this is impossible because $P' \subseteq P$, consequently $P' \setminus P = \emptyset$. Thus, the vertex sets of $G(\mathcal{L})$ and $G(\mathcal{L}')$ are the same.

Secondly we show that the edge sets are the same. This can be done in a similar way to the vertex sets. Thus, the claim follows.

By induction hypothesis, $G(\mathcal{L}')$ is connected. Then from the claim above, we conclude that $G(\mathcal{L})$ is connected.

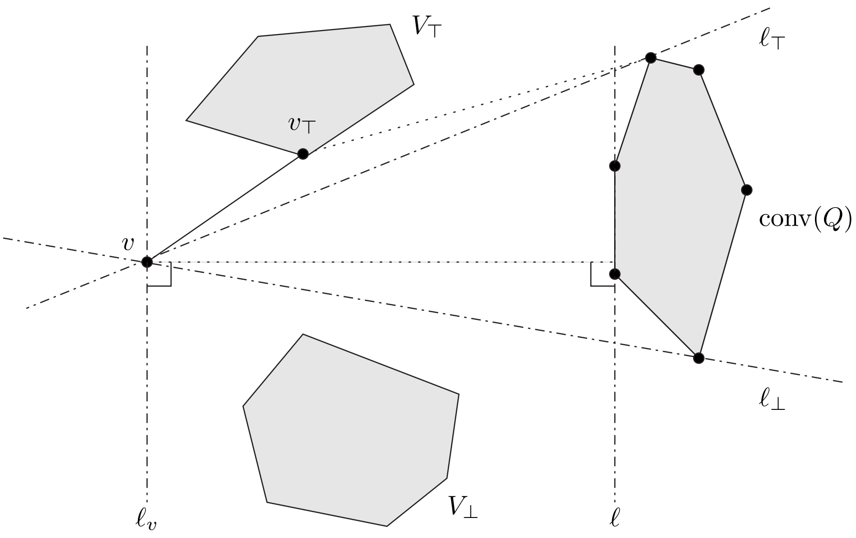

Figure 2: v is not an isolated vertex.

Case 2: v is a vertex of $G(\mathcal{L})$. In this case, we introduce the following notation. Let ℓ be a line supporting $\operatorname{conv}(Q)$ and perpendicular to the line spanned by v and the point in $\operatorname{conv}(Q)$ closest to v. Further, let ℓ_v be a line parallel to ℓ and passing through v. Denote by $\ell_\top$ and $\ell_\perp$ the lines supporting $\operatorname{conv}(\{v\} \cup Q)$ and passing through v. These lines ℓ, ℓ_v, $\ell_\top$ and $\ell_\perp$ are well-defined since $\operatorname{conv}(P) \cap \operatorname{conv}(Q) = \emptyset$. See Figure 2. Note that $\ell_\top$ and $\ell_\perp$ coincide when $|Q| = 1$. By an argument similar to the first case, we can observe that $G(\mathcal{L}') = G(\mathcal{L}) - v$.

Now, by the induction hypothesis, $G(\mathcal{L}')$ is connected. Therefore, it suffices to show that v is not an isolated vertex of $G(\mathcal{L})$.

From our choices, the vertices of $G(\mathcal{L})$ other than v should lie either in the space bounded by ℓ_v and $\ell_\top$ or in the space bounded by ℓ_v and $\ell_\perp$. Let $V_\top$ (and $V_\perp$) be the set of vertices of $G(\mathcal{L})$ lying in the former (and latter, respectively) space, as in Figure 2. Note that at least one of the two is nonempty since the number of vertices of $G(\mathcal{L})$ is more than one. Assume that $V_\top$ is nonempty, without loss of generality. Then choose a vertex in $V_\top$ which is closest to $\ell_\top$ and name it $v_\top$. We can see that $P \cap \operatorname{conv}(\{v, v_\top\} \cup Q) = \{v, v_\top\}$ because of our choices. This means that $\{v, v_\top\}$ forms an edge in $G(\mathcal{L})$, thus v is not an isolated vertex of $G(\mathcal{L})$. □

3.3 Chordality of the graph

Next, we show the chordality of $G(\mathcal{L})$. A graph is *chordal* if it has no induced cycle of length more than three.

Lemma 10. *In the setup above, $G(\mathcal{L})$ is chordal.*

Proof. Suppose, for the contradiction, that $G(\mathcal{L})$ has an induced cycle of length more than 3. Choose such an induced cycle C arbitrarily, and denote by V_C the set of vertices of C.

The convex hull of V_C and the convex hull of Q have two outer common tangents ℓ_1 and ℓ_2.[2] Choose $v_1 \in V_C \cap \ell_1$ and $v_2 \in V_C \cap \ell_2$ arbitrarily.

We observe that $v_1 \neq v_2$. To show that, suppose not. Then, since ℓ_1 and ℓ_2 are outer common tangents of $\operatorname{conv}(V_C)$ and $\operatorname{conv}(Q)$, all points of V_C must be contained in $\operatorname{conv}(\{v_1\} \cup Q)$.

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[2] Here, an *outer common tangent* of two convex sets A and B is a line ℓ which touches A, B and determines a halfplane containing both of A and B.

However, this is a contradiction to the fact that v_1 is a vertex of $G(\mathcal{L})$. Therefore, v_1 is distinct from v_2.

Now, we have two cases.

Case 1: $\{v_1, v_2\}$ is an edge of C. In the cycle C, two vertices v_1 and v_2 are joined by two distinct paths. By our assumption, one of them is v_1v_2, namely a path of length one. Let $v_1u_1 \cdots u_kv_2$ be the other path. Since the length of C is more than three, it holds that $k \geq 2$.

Since $\{v_1, v_2\}$ is an edge of $G(\mathcal{L})$, by Lemma 8 it follows that $\mathrm{conv}(\{v_1, v_2\} \cup Q)$ contains no point of $P \setminus \{v_1, v_2\}$, in particular none of $\{u_1, \ldots, u_k\}$. Since we chose v_1 and v_2 via the outer common tangents of $\mathrm{conv}(V_C)$ and $\mathrm{conv}(Q)$, this implies that all points of $\{u_1, \ldots, u_k\}$ lie in the region bounded by ℓ_1, ℓ_2 and the line spanned by v_1, v_2. Take a point $u_i \in \{u_1, \ldots, u_k\}$ which is closest to the line segment $\overline{v_1v_2}$. Since $k \geq 2$, at least one of $\{v_1, u_i\}$ and $\{v_2, u_i\}$ is not an edge of $G(\mathcal{L})$. Without loss of generality, assume that $\{v_1, u_i\}$ is not an edge. Since all points of $\{u_1, \ldots, u_k\}$ lie in the region bounded by ℓ_1, ℓ_2 and the line spanned by v_1, v_2, we have $\mathrm{conv}(\{v_1, v_2\} \cup Q) \subseteq \mathrm{conv}(\{v_1, v_2, u_i\} \cup Q)$. Since $\{v_1, u_i\}$ is not an edge of $G(\mathcal{L})$, by Lemma 8 there must exist a point $p \in \mathrm{conv}(\{v_1, u_i\} \cup Q)$. However, $\{v_1, u_1\}, \{u_1, u_2\}, \ldots, \{u_{i-1}, u_i\}$ are edges of $G(\mathcal{L})$ and we have $\mathrm{conv}(\{v_1, u_i\} \cup Q) \subseteq \bigcup_{j=0}^{i-1} \mathrm{conv}(\{u_j, u_{j+1}\} \cup Q)$ by our choices, where u_0 is set to v_1. This means that there exists some index $j \in \{0, \ldots, i-1\}$ such that the set $\mathrm{conv}(\{u_j, u_{j+1}\} \cup Q)$ contains p. Lemma 8 implies that $\{u_j, u_{j+1}\}$ is not an edge of $G(\mathcal{L})$. This is a contradiction.

Case 2: $\{v_1, v_2\}$ is not an edge of C. By Lemma 8, there must exist a point of $P \setminus \{v_1, v_2\}$ belonging to $\mathrm{conv}(\{v_1, v_2\} \cup Q)$. Let p be the furthest point from the line spanned by v_1 and v_2 among all such points in $P \setminus \{v_1, v_2\}$. Consider a path in C joining v_1 and v_2, and denote it by $v_1u_1 \cdots u_kv_2$. Since $\{v_1, v_2\}$ is not an edge, we have $k \geq 1$.

Now we claim that this path has p as a vertex. To show that, denote by ℓ the line spanned by v_1 and v_2 and further denote by ℓ_p the line parallel to ℓ which passes the point p. Because of our choice, the points $u_1, \ldots, u_k$ must lie in the region bounded by ℓ, ℓ_p, ℓ_1 and ℓ_2. Then, we can see that $\bigcup_{j=0}^{k} \mathrm{conv}(\{u_j, u_{j+1}\} \cup Q)$ contains p, where u_0 and u_{k+1} are set to v_1 and v_2 respectively. This implies the existence of some $j \in \{0, \ldots, k\}$ such that $\mathrm{conv}(\{u_j, u_{j+1}\} \cup Q)$ contains p. This contradicts the fact that $\{u_j, u_{j+1}\}$ is an edge of $G(\mathcal{L})$. Thus the claim is proved.

Now, we know that a path in C joining v_1 and v_2 passes p. However, we have two such paths in C. Since they must not share a vertex other than v_1 and v_2, this is a contradiction. □

Then we observe the next lemma.

Lemma 11. *The clique complex of a connected chordal graph is homotopy equivalent to a single point.*

Proof. We prove it by induction on the number of vertices. If a graph has only one vertex, it is always connected and chordal, and the clique complex consists of a single point. So the statement is true.

Assume that a connected chordal graph G has at least two vertices. Then we use a useful property of chordal graphs due to Dirac [3]: Every chordal graph has a vertex whose neighbors form a clique. Let us take such a vertex and name v. Then v and its neighbors form a clique in G. Since G is connected, the neighborhood of v is not empty. Now remove v from G to obtain a smaller graph $G - v$. Since $G - v$ is also connected and chordal, the clique complex of $G - v$ is homotopy equivalent to a single point by the induction hypothesis. Then we put v back to G.

This corresponds to gluing the clique complex of $G - v$ and a simplex by a facet of the simplex. So the result is also homotopy equivalent to a single point. $\square$

Therefore, from Lemmas 10 and 11, we immediately obtain the following.

Corollary 12. *The free complex* $\mathrm{Free}(\mathcal{L})$ *of a two-dimensional generalized convex shelling is homotopy equivalent to a single point.*

Note that Corollary 12 holds for all d-dimensional generalized convex shellings even if $Q = \emptyset$. This has been shown by Edelman & Reiner [5] (based on a theorem by Edelman & Jamison [4]) as a statement that the free complex of an arbitrary convex geometry is homotopy equivalent to a single point. We may notice that Corollary 12 and their statement are linked via the affine representation theorem for convex geometries by Kashiwabara, Nakamura & Okamoto [8]. However, our approach is discrete-geometric while they used tools from topological combinatorics.

Since an induced subgraph of a chordal graph is also chordal, we can immediately see the following.

Lemma 13. *Let* x *be a vertex of* $G(\mathcal{L})$ *and* c_x *be the number of connected components of* $G(\mathcal{L}){-}x$*. Then* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *is homotopy equivalent to* c_x *distinct points.*

Therefore, in order to prove Theorem 1, we only have to show the following two lemmas.

3.4 Relationship of a cut-vertex and a dependency set

Lemma 14. *Let* x *be a cut-vertex of* $G(\mathcal{L})$*. Then* $G(\mathcal{L}){-}x$ *has exactly two connected components.*

Proof. Since x is a vertex of $G(\mathcal{L})$, we have $P \cap \mathrm{conv}(\{x\} \cup Q) = \{x\}$. Consider two connected components C_1 and C_2 of $G(\mathcal{L}){-}x$. Choose $u \in V(C_1)$ and $v \in V(C_2)$ such that $\{x, u\}$ and $\{x, v\}$ are edges of $G(\mathcal{L})$. Since $\{u, v\}$ is not an edge of $G(\mathcal{L})$, it should hold that $P \cap \mathrm{conv}(\{u, v\} \cup Q) \neq \{u, v\}$. Let $P' := (P \cap \mathrm{conv}(\{u, v\} \cup Q)) \setminus \{u, v\}$. From the observation above, $P' \neq \emptyset$. We claim that $x \in P'$. To show that, suppose that $x \notin P'$ for the sake of contradiction. Let P'' be the set of vertices of $G(\mathcal{L})$ which also belong to P', namely $P'' := V \cap P'$. (Note that $P'' \neq \emptyset$.) Then each $y \in P''$ lies in either

(1) $\mathrm{conv}(\{u\} \cup Q)$,

(2) $\mathrm{conv}(\{v\} \cup Q)$, or

(3) $\mathrm{conv}(\{u, v\} \cup Q) \setminus (\mathrm{conv}(\{u\} \cup Q) \cup \mathrm{conv}(\{v\} \cup Q))$.

When (1) or (2) happens, u or v cannot be a vertex of $G(\mathcal{L})$ by Lemma 8, respectively. This is a contradiction. Therefore, it holds that $P'' \subseteq \mathrm{conv}(\{u, v\} \cup Q) \setminus (\mathrm{conv}(\{u\} \cup Q) \cup \mathrm{conv}(\{v\} \cup Q))$. Now, let us take the convex hull of $P'' \cup \{u, v\}$, and it has two chains of edges connecting u and v. By our assumption, one is the edge $\{u, v\}$ and the other consists of at least two edges. Consider the latter one. (In Figure 3, the gray region is the convex hull of $P'' \cup \{u, v\}$.) Then this chain corresponds to a path from u to v in $G(\mathcal{L})$. However, this means that C_1 and C_2 are not distinct connected component of $G(\mathcal{L}){-}x$, which gives a contradiction. Thus, we have $x \in P'$.

Now, suppose that $G(\mathcal{L}){-}x$ has at least three connected components, say C_1, C_2, C_3. As before, choose $u \in V(C_1), v \in V(C_2), w \in V(C_3)$ such that $\{x, u\}$, $\{x, v\}$ and $\{x, w\}$ are edges of $G(\mathcal{L})$. Consider two outer common tangents ℓ_1, ℓ_2 of $\mathrm{conv}(\{u, v, w\})$ and $\mathrm{conv}(Q)$. Without loss of generality, let u be the intersection of ℓ_1 and $\mathrm{conv}(\{u, v, w\})$, and v be the intersection of ℓ_2 and $\mathrm{conv}(\{u, v, w\})$. Note that these intersection points must be distinct by the same reason as in the proof of Lemma 10. We have two cases. Let ℓ be the line spanned by u and v.

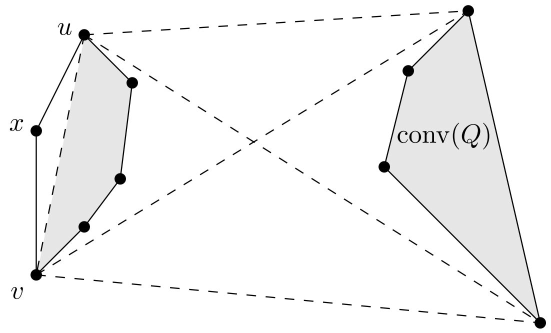

Figure 3: Where does x lie?

Case 1: w and Q lie on the same side of ℓ. In this case, we can see that $\mathrm{conv}(\{w\} \cup Q)$ is identical to the intersection of $\mathrm{conv}(\{u, v\} \cup Q)$, $\mathrm{conv}(\{v, w\} \cup Q)$ and $\mathrm{conv}(\{u, w\} \cup Q)$. By the claim above, x belongs to all of these three sets. Therefore, x belongs to $\mathrm{conv}(\{w\} \cup Q)$. However, since w is a vertex of $G(\mathcal{L})$, this contradicts Lemma 8.

Case 2: w and Q lie on the different sides of ℓ. By an argument similar to Case 1, we can observe that x belongs to $\mathrm{conv}(\{w\} \cup Q)$, which is again a contradiction. ☐

Lemma 15. *Let x be a vertex of $G(\mathcal{L})$. If x is a cut-vertex of $G(\mathcal{L})$, then* $\mathrm{Dep}_{\mathcal{L}}(x) = P$.

Proof. Assume that x is a cut-vertex of $G(\mathcal{L})$. We have to show that $\mathrm{Dep}_{\mathcal{L}}(x) = P$, namely, for every $y \in P$ there exists a set $A \subseteq P$ such that

(1) $\mathrm{ex}_{\mathcal{L}}(A) = A$,

(2) $y \in A$,

(3) $x \in \tau_{\mathcal{L}}(A)$, and

(4) $x \notin \tau_{\mathcal{L}}(A \setminus \{y\})$.

Fix $y \in P$ arbitrarily. According to the position of y, we have several cases. Let $\ell_\top$ and $\ell_\perp$ be lines supporting $\mathrm{conv}(\{x\} \cup Q)$ which pass through x. (In case $|Q| = 1$, they coincide.) Denote by $\ell_\top^{\supseteq}$ the closed halfplane determined by $\ell_\top$ which contains Q, and by $\ell_\top^{\not\supseteq}$ the closed halfplane determined by $\ell_\top$ which does not contain Q. We define $\ell_\perp^{\supseteq}$ and $\ell_\perp^{\not\supseteq}$ analogously. Then, the whole plane is decomposed into four parts:

$$\begin{aligned} R_1 &:= \ell_\top^{\supseteq} \cap \ell_\perp^{\supseteq}, \\ R_2 &:= \ell_\top^{\supseteq} \cap \ell_\perp^{\not\supseteq}, \\ R_3 &:= \ell_\top^{\not\supseteq} \cap \ell_\perp^{\supseteq}, \\ R_4 &:= \ell_\top^{\not\supseteq} \cap \ell_\perp^{\not\supseteq}. \end{aligned}$$

Figure 4 illustrates this decomposition.

First, let us observe that R_1 contains no point from $P \setminus \{x\}$. To show that, suppose that it contains a point p. If it lies in "front" of $\mathrm{conv}(Q)$ (i.e., the bounded region determined by $\ell_\top$, $\ell_\perp$ and $\mathrm{conv}(Q)$), then it holds that $p \in \mathrm{conv}(\{x\} \cup Q)$. However, this means that x is not a vertex of $G(\mathcal{L})$ by Lemma 8, which is a contradiction. Otherwise, the line segment connecting p and x intersects $\mathrm{conv}(Q)$. However, this implies that $\mathrm{conv}(P) \cap \mathrm{conv}(Q)$ is not empty, which is also a contradiction. Thus, R_1 contains no point from $P \setminus \{x\}$.

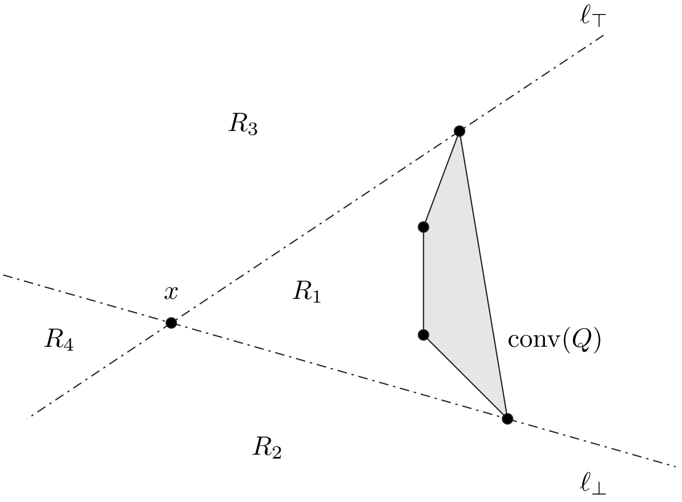

Figure 4: The whole plane is divided into four parts.

Hence we obtain three cases to consider about the position of y. However, the cases of R_2 and R_3 are symmetric. So the essential cases are the following two.

Case 1: y lies in R_4. In this case, we can choose $\{y\}$ as A. We claim that this A satisfies the conditions (1)–(4) above. Since y is an extreme point of $\mathrm{conv}(\{y\} \cup Q)$, by Lemma 5 the first condition is fulfilled. The second condition is true by definition. The third and fourth conditions can be verified via Lemma 5. This case is done.

Case 2: y lies in R_3. From the argument in the proof of Lemma 14, we can see that one component $G_\top$ of $G(\mathcal{L})-x$ lies in R_3 and the other component $G_\perp$ of $G(\mathcal{L})-x$ is contained in R_2. Both of them are non-empty. Now, let A be the set of points of P which are moreover the extreme points of $\mathrm{conv}(\{y\} \cup V(G_\perp) \cup Q)$. We claim that this A satisfies the conditions (1)–(4) above.

By Lemma 5, the condition (1) is clear. Since y lies on the different side of $\ell_\top$ than Q and $V(G_\perp)$, we can see that y is an extreme point of $\mathrm{conv}(\{y\} \cup V(G_\perp) \cup Q)$. Hence, the condition (2) is fulfilled. Since Q and $V(G_\perp)$ lie on different sides of $\ell_\perp$, and no vertex of $G_\perp$ lies on ℓ (because of Lemma 8), we can see that $x \notin \mathrm{conv}(V(G_\perp) \cup Q)$, which means that the condition (4) is satisfied.

To verify the condition (3), we use the following property of the closure operator [4].

Anti-exchange property: Let $A \subseteq E$ be a set and $e, f \in E$ be two distinct elements such that $e, f \notin \tau_{\mathcal{L}}(A)$. If $f \in \tau_{\mathcal{L}}(A \cup \{e\})$ then $e \notin \tau_{\mathcal{L}}(A \cup \{f\})$.

Take any vertex v of $G_\perp$. By the anti-exchange property and Lemma 5, we can find a point $z \in \mathrm{conv}(\{y\} \cup Q)$ which is a vertex of $G_\top$. Since x is a cut-vertex of $G(\mathcal{L})$, $\{z, v\}$ is not an edge of $G(\mathcal{L})$. Then, by Lemma 8 and the fact that x is a cut-vertex, we see that $\mathrm{conv}(\{z, v\} \cup Q)$ contains x. Namely, we have

$$x \in \mathrm{conv}(\{z, v\} \cup Q) \subseteq \mathrm{conv}(\{y, v\} \cup Q) \subseteq \mathrm{conv}(\{y\} \cup V(G_\perp) \cup Q),$$

which implies that the condition (3) holds by Lemma 5. In this way, the whole proof is completed. $\square$

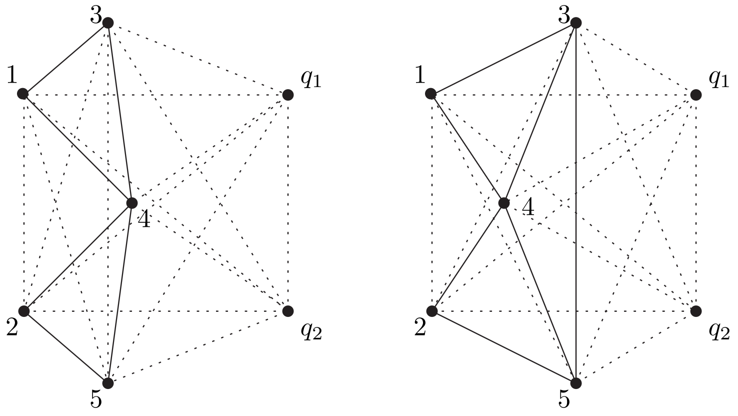

Figure 5: Examples.

Thus, we are able to conclude Theorem 1, namely, for the 2-dimensional generalized convex shelling $\mathcal{L}$ on P with respect to Q and an element $x \in P$, if $\mathrm{Dep}_{\mathcal{L}}(x) \neq P$ then $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible, and if $\mathrm{Dep}_{\mathcal{L}}(x) = P$ then $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible or homotopy equivalent to a 0-dimensional sphere. Furthermore, note that we proved a stronger statement than Theorem 1 on the way, i.e., we proved that $G(\mathcal{L})$ is chordal and it has some special properties indicated in Lemmas 14 and 15.

4 Examples

In this section, we show that both cases in Theorem 1 can really occur by exhibiting such examples.

Look at Figure 5. In both of the examples, $P = \{1, 2, 3, 4, 5\}$ and $Q = \{q_1, q_2\}$. Let $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. The solid lines show the edges of $G(\mathcal{L})$, and the dotted lines are just used for the clarification of the placement of points.

In both cases, we can observe that $\mathrm{Dep}_{\mathcal{L}}(4) = P$. In the left case, the deletion of 4 from $G(\mathcal{L})$ results in a disconnected graph, therefore $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(4)$ is homotopy equivalent to two distinct points. However, in the right case, the deletion of 4 from $G(\mathcal{L})$ results in a connected graph, therefore $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(4)$ is contractible.

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Note added in proof: The conjecture by Edelman and Reiner has recently been resolved by Hachimori and Kashiwabara in the following article:

> M. Hachimori and K. Kashiwabara. On the topology of the free complexes of convex geometries. Discrete Mathematics **307** (2007) 274–279.