The Affine Representation Theorem for Abstract Convex Geometries*

Kenji Kashiwabara† Masataka Nakamura‡ Yoshio Okamoto§¶

Abstract

A convex geometry is a combinatorial abstract model introduced by Edelman and Jamison which captures a combinatorial essence of "convexity" shared by some objects including finite point sets, partially ordered sets, trees, rooted graphs. In this paper, we introduce a generalized convex shelling, and show that every convex geometry can be represented as a generalized convex shelling. This is "the representation theorem for convex geometries" analogous to "the representation theorem for oriented matroids" by Folkman and Lawrence. An important feature is that our representation theorem is affine-geometric while that for oriented matroids is topological. Thus our representation theorem indicates the intrinsic simplicity of convex geometries, and opens a new research direction in the theory of convex geometries.
Keywords: antimatroid, convex geometry, generalized convex shelling

1 Introduction

Some abstract models of geometric concepts are known to be useful. For example, a matroid is considered as the abstraction of linear and affine dependence [21], and plays an important role in finite geometry and coding theory, and also in systems analysis [19] and combinatorial optimization [22]. Another example is an oriented matroid, which is also considered as the abstraction of linear and affine dependence and which captures essences of convex polytopes, point configurations, and hyperplane arrangements [1]. Oriented matroids play an important role in the theory of convex polytopes, discrete geometry, computational geometry and linear programming, and they are known to be quite powerful models.

One of the most important theorems in oriented matroid theory is the "topological representation theorem" by Folkman and Lawrence [9]. The topological representation theorem states that: every simple oriented matroid can be represented as a "pseudohyperplane arrangement." So, in principle, when we investigate an oriented matroid, we only have to look at the corresponding pseudohyperplane arrangement. A recent study by Swartz [24] revealed the topological representation of matroids, saying that every simple matroid can be represented as the arrangement of homotopy spheres.

*The abstract version of the paper has appeared as "Affine Representations of Abstract Convex Geometries" in Abstracts of 19th European Workshop on Computational Geometry (2003) pp. 77–80.

†Department of Systems Science, Graduate School of Arts and Sciences, The University of Tokyo, 3–8–1, Komaba, Meguro, Tokyo, 153–8902, Japan. E-mail: kashiwa@idea.c.u-tokyo.ac.jp.

‡Department of Systems Science, Graduate School of Arts and Sciences, The University of Tokyo, 3–8–1, Komaba, Meguro, Tokyo, 153–8902, Japan. E-mail: nakamura@klee.c.u-tokyo.ac.jp.

§Institute of Theoretical Computer Science, Department of Computer Science, ETH Zurich, CH-8092 Zurich, Switzerland. E-mail: okamotoy@inf.ethz.ch. He is supported by the Berlin-Zürich Joint Graduate Program "Combinatorics, Geometry, and Computation" (CGC), financed by ETH Zürich and the German Science Foundation (DFG).

¶Corresponding author.

In this paper, we study yet another example of combinatorial abstraction of geometric concepts, namely a convex geometry. A convex geometry was introduced by Edelman and Jamison [7] as an abstraction of convexity, and it can be seen as a "dual" (or a "polar" or a "complement") of an antimatroid [5]. (Therefore, we sometimes use the word "antimatroid" instead of "convex geometry" to express the same object.) A convex geometry and an antimatroid have been appearing in papers not only on discrete geometry but also on some other areas like social choice theory [13, 17, 18], knowledge spaces in mathematical psychology [6], the discrete-event system [11], semimodular lattices [23]. Furthermore, convex geometries form a greedily solvable special case of a certain optimization problem [3], and a recent development has uncovered the relationship of convex geometries with submodular-type optimization [10, 16]. From the opposite side of view, the convex geometries form a special subclass of the closure spaces, and the antimatroids form a subclass of the greedoids [2, 15].

In this paper, we prove a representation theorem for convex geometries. The theorem states that every convex geometry can be represented as a "generalized convex shelling." Since a generalized convex shelling is defined in a purely affine-geometric manner, this theorem gives an affine-geometric representation of a convex geometry. Since an affine-geometric representation theorem does exist neither for matroids nor for oriented matroids, our affine-geometric representation theorem for convex geometries indicates the intrinsic simplicity of convex geometries. Just as the topological representation theorem for oriented matroids plays a significant role in the theory of oriented matroids, our theorem should play a similar role in the theory of convex geometries.

Organization In Section 2, we give a definition of a convex geometry and state our theorem precisely. The proof of the theorem is constructive. In Section 3, we give a construction for the proof. In Section 4, we collect facts on convex geometries which will be used for showing the validity of the construction. In Section 5, we conclude the proof. Section 6 summarizes the paper and gives some recent progresses to which our paper has opened the direction.

Even though some of the lemmas in this paper have been known, we try to put complete proofs for all of them in order to make the paper self-contained.

Notation The set of nonnegative real numbers and the set of positive real numbers are denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively. For a set X of points in $\mathbb{R}^d$, $\mathrm{conv}(X)$ represents the convex hull of X, i.e., the minimal convex set containing X. For a finite set X, we denote by $|X|$ the size of X, i.e., the number of elements in X.

2 Convex geometries and the representation theorem

In this section, we will give a definition of a convex geometry, which was introduced by Edelman and Jamison [7], and will state our theorem precisely.

Let E be a nonempty finite set. A family $\mathcal{L}$ of subsets of E is called a *convex geometry* on E if $\mathcal{L}$ satisfies the following three axioms:

(L1) $\emptyset \in \mathcal{L}$ and $E \in \mathcal{L}$;
(L2) if $X, Y \in \mathcal{L}$, then $X \cap Y \in \mathcal{L}$;
(L3) if $X \in \mathcal{L} \setminus \{E\}$ then there exists some $e \in E \setminus X$ such that $X \cup \{e\} \in \mathcal{L}$.

A member of a convex geometry $\mathcal{L}$ is called a *convex set*. Two convex geometries $\mathcal{L}_1$ on E_1 and $\mathcal{L}_2$ on E_2 are *isomorphic* if there exists a bijection $\psi : E_1 \to E_2$ such that $\psi(X) \in \mathcal{L}_2$ if and only if $X \in \mathcal{L}_1$.

Let us look at some examples of convex geometries.

Example 2.1 (convex shelling). Let P be a finite set of distinct points in $\mathbb{R}^d$, and define

$$\mathcal{L} = \{X \subseteq P : \mathrm{conv}(X) \cap P = X\}.$$

Then, we can see that $\mathcal{L}$ is a convex geometry on P, and we say this kind of convex geometries is a *convex shelling* on P. A convex geometry isomorphic to the convex shelling on some finite point set P is also called a convex shelling.

Example 2.2 (poset shelling). Let E be a partially ordered set endowed with a partial order $\preceq$, and define $\mathcal{L} = \{X \subseteq E : e \in X \text{ and } f \preceq e \text{ imply } f \in X\}$. (Namely, $\mathcal{L}$ is the family of order ideals of E.) Then we can see that $\mathcal{L}$ is a convex geometry on E, and we say this kind of convex geometries is a *poset shelling* on E.

Example 2.3 (tree shelling). Let V be the vertex set of a (graph-theoretic) tree T, and define $\mathcal{L} = \{X \subseteq V : \text{the subgraph induced by } X \text{ is connected}\}$. Then we can see that $\mathcal{L}$ is a convex geometry on V, and we say this kind of convex geometries is a *tree shelling*.

Example 2.4 (graph search). Let $G = (V, E)$ be a connected graph with root $r \in V$, and define $\mathcal{L} = \{X \subseteq V \setminus \{r\} : \text{the subgraph induced by } V \setminus X \text{ is connected}\}$. Then we can see that $\mathcal{L}$ is a convex geometry on $V \setminus \{r\}$, and we say this kind of convex geometries is a *graph search*.

In the literature [7, 15], we can find more examples of convex geometries arising from various objects.

Now we introduce yet another example of convex geometries, which is so far not mentioned explicitly.

Example 2.5 (generalized convex shelling). Let P and Q be finite point sets in $\mathbb{R}^d$ such that $P \cap \mathrm{conv}(Q) = \emptyset$. (In particular, $P \cap Q = \emptyset$.) Then define

$$\mathcal{L} = \{X \subseteq P : \mathrm{conv}(X \cup Q) \cap P = X\}.$$

We say $\mathcal{L}$ is the *generalized convex shelling on* P *with respect to* Q. If $Q = \emptyset$, this just gives a convex shelling on P. So, as the name indicates, a generalized convex shelling is a generalization of a convex shelling. While at first sight it is not obvious that a generalized convex shelling is indeed a convex geometry, later we will prove that as Lemma 2.2.

A generalized convex shelling is related to a minor of a convex geometry. Let $\mathcal{L}$ be a convex geometry and $A, B \in \mathcal{L}$ such that $A \subseteq B$. Then, define

$$\mathcal{L}[A, B] = \{X \subseteq B \setminus A : X \cup A \in \mathcal{L}\}.$$

As in the following lemma, it is known that $\mathcal{L}[A, B]$ is a convex geometry on $B \setminus A$ and it is called a *minor* of $\mathcal{L}$. (Remark that the definition of a minor is different from that in a paper of Edelman and Jamison [7]. Rather, our definition obeys that in the book by Korte, Lovász and Schrader [15].)

Lemma 2.1. *Let $\mathcal{L}$ be a convex geometry on* E *and* $A, B \in \mathcal{L}$ *satisfy* $A \subseteq B \subseteq E$. *Then* $\mathcal{L}[A, B]$ *is a convex geometry on* $B \setminus A$.

Proof. We only have to check that $\mathcal{L}[A, B]$ satisfies (L1), (L2) and (L3). Let us check (L1) first. Since $A \in \mathcal{L}$, we have $\emptyset \cup A = A \in \mathcal{L}$. Hence $\emptyset \in \mathcal{L}[A, B]$. Similarly, since $B \in \mathcal{L}$, we have $(B \setminus A) \cup A = B \in \mathcal{L}$. Hence $B \setminus A \in \mathcal{L}[A, B]$.

Secondly, we will check (L2). Choose $X, Y \in \mathcal{L}[A, B]$ arbitrarily. Then, it follows that $X \cup A, Y \cup A \in \mathcal{L}$. Using (L2) for $\mathcal{L}$, we get $(X \cup A) \cap (Y \cup A) \in \mathcal{L}$, namely $(X \cap Y) \cup A \in \mathcal{L}$. Therefore, it holds that $X \cap Y \in \mathcal{L}[A, B]$.

Finally, we will check (L3). Choose $X \in \mathcal{L}[A, B] \setminus \{B \setminus A\}$ arbitrarily. Then we have $X \cup A \in \mathcal{L}$, $X \cap A = \emptyset$ and $X \cup A \subsetneq B$. Applying (L3) to $X \cup A$ many times, we can find a sequence $e_1, e_2, \ldots, e_k \in E \setminus (X \cup A)$ of elements such that $(X \cup A) \cup \{e_1, \ldots, e_i\} \in \mathcal{L}$ for all $i = 1, \ldots, k$ and $(X \cup A) \cup \{e_1, \ldots, e_k\} = E$. Let i^* be the minimal index in $\{1, \ldots, k\}$ such that $e_{i^*} \in B \setminus (X \cup A)$. Then we can see that $((X \cup A) \cup \{e_1, \ldots, e_{i^*}\}) \cap B = (X \cup A) \cup \{e_{i^*}\}$ and from (L2) we can also see that this belongs to $\mathcal{L}$. Thus we have found $e_{i^*} \in B \setminus (X \cup A)$ such that $(X \cup A) \cup \{e_{i^*}\} \in \mathcal{L}$, namely $X \cup \{e_{i^*}\} \in \mathcal{L}[A, B]$. ☐

In this proof, we have used the "chain argument," which is useful in the theory of convex geometries, and will be used again in the rest of this paper.

The next lemma shows that a generalized convex shelling is a minor of some convex shelling. This implies that a generalized convex shelling is a convex geometry, together with Lemma 2.1.

Lemma 2.2. *Let* P *and* Q *be finite point sets in* $\mathbb{R}^d$ *such that* $P \cap \mathrm{conv}(Q) = \emptyset$. *Also let* $\mathcal{L}$ *be the generalized convex shelling on* P *with respect to* Q, *and* $\tilde{\mathcal{L}}$ *be the convex shelling on* $P \cup Q$. *Then we have* $\mathcal{L} = \tilde{\mathcal{L}}[Q, P \cup Q]$.

Proof. First, because of the condition that $P \cap \mathrm{conv}(Q) = \emptyset$, it follows that $Q \in \tilde{\mathcal{L}}$. So, $\tilde{\mathcal{L}}[Q, P \cup Q]$ is well-defined. Since $\tilde{\mathcal{L}} = \{X \subseteq P \cup Q : \mathrm{conv}(X) \cap (P \cup Q) = X\}$, we have

$$
\begin{aligned}
\tilde{\mathcal{L}}[Q, P \cup Q] &= \{X \subseteq (P \cup Q) \setminus Q : X \cup Q \in \tilde{\mathcal{L}}\} \\
&= \{X \subseteq P : X \cup Q \in \tilde{\mathcal{L}}\} \\
&= \{X \subseteq P : \mathrm{conv}(X \cup Q) \cap (P \cup Q) = X \cup Q\} \\
&= \{X \subseteq P : \mathrm{conv}(X \cup Q) \cap P = X\} \\
&= \mathcal{L}.
\end{aligned}
$$

Notice that the derivations of the second equality and the fourth equality use the assumption that $P \cap \mathrm{conv}(Q) = \emptyset$, in particular $P \cap Q = \emptyset$. This concludes the proof. ☐

We are ready to state our main theorem. This states that the class of convex geometries coincides with the class of generalized convex shellings, although convex geometries arise from diverse objects as we have seen.

Theorem 2.3. *Every convex geometry is isomorphic to a generalized convex shelling.*

The main concern of this paper is the proof of Theorem 2.3. For the proof of Theorem 2.3, in the next section we construct finite sets P_0 and Q_0 of points from a given convex geometry $\mathcal{L}$ so that $\mathcal{L}$ can be isomorphic to the generalized convex shelling on P_0 with respect to Q_0. In Section 4, we prepare more concepts from convex geometries which are needed in the proof. Section 5 completes the proof of the validity of the construction.

3 Construction of point sets

For our construction, we use rooted circuits of a convex geometry. So, at the beginning of this section, we define rooted circuits. A rooted circuit of a convex geometry was originally introduced by Korte and Lovász [14].

In order to define a rooted circuit, we need other technical terms. For a convex geometry $\mathcal{L}$ on E and $A \subseteq E$, the *trace* of $\mathcal{L}$ on A is defined as $\mathrm{Tr}(\mathcal{L}, A) = \{X \cap A : X \in \mathcal{L}\}$. A *rooted set* is a pair (X, r) of a set X and an element r of X. A *rooted subset* of E is a rooted set (X, r) such that $X \subseteq E$.

Here comes the definition of a rooted circuit. Let $\mathcal{L}$ be a convex geometry on E. A rooted subset (C, r) of E is called a *rooted circuit* of $\mathcal{L}$ if $\mathrm{Tr}(\mathcal{L}, C) = 2^C \setminus \{C \setminus \{r\}\}$. We denote the family of rooted circuits of a convex geometry $\mathcal{L}$ by $\mathcal{C}(\mathcal{L})$.

Now we are ready for our construction. We construct point sets P_0 and Q_0 from a given convex geometry $\mathcal{L}$ on E so that $\mathcal{L}$ can be isomorphic to the generalized convex shelling on P_0 with respect to Q_0.

Let us say that $|E| = n$. We use the $(n-1)$-dimensional space $\mathbb{R}^{n-1}$. For each element $e \in E$, we take a point $\mathbf{p}(e) \in \mathbb{R}^{n-1}$ such that the points $\{\mathbf{p}(e) \in \mathbb{R}^{n-1} : e \in E\}$ form an affine basis of $\mathbb{R}^{n-1}$. (Namely, they are the vertex set of an $(n-1)$-dimensional simplex.) Furthermore, for each rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ we put a point $\mathbf{q}(C, r) \in \mathbb{R}^{n-1}$ determined as

$$\mathbf{q}(C, r) = |C|\mathbf{p}(r) - \sum_{e \in C \setminus \{r\}} \mathbf{p}(e). \tag{1}$$

Note that $\mathbf{p}(r)$ lies in the relative interior of $\mathrm{conv}(\{\mathbf{p}(e) : e \in C \setminus \{r\}\} \cup \{\mathbf{q}(C, r)\})$ for any rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$. Actually, this property is all that is needed in the construction. The definition of $\mathbf{q}(C, r)$ above is just one of such choices, but it eases the later calculation. Thus, we have set up $|E| + |\mathcal{C}(\mathcal{L})|$ points in $\mathbb{R}^{n-1}$.

Let $P_0 = \{\mathbf{p}(e) : e \in E\}$ and $Q_0 = \{\mathbf{q}(C, r) : (C, r) \in \mathcal{C}(\mathcal{L})\}$. Then it holds that $P_0 \cap Q_0 = \emptyset$. Now our claim is as follows.

Claim 3.1. *For P_0 and Q_0 constructed above, the generalized convex shelling on P_0 with respect to Q_0 is isomorphic to $\mathcal{L}$.*

This claim proves Theorem 2.3.

To illustrate the construction, let us look at examples for $n = 3$. Below we enumerate all of the six non-isomorphic convex geometries on $E = \{1, 2, 3\}$ together with their rooted circuits.

$$\begin{aligned}
&\mathcal{L}_1 = 2^{\{1,2,3\}}, && \mathcal{C}(\mathcal{L}_1) = \emptyset,\\
&\mathcal{L}_2 = \mathcal{L}_1 \setminus \{\{1,3\}\}, && \mathcal{C}(\mathcal{L}_2) = \{(\{1,2,3\}, 2)\},\\
&\mathcal{L}_3 = \mathcal{L}_2 \setminus \{\{3\}\}, && \mathcal{C}(\mathcal{L}_3) = \{(\{2,3\}, 2)\},\\
&\mathcal{L}_4 = \mathcal{L}_3 \setminus \{\{2,3\}\}, && \mathcal{C}(\mathcal{L}_4) = \{(\{1,3\}, 1), (\{2,3\}, 2)\},\\
&\mathcal{L}_5 = \mathcal{L}_3 \setminus \{\{1\}\}, && \mathcal{C}(\mathcal{L}_5) = \{(\{1,2\}, 2), (\{2,3\}, 2)\},\\
&\mathcal{L}_6 = \mathcal{L}_4 \setminus \{\{2\}\}, && \mathcal{C}(\mathcal{L}_6) = \{(\{1,2\}, 1), (\{1,3\}, 1), (\{2,3\}, 2)\}.
\end{aligned}$$

Figure 1 depicts the construction of the point sets for these examples.

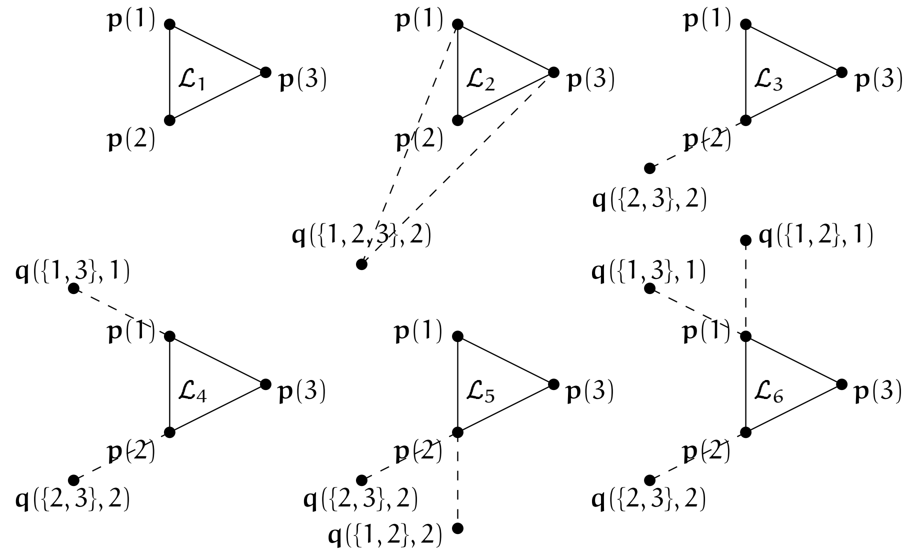

Figure 1: Construction of the point sets for $n = 3$.

4 More concepts from convex geometries

In this section, we introduce more concepts from the theory of convex geometries, which will be needed in the proof of Claim 3.1 (i.e., Theorem 2.3). In the literature [2, 5, 7, 15] the reader can find more theory of convex geometries (or antimatroids, equivalently). For the sake of completeness of the paper, we will include the proofs of most of the lemmas so that we can get some intuitions about these concepts with which the reader may be unfamiliar. (Some of them have already appeared in the literature, but here they will be proved in the setting of convex geometries, not in the setting of antimatroids as in a book by Korte, Lovász and Schrader [15], and also some proofs would be simpler or more concise.) The reader is encouraged to interpret these concepts and lemmas with the examples in Section 2.

Let $\mathcal{L}$ be a convex geometry on E. Then the *closure operator* of $\mathcal{L}$ is a map $\tau_{\mathcal{L}} : 2^E \to 2^E$ defined as $\tau_{\mathcal{L}}(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}$ for $A \subseteq E$. By (L2) in the definition of a convex geometry, we can see that $\tau_{\mathcal{L}}(A) \in \mathcal{L}$ for any $A \subseteq E$. Furthermore, from the definition of a closure operator, we can prove the following facts.

Lemma 4.1. *Let $\mathcal{L}$ be a convex geometry on E, and $\tau_{\mathcal{L}}$ the closure operator of $\mathcal{L}$.*

- **(T1)** *(Characterization of convex sets) For $X \subseteq E$, it holds that $X \in \mathcal{L}$ if and only if $\tau_{\mathcal{L}}(X) = X$.*
- **(T2)** *(Extensionality) $A \subseteq \tau_{\mathcal{L}}(A)$ for $A \subseteq E$.*
- **(T3)** *(Idempotence) $\tau_{\mathcal{L}}(\tau_{\mathcal{L}}(A)) = \tau_{\mathcal{L}}(A)$ for $A \subseteq E$.*
- **(T4)** *(Monotonicity) $A \subseteq B$ implies $\tau_{\mathcal{L}}(A) \subseteq \tau_{\mathcal{L}}(B)$.*
- **(T5)** *(Antiexchange property) Let $A \subseteq E$ and $e, f \in E$ such that $e \neq f$ and $e, f \notin \tau_{\mathcal{L}}(A)$. If $f \in \tau_{\mathcal{L}}(A \cup \{e\})$ then $e \notin \tau_{\mathcal{L}}(A \cup \{f\})$.*

Proof. The properties (T1)–(T4) are immediate from the definitions. The proof of the antiexchange property (T5) goes as follows.

Let A, e and f be as in the description of (T5). Further, let X be a set such that $X \subseteq E \setminus \{e\}$, $X \in \mathcal{L}$ and X is maximal (in the sense of set-inclusion) with respect to these two properties. Since $\tau_{\mathcal{L}}(A) \subseteq E \setminus \{e\}$ and $\tau_{\mathcal{L}}(A) \in \mathcal{L}$, such a set X always exists, and we have $A \subseteq \tau_{\mathcal{L}}(A) \subseteq X$. By (L3) in the definition of a convex geometry, there exists some element $e' \in E \setminus X$ such that $X \cup \{e'\} \in \mathcal{L}$. If $e' \neq e$, then $X \cup \{e'\} \subseteq E \setminus \{e\}$. This contradicts the maximality of X. So we have $e' = e$. This means that $X \cup \{e\} \in \mathcal{L}$.

Assume that $f \in \tau_{\mathcal{L}}(A \cup \{e\})$. Since $A \cup \{e\} \subseteq X \cup \{e\}$ and $X \cup \{e\} \in \mathcal{L}$, it holds that $f \in \tau_{\mathcal{L}}(A \cup \{e\}) \subseteq \tau_{\mathcal{L}}(X \cup \{e\}) = X \cup \{e\}$. (Here, we have used (T4) and (T1).) Since $e \neq f$, we have $f \in X$. This means that $X \cup \{f\} = X$. Therefore, it follows that $\tau_{\mathcal{L}}(X \cup \{f\}) = \tau_{\mathcal{L}}(X) = X \not\ni e$. (Here again we have used (T1).) By the monotonicity (T4) we have that $\tau_{\mathcal{L}}(A \cup \{f\}) \subseteq \tau_{\mathcal{L}}(X \cup \{f\})$. Hence it holds that $e \notin \tau_{\mathcal{L}}(A \cup \{f\})$. □

Note that the properties (T1)–(T4) of Lemma 4.1 hold for more general "closure spaces" [7, 15]. Indeed, the antiexchange property (T5) characterizes a convex geometry in the following sense: a map $\tau : 2^E \to 2^E$ satisfying extensionality, idempotence, monotonicity and also $\tau(\emptyset) = \emptyset$ is the closure operator of some convex geometry if and only if τ additionally satisfies the antiexchange property [7, 15].

In the following lemma, we can see that a trace of a convex geometry is again a convex geometry and that the closure operator of a trace is nicely combined with that of the original convex geometry.

Lemma 4.2. *Let $\mathcal{L}$ be a convex geometry on E, and $\tau_{\mathcal{L}}$ the closure operator of $\mathcal{L}$. Then, $\mathrm{Tr}(\mathcal{L}, A)$ is a convex geometry on A for every $A \subseteq E$. Moreover, the closure operator $\tau_{\mathrm{Tr}(\mathcal{L},A)} : 2^A \to 2^A$ of $\mathrm{Tr}(\mathcal{L}, A)$ is derived as $\tau_{\mathrm{Tr}(\mathcal{L},A)}(B) = \tau_{\mathcal{L}}(B) \cap A$ for $B \subseteq A$.*

Proof. The proof is a routine. To check that $\mathrm{Tr}(\mathcal{L}, A)$ satisfies (L3), we may use the chain argument (as the proof of Lemma 2.1). □

Now, we will look at how a closure operator reveals properties of rooted circuits.

Lemma 4.3. *Let $\mathcal{L}$ be a convex geometry on E. If (C, r) is a rooted circuit of $\mathcal{L}$, then $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$.*

Proof. Assume that $(C, r) \in \mathcal{C}(\mathcal{L})$. This means that $\mathrm{Tr}(\mathcal{L}, C) = 2^C \setminus \{C \setminus \{r\}\}$. Since $\tau_{\mathcal{L}}(C \setminus \{r\}) = \bigcap\{X \in \mathcal{L} : C \setminus \{r\} \subseteq X\}$ by definition, in order to show that $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$ we only have to check that $r \in X$ for all $X \in \mathcal{L}$ such that $C \setminus \{r\} \subseteq X$. Take such a set X arbitrarily. Now observe that

$$X \cap C = \begin{cases} C & (r \in X), \\ C \setminus \{r\} & (r \notin X). \end{cases}$$

However, if $X \cap C = C \setminus \{r\}$, one would conclude that $C \setminus \{r\} \in \mathrm{Tr}(\mathcal{L}, C)$. (Recall the definition of a trace: $\mathrm{Tr}(\mathcal{L}, C) = \{X \cap C : X \in \mathcal{L}\}$.) This contradicts our assumption. So it should hold that $X \cap C = C$, which means $r \in X$. □

Here is another lemma.

Lemma 4.4. *Let $\mathcal{L}$ be a convex geometry on E, and $r \notin X \subseteq E$. Then $r \in \tau_{\mathcal{L}}(X) \setminus X$ if and only if there exists $C \subseteq X \cup \{r\}$ such that (C, r) is a rooted circuit of $\mathcal{L}$.*

Proof. First we will prove the if-part. Assume that there exists $C \subseteq X \cup \{r\}$ such that $(C, r) \in \mathcal{C}(\mathcal{L})$. Then, from Lemma 4.3, we can see that $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$. Combining this with $\tau_{\mathcal{L}}(C \setminus \{r\}) \subseteq \tau_{\mathcal{L}}(X)$ (following by the monotonicity (T4)) and $r \notin X$, we have $r \in \tau_{\mathcal{L}}(X) \setminus X$.

To prove the converse, assume that $r \in \tau_{\mathcal{L}}(X) \setminus X$. Let $D \subseteq X$ be a minimal subset of X satisfying $r \in \tau_{\mathcal{L}}(D)$. Remark that such a set D always exists because X itself satisfies $r \in \tau_{\mathcal{L}}(X)$. Now we claim that $(D \cup \{r\}, r)$ is a rooted circuit.

Let $e \in D$ be an arbitrary element. By the minimality of D, we can observe that $r \notin \tau_{\mathcal{L}}(D \setminus \{e\})$. Moreover, we claim that $e \notin \tau_{\mathcal{L}}(D \setminus \{e\})$. To appreciate this, suppose the contrary, namely, $e \in \tau_{\mathcal{L}}(D \setminus \{e\})$. Then, using monotonicity (T4), we have that $D = (D \setminus \{e\}) \cup \{e\} \subseteq \tau_{\mathcal{L}}(D \setminus \{e\}) \cup \{e\} = \tau_{\mathcal{L}}(D \setminus \{e\})$. By monotonicity (T4) and idempotence (T3), we can observe that $\tau_{\mathcal{L}}(D) \subseteq \tau_{\mathcal{L}}(\tau_{\mathcal{L}}(D \setminus \{e\})) = \tau_{\mathcal{L}}(D \setminus \{e\})$. On the other hand, we have that $\tau_{\mathcal{L}}(D \setminus \{e\}) \subseteq \tau_{\mathcal{L}}(D)$ again by the monotonicity (T4). Therefore, it holds that $\tau_{\mathcal{L}}(D) = \tau_{\mathcal{L}}(D \setminus \{e\})$. However, since $r \in \tau_{\mathcal{L}}(D)$, this would imply that $r \in \tau_{\mathcal{L}}(D \setminus \{e\})$, which is a contradiction. Thus, the claim is proved.

From the claim above, we can see that $D \setminus \{e\} = (D \cup \{r\}) \cap \tau_{\mathcal{L}}(D \setminus \{e\}) \in \mathsf{Tr}(\mathcal{L}, D \cup \{r\})$. (Remember that $\tau_{\mathcal{L}}(A) \in \mathcal{L}$ for all $A \subseteq E$.) Furthermore, we have that $(D \setminus \{e\}) \cup \{r\} = ((D \setminus \{e\}) \cup \{r\}) \cap \tau_{\mathcal{L}}(D) \in \mathsf{Tr}(\mathcal{L}, D \cup \{r\})$. Since these hold for all $e \in D$, by using (L2) we can see that $\mathsf{Tr}(\mathcal{L}, D \cup \{r\}) = 2^{D \cup \{r\}} \setminus \{D\}$. $\square$

The following lemma due to Korte and Lovász [14] says that the family of rooted circuits of a convex geometry determines it uniquely.

Lemma 4.5. *Let $\mathcal{C}(\mathcal{L})$ be the family of rooted circuits of a convex geometry $\mathcal{L}$ on E. Then we have*

$$\mathcal{L} = \{X \subseteq E : (E \setminus X) \cap C \neq \{r\} \text{ for all } (C, r) \in \mathcal{C}(\mathcal{L})\}.$$

Proof. First we show that $\mathcal{L} \subseteq \{X \subseteq E : (E \setminus X) \cap C \neq \{r\} \text{ for all } (C, r) \in \mathcal{C}(\mathcal{L})\}$. Choose $X \in \mathcal{L}$ arbitrarily, and suppose that there exists some rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$ such that $(E \setminus X) \cap C = \{r\}$. Then we have $X \cap C = C \setminus \{r\}$. However, this means that $C \setminus \{r\} \in \mathsf{Tr}(\mathcal{L}, C)$, which is a contradiction to the definition of a rooted circuit. So it should hold that $(E \setminus X) \cap C \neq \{r\}$ for all $(C, r) \in \mathcal{C}(\mathcal{L})$.

Let us show the other direction. Choose $X \notin \mathcal{L}$ arbitrarily. This means $X \subsetneq \tau_{\mathcal{L}}(X)$ by (T1) and (T2). So there exists $r \in \tau_{\mathcal{L}}(X) \setminus X$. By Lemma 4.3, we have a set $C \subseteq X \cup \{r\}$ such that (C, r) is a rooted circuit of $\mathcal{L}$. So it follows that $(E \setminus X) \cap C = \{r\}$, concluding $\mathcal{L} \supseteq \{X \subseteq E : (E \setminus X) \cap C \neq \{r\} \text{ for all } (C, r) \in \mathcal{C}(\mathcal{L})\}$. $\square$

The next lemma shows that a rooted circuit is minimal in a certain sense.

Lemma 4.6. *Let $\mathcal{L}$ be a convex geometry on E, and (C, r) a rooted circuit of $\mathcal{L}$. Then $\mathsf{Tr}(\mathcal{L}, D) = 2^D$ for any proper subset $D \subsetneq C$.*

Proof. Observe that

$$\begin{aligned}
\mathsf{Tr}(\mathcal{L}, D) &= \{X \cap D : X \in \mathcal{L}\} \\
&= \{(X \cap C) \cap D : X \in \mathcal{L}\} \\
&= \{Y \cap D : Y \in \mathsf{Tr}(\mathcal{L}, C)\} \\
&= \{Y \cap D : Y \in 2^C \setminus \{C \setminus \{r\}\}\}.
\end{aligned}$$

Here, the first and the third identities are due to the definition of a trace. The second one comes from the assumption that $D \subsetneq C$, and the last one from the definition of a rooted circuit. First consider the case in which $D \neq C \setminus \{r\}$. In this case, all subsets of D belong to $2^C \setminus \{C \setminus \{r\}\}$. So $\mathrm{Tr}(\mathcal{L}, D) = 2^D$. Next consider the case in which $D = C \setminus \{r\}$. In this case, $C \cap D = C \setminus \{r\}$ and every proper subset of D belongs to $2^C \setminus \{C \setminus \{r\}\}$. Therefore, we also have that $\mathrm{Tr}(\mathcal{L}, D) = 2^D$. □

Here are more properties of rooted circuits.

Lemma 4.7. *Let $\mathcal{L}$ be a convex geometry on E, and $\mathcal{C}$ be the family of rooted circuits of $\mathcal{L}$. Then the following properties hold.*

(C1) *If $(C_1, r), (C_2, r) \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C2) *If $(C_1, r_1), (C_2, r_2) \in \mathcal{C}$ and $r_1 \in C_2 \setminus \{r_2\}$, then there exists $(C_3, r_2) \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 \setminus \{r_1\}$.*

Proof. Let us first prove (C1). Suppose $C_1 \subsetneq C_2$. Then, using Lemma 4.6, we can see that $\mathrm{Tr}(\mathcal{L}, C_1) = 2^{C_1}$. This is a contradiction to the assumption that (C_1, r) is a rooted circuit. Hence it follows that $C_1 = C_2$.

Next we prove (C2). Let $X = (C_1 \cup C_2) \setminus \{r_1, r_2\}$. Since $C_2 \setminus \{r_2\} \subseteq (C_1 \cup C_2) \setminus \{r_2\} = ((C_1 \cup C_2) \setminus \{r_1, r_2\}) \cup \{r_1\} \subseteq X \cup \{r_1\}$, we have $r_2 \in \tau_{\mathcal{L}}(C_2 \setminus \{r_2\}) \subseteq \tau_{\mathcal{L}}(X \cup \{r_1\})$ by Lemma 4.3 and the monotonicity (T4) of $\tau_{\mathcal{L}}$. Similarly, we have $r_1 \in \tau_{\mathcal{L}}(C_1 \setminus \{r_1\}) \subseteq \tau_{\mathcal{L}}(X \cup \{r_2\})$. Therefore by the antiexchange property (T5), we have $r_1 \in \tau_{\mathcal{L}}(X)$ or $r_2 \in \tau_{\mathcal{L}}(X)$. If $r_1 \in \tau_{\mathcal{L}}(X)$, then we have $C_2 \setminus \{r_2\} \subseteq \tau_{\mathcal{L}}(X)$. So it should hold that $r_2 \in \tau_{\mathcal{L}}(C_2 \setminus \{r_2\}) \subseteq \tau_{\mathcal{L}}(\tau_{\mathcal{L}}(X)) = \tau_{\mathcal{L}}(X)$ by Lemma 4.3 and the idempotence (T3) of $\tau_{\mathcal{L}}$. Hence in both cases we have $r_2 \in \tau_{\mathcal{L}}(X) \setminus X$. By Lemma 4.4, there exists $C_3 \subseteq X \cup \{r_2\}$ such that (C_3, r_2) is a rooted circuit of $\mathcal{L}$. This is what we have wanted. □

Note that (C1) and (C2) in Lemma 4.7 characterize the family of rooted circuits of a convex geometry among families of rooted subsets, that is, a given family $\mathcal{C}$ of rooted subsets of E satisfies (C1) and (C2) if and only if $\mathcal{C}$ is the family of rooted circuits of some convex geometry on E. This characterization is due to Dietrich [4, 5].

Here, we observe a relation of a rooted circuit with a closure operator.

Lemma 4.8. *Let $\mathcal{L}$ be a convex geometry on E. Then $(C, r) \in \mathcal{C}(\mathcal{L})$ if and only if $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$ and $r \notin \tau_{\mathcal{L}}(D \setminus \{r\})$ for every proper subset $D \subsetneq C$.*

Proof. Assume that $(C, r) \in \mathcal{C}(\mathcal{L})$. From Lemma 4.3 it follows that $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$. Now we show that $r \notin \tau_{\mathcal{L}}(D \setminus \{r\})$ for every proper subset $D \subsetneq C$. Take a proper subset $D \subsetneq C$ arbitrarily. Then Lemma 4.2 tells us $\tau_{\mathrm{Tr}(\mathcal{L}, C)}(D \setminus \{r\}) = \tau_{\mathcal{L}}(D \setminus \{r\}) \cap C$. Since $D \setminus \{r\} \in \mathrm{Tr}(\mathcal{L}, C)$ and $D \setminus \{r\} \subseteq C$, we have $\tau_{\mathcal{L}}(D \setminus \{r\}) = D \setminus \{r\}$. (Recall (T1) in Lemma 4.1.) Thus, it follows that $r \notin \tau_{\mathcal{L}}(D \setminus \{r\})$.

Next, we prove that if $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$ and $r \notin \tau_{\mathcal{L}}(D \setminus \{r\})$ for any proper subset $D \subsetneq C$ then $(C, r) \in \mathcal{C}(\mathcal{L})$. Since $r \in \tau_{\mathcal{L}}(C \setminus \{r\})$ (the assumption) and $r \notin C \setminus \{r\}$ (clear), we have $r \in \tau_{\mathcal{L}}(C \setminus \{r\}) \setminus (C \setminus \{r\})$. Therefore, by Lemma 4.4, there exists $C' \subseteq (C \setminus \{r\}) \cup \{r\} = C$ such that $(C', r) \in \mathcal{C}(\mathcal{L})$. By Lemma 4.3, we have $r \in \tau_{\mathcal{L}}(C' \setminus \{r\})$. Since we have assumed that $r \notin \tau_{\mathcal{L}}(D \setminus \{r\})$ for any proper subset $D \subsetneq C$, it should hold that $C' = C$. This implies that $(C, r) \in \mathcal{C}(\mathcal{L})$. □

Now, we will determine the closure operator of a generalized convex shelling.

Lemma 4.9. *Let P and Q be finite point sets in $\mathbb{R}^d$, and $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. Then $\tau_{\mathcal{L}}(A) = \mathrm{conv}(A \cup Q) \cap P$ for $A \subseteq P$.*

To prove Lemma 4.9, we will use the following lemma.

Lemma 4.10. *Let $\mathcal{L}$ be a convex geometry on* E*, and* $S \subseteq E$*. Consider the minor* $\mathcal{L}' = \mathcal{L}[S, E]$*. Then, we have that* $\tau_{\mathcal{L}'}(T) = \tau_{\mathcal{L}}(T \cup S) \setminus S$ *for each* $T \subseteq E \setminus S$.

Proof. From the definitions of a closure operator and a minor, it holds that

$$\begin{aligned}
\tau_{\mathcal{L}'}(T) &= \bigcap\{X \in \mathcal{L}' : T \subseteq X\} \\
&= \bigcap\{X \subseteq E : X \cup S \in \mathcal{L}, T \subseteq X\} \\
&= \bigcap\{Y \setminus S : Y \in \mathcal{L}, T \cup S \subseteq Y\} \\
&= \left(\bigcap\{Y \in \mathcal{L} : T \cup S \subseteq Y\}\right) \setminus S \\
&= \tau_{\mathcal{L}}(T \cup S) \setminus S.
\end{aligned}$$

At the third identity, we replaced $X \cup S$ by Y. □

Proof of Lemma 4.9. First observe that the convex shelling $\mathcal{L}^*$ on $P \cup Q$ has the closure operator $\tau_{\mathcal{L}^*}$ as $\tau_{\mathcal{L}^*}(B) = \mathrm{conv}(B) \cap (P \cup Q)$ for each $B \subseteq P \cup Q$. From Lemma 2.2, the generalized convex shelling $\mathcal{L}$ on P with respect to Q is the same as $\mathcal{L}^*[Q, P \cup Q]$. Therefore, from Lemma 4.10, we have that

$$\begin{aligned}
\tau_{\mathcal{L}}(A) &= \tau_{\mathcal{L}^*}(A \cup Q) \setminus Q \\
&= (\mathrm{conv}(A \cup Q) \cap (P \cup Q)) \setminus Q \\
&= \mathrm{conv}(A \cup Q) \cap P.
\end{aligned}$$

This concludes the proof. □

Combining Lemmas 4.8 and 4.9, we can obtain a characterization of the family of rooted circuits of a generalized convex shelling.

Lemma 4.11. *Let $\mathcal{L}$ denote the convex shelling on* P *with respect to* Q*, and let* $C \subseteq P$ *and* $r \in C$*. Then* $(C, r) \in \mathcal{C}(\mathcal{L})$ *if and only if* $r \in \mathrm{conv}((C \setminus \{r\}) \cup Q)$ *and* $r \notin \mathrm{conv}((D \setminus \{r\}) \cup Q)$ *for any proper subset* $D \subsetneq C$.

Proof. This is a direct consequence of Lemmas 4.8 and 4.9. □

5 Proof of the main theorem

First we have to check that P_0 and Q_0 satisfy the precondition of a generalized convex shelling, namely $P_0 \cap \mathrm{conv}(Q_0) = \emptyset$.

Lemma 5.1. *For* P_0 *and* Q_0 *constructed in Section 3, it holds that* $P_0 \cap \mathrm{conv}(Q_0) = \emptyset$.

To show Lemma 5.1, the next fact is useful, which will be used later again and again.

Lemma 5.2. *Let* V *be a set of affinely independent points in* $\mathbb{R}^d$ *and* $V_1, V_2 \subseteq V$*. If there exist sets* $\{\alpha_v \in \mathbb{R}_{>0} : v \in V_1\}$ *and* $\{\beta_v \in \mathbb{R}_{>0} : v \in V_2\}$ *of positive numbers such that*

$$\sum_{v \in V_1} \alpha_v = \sum_{v \in V_2} \beta_v \quad \text{and} \quad \sum_{v \in V_1} \alpha_v v = \sum_{v \in V_2} \beta_v v,$$

then it holds that $V_1 = V_2$.

Proof. This is a direct consequence of the affine independence of the points in V. $\square$

Now we will show Lemma 5.1 with Lemma 5.2.

Proof of Lemma 5.1. Suppose that there exist some element $\overline{e} \in E$ and rooted circuits $(C_1, r_1), \ldots, (C_k, r_k) \in \mathcal{C}(\mathcal{L})$ such that $\mathbf{p}(\overline{e})$ lies in the relative interior of $\mathrm{conv}(\{\mathbf{q}(C_i, r_i) : i = 1, \ldots, k\})$, namely, there exist some positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ such that

$$\sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i \mathbf{q}(C_i, r_i) = \mathbf{p}(\overline{e}).$$

By the construction of Q_0, we have that

$$\sum_{i=1}^{k} \lambda_i |C_i| \mathbf{p}(r_i) = \mathbf{p}(\overline{e}) + \sum_{i=1}^{k} \lambda_i \left(\sum_{f \in C_i \setminus \{r_i\}} \mathbf{p}(f) \right).$$

Since $\mathbf{p}(e)$'s are affinely independent, we have that

$$\{r_i : i = 1, \ldots, k\} = \{\overline{e}\} \cup \bigcup_{i=1}^{k} (C_i \setminus \{r_i\})$$

using Lemma 5.2. Let us denote $R = \{r_i : i = 1, \ldots, k\}$. Then the identity above implies that

$$R = R \cup \bigcup_{i=1}^{k} (C_i \setminus \{r_i\}) = \bigcup_{i=1}^{k} C_i. \tag{2}$$

By the conditions (L1) and (L3) in the definition of a convex geometry, there exists a subfamily $\{X_j : j = 0, 1, \ldots, n\} \subseteq \mathcal{L}$ such that $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$ and $|X_i| = i$ for each $i = 0, 1, \ldots, n$. Especially, $X_0 = \emptyset$ and $X_n = E$. Fix such a subfamily $\{X_j : j = 0, \ldots, n\}$. Then there exists a unique index j^* such that $|(E \setminus X_{j^*}) \cap R| = 1$. Let us say $(E \setminus X_{j^*}) \cap R = \{r\}$. From the identity (2), there exists a rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$ such that $C \subseteq R$ since $r \in R$. Then we have $(E \setminus X_{j^*}) \cap C = \{r\}$. However this implies that $X_{j^*} \notin \mathcal{L}$ by Lemma 4.5, which is a contradiction. $\square$

Remark that we can even show that $\mathrm{conv}(P_0) \cap \mathrm{conv}(Q_0) = \emptyset$, but this fact is not needed in our proof.

In the rest of this section, for P_0 and Q_0 constructed in Section 3, we denote by $\mathcal{L}'$ the generalized convex shelling on P_0 with respect to Q_0. Lemma 5.1 tells us that $\mathcal{L}'$ is well-defined. In order to prove Claim 3.1, we only have to show that $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}')$ due to Lemma 4.5. Namely, we want a bijection $\psi : E \to P_0$ such that $(\psi(C), \psi(r)) \in \mathcal{C}(\mathcal{L}')$ if and only if $(C, r) \in \mathcal{C}(\mathcal{L})$. In our case, the natural bijection $\psi : E \to P_0$ is as follows: $\psi(e) = \mathbf{p}(e)$ for $e \in E$. Thus we only have to show the next lemma.

Lemma 5.3. *In the setting above, it holds that*

$$\mathcal{C}(\mathcal{L}') = \{(\psi(C), \psi(r)) : (C, r) \in \mathcal{C}(\mathcal{L})\}.$$

This lemma follows from the following two lemmas (Lemmas 5.4 and 5.5) and (C1) in Lemma 4.7.

Lemma 5.4. *In the setting above, for every rooted circuit* $(C, r) \in \mathcal{C}(\mathcal{L})$*, there exists* $(C', \mathbf{r}') \in \mathcal{C}(\mathcal{L}')$ *such that* $C' \subseteq \psi(C)$ *and* $\mathbf{r}' = \psi(r)$.

Lemma 5.5. *In the setting above, for every rooted circuit* $(C', \mathbf{r}') \in \mathcal{C}(\mathcal{L}')$*, there exists* $(C, r) \in \mathcal{C}(\mathcal{L})$ *such that* $C \subseteq \psi^{-1}(C')$ *and* $r = \psi^{-1}(\mathbf{r}')$.

Before proving Lemmas 5.4 and 5.5, let us show how Lemma 5.3 can be derived from them.

Proof of Lemma 5.3. First we prove that if $(C, r) \in \mathcal{C}(\mathcal{L})$ then $(\psi(C), \psi(r)) \in \mathcal{C}(\mathcal{L}')$. Take an arbitrary $(C, r) \in \mathcal{C}(\mathcal{L})$. Then from Lemma 5.4, there exists some $(C', \mathbf{r}') \in \mathcal{C}(\mathcal{L}')$ such that $C' \subseteq \psi(C)$ and $\mathbf{r}' = \psi(r)$. Note that $r = \psi^{-1}(\mathbf{r}')$ since ψ is a bijection. Then from Lemma 5.5, there exists some $(\tilde{C}, \tilde{r}) \in \mathcal{C}(\mathcal{L})$ such that $\tilde{C} \subseteq \psi^{-1}(C')$ and $\tilde{r} = \psi^{-1}(\mathbf{r}')$. So we have $r = \psi^{-1}(\mathbf{r}') = \tilde{r}$.

Now using (C1) in Lemma 4.7, we have $(C, r) = (\psi^{-1}(C'), \psi^{-1}(\mathbf{r}')) = (\tilde{C}, \tilde{r})$. Since ψ is a bijection, we also have $(\psi(C), \psi(r)) = (C', \mathbf{r}') = (\psi(\tilde{C}), \psi(\tilde{r}))$. Therefore, we have $(\psi(C), \psi(r)) \in \mathcal{C}(\mathcal{L}')$ since $(C', \mathbf{r}') \in \mathcal{C}(\mathcal{L}')$.

Similarly, we can show that if $(C', \mathbf{r}') \in \mathcal{C}(\mathcal{L}')$ then $(\psi^{-1}(C'), \psi^{-1}(\mathbf{r}')) \in \mathcal{C}(\mathcal{L})$. □

To prove Lemma 5.4, we will use Lemma 4.11.

Proof of Lemma 5.4. Take an arbitrary rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$. From our construction, we have $\mathbf{p}(r) \in \mathrm{conv}(\{\mathbf{p}(e) : e \in C \setminus \{r\}\} \cup \{\mathbf{q}(C, r)\})$, which implies $\psi(r) \in \mathrm{conv}(\psi(C \setminus \{r\}) \cup Q_0)$. Take a subset $C' \subseteq \psi(C)$ such that $\psi(r) \in \mathrm{conv}((C' \setminus \{\psi(r)\}) \cup Q_0)$ and $\psi(r) \notin \mathrm{conv}((D' \setminus \{\psi(r)\}) \cup Q_0)$ for any proper subset $D' \subsetneq C'$. (Note that such a set C' exists because if $\psi(r) \in A \subseteq B$ and $\psi(r) \in \mathrm{conv}((A \setminus \{\psi(r)\}) \cup Q_0)$ then $\psi(r) \in \mathrm{conv}((B \setminus \{\psi(r)\}) \cup Q_0)$.) From Lemma 4.11, it follows that $(C', \psi(r)) \in \mathcal{C}(\mathcal{L}')$. □

In order to prove Lemma 5.5, we will prepare another lemma.

Lemma 5.6. *In the setting above, let* $\overline{e} \in E$ *be an element such that* $\mathbf{p}(\overline{e})$ *lies in the relative interior of* $\mathrm{conv}(\{\mathbf{p}(f) : f \in F\} \cup \{\mathbf{q}(C_i, r_i) : i = 1, \ldots, k\})$ *for some* $F \subseteq E \setminus \{\overline{e}\}$ *and some* $(C_1, r_1), (C_2, r_2), \ldots, (C_k, r_k) \in \mathcal{C}(\mathcal{L})$.

(1) It holds that

$$F \cup \{r_i : i = 1, \ldots, k\} = \{\overline{e}\} \cup \bigcup_{i=1}^{k} (C_i \setminus \{r_i\}).$$

(2) It holds that $\overline{e} \in \tau_{\mathcal{L}}(F)$.

Proof. Let us first prove (1). From the assumption, we have the following convex combination, namely there exist some $\{\mu_f \in \mathbb{R}_{>0} : f \in F\}$ and $\{\lambda_i \in \mathbb{R}_{>0} : i = 1, \ldots, k\}$ such that

$$\sum_{f \in F} \mu_f + \sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad \sum_{f \in F} \mu_f \mathbf{p}(f) + \sum_{i=1}^{k} \lambda_i \mathbf{q}(C_i, r_i) = \mathbf{p}(\overline{e}).$$

From the construction of Q_0, we have that

$$\mathbf{p}(\overline{e}) = \sum_{f \in F} \mu_f \mathbf{p}(f) + \sum_{i=1}^{k} \lambda_i \left(|C_i| \mathbf{p}(r_i) - \sum_{e \in C_i \setminus \{r_i\}} \mathbf{p}(e) \right),$$

meaning that

$$\sum_{f\in F}\mu_f\mathbf{p}(f)+\sum_{i=1}^{k}\lambda_i|C_i|\mathbf{p}(r_i)=\mathbf{p}(\overline{e})+\sum_{i=1}^{k}\sum_{e\in C_i\setminus\{r_i\}}\mathbf{p}(e).$$

By Lemma 5.2, it holds that

$$F\cup\{r_i : i=1,\dots,k\}=\{\overline{e}\}\cup\bigcup_{i=1}^{k}(C_i\setminus\{r_i\}).$$

Thus, the part (1) is proved.

For the part (2), set

$$R=\{r_i : i=1,\dots,k\}\quad\text{and}\quad F^*=\left(\left(\bigcup_{i=1}^{k}(C_i\setminus\{r_i\})\right)\cup\{\overline{e}\}\right)\setminus R.$$

By the part (1) of this lemma, we have that $F^*\subseteq F$. Moreover, by the part (1) again, we have that $\overline{e}\in R$. Therefore, F^* can be represented as

$$F^*=\left(\bigcup_{i=1}^{k}C_i\right)\setminus R.$$

Now we claim that for every $X\in\mathcal{L}$ satisfying $F^*\subseteq X$ it holds that $\overline{e}\in X$. To show that by a contradiction, we suppose that there exists $X^*\in\mathcal{L}$ such that $F^*\subseteq X^*$ and $\overline{e}\notin X^*$. Since $\overline{e}\in R$ and $\overline{e}\notin X^*$, we have $\overline{e}\in(E\setminus X^*)\cap R$. This implies that $|(E\setminus X^*)\cap R|\geq 1$. So there exists $Z\in\mathcal{L}$ such that $|(E\setminus Z)\cap R|=1$ and $Z\supseteq X^*$. (Here we have used (L3) in the definition of a convex geometry.) Let us say that $(E\setminus Z)\cap R=\{r_1\}$, without loss of generality. Since $F^*\subseteq X^*\subseteq Z$ we have $(E\setminus Z)\cap F^*=\emptyset$. Therefore, it follows that

$$\begin{aligned}(E\setminus Z)\cap\left(\bigcup\{C_i : i=1,\dots,k\}\right)&=(E\setminus Z)\cap(F^*\cup R)\\&=((E\setminus Z)\cap F^*)\cup((E\setminus Z)\cap R)\\&=\emptyset\cup\{r_1\}\\&=\{r_1\}.\end{aligned}$$

Then we have $(E\setminus Z)\cap C_1=\{r_1\}$. However this implies that $Z\notin\mathcal{L}$, together with Lemma 4.5. This is a contradiction.

Now consider $\tau_{\mathcal{L}}(F^*)$. Since $F^*\subseteq\tau_{\mathcal{L}}(F^*)\in\mathcal{L}$ (the extensionality of $\tau_{\mathcal{L}}$), we have that $\overline{e}\in\tau_{\mathcal{L}}(F^*)$. (Here, we have used the claim above.) By the monotonicity (T4) of $\tau_{\mathcal{L}}$ we have that $\tau_{\mathcal{L}}(F^*)\subseteq\tau_{\mathcal{L}}(F)$. From this we conclude that $\overline{e}\in\tau_{\mathcal{L}}(F)$. □

Now we are ready to prove Lemma 5.5.

Proof of Lemma 5.5. Let $(C',\mathbf{r}')\in\mathcal{C}(\mathcal{L}')$. Lemma 4.11 tells us that $\mathbf{r}'\in\mathrm{conv}((C'\setminus\{\mathbf{r}'\})\cup Q_0)$ and $\mathbf{r}'\notin\mathrm{conv}((D'\setminus\{\mathbf{r}'\})\cup Q_0)$ for any proper subset $D'\subsetneq C'$. Let us observe the following.

> There exists some subset $Q_1\subseteq Q_0$ such that $\mathbf{r}'$ lies in the relative interior of $\mathrm{conv}((C'\setminus\{\mathbf{r}'\})\cup Q_1)$.

To see this, suppose contrarily that there exists no such set. Namely, $\mathbf{r}'$ does not lie in the relative interior of $\mathrm{conv}((C' \setminus \{\mathbf{r}'\}) \cup Q_1)$ for any subset $Q_1 \subseteq Q_0$. If we take $Q_1 = \emptyset$, this particularly means that $\mathbf{r}'$ does not lie in the relative interior of $\mathrm{conv}(C' \setminus \{\mathbf{r}'\})$. Therefore, $\mathbf{r}'$ lies on a proper face of $\mathrm{conv}(C' \setminus \{\mathbf{r}'\})$. Let $F \subsetneq C' \setminus \{\mathbf{r}'\}$ be a unique minimal set such that $\mathbf{r}'$ lies in the relative interior of $\mathrm{conv}(F)$. Then, it holds that $\mathbf{r}' \in \mathrm{conv}(F) \subseteq \mathrm{conv}(F \cup Q_0)$. However, this contradicts the assumption that $\mathbf{r}' \notin \mathrm{conv}((D' \setminus \{\mathbf{r}'\}) \cup Q_0)$ for any proper subset $D' \subsetneq C'$. The claim is proved.

Using this observation together with Lemma 5.6(2), we have that $\psi^{-1}(\mathbf{r}') \in \tau_{\mathcal{L}}(\psi^{-1}(C' \setminus \{\mathbf{r}'\}))$. Choose $C \subseteq \psi^{-1}(C')$ such that $\psi^{-1}(\mathbf{r}') \in \tau_{\mathcal{L}}(C \setminus \{\psi^{-1}(\mathbf{r}')\})$ and $\psi^{-1}(\mathbf{r}') \notin \tau_{\mathcal{L}}(D \setminus \{\psi^{-1}(\mathbf{r}')\})$ for any proper subset $D \subsetneq C$. (Note that such a set C exists because of the same reason as in the proof of Lemma 5.4.) By Lemma 4.8, it follows that $(C, \psi^{-1}(\mathbf{r}')) \in \mathcal{C}(\mathcal{L})$. □

This completes the whole proof. Q.E.D.

6 Conclusion

In this paper, we have provided the affine representation theorem for (abstract) convex geometries. This should be useful as the representation theorem for oriented matroids by Folkman and Lawrence [9]. Actually, the theorem has opened several new directions of research. We indicate some of them here.

1. Our theorem makes it possible to discuss the dimension of the space in which a given convex geometry can be realized. Hachimori and Nakamura [12] studied stem clutters of a convex geometry which can be realized in the 2-dimensional space. They gave a characterization of a stem clutter in dimension 2 with the max-flow min-cut property.
2. Okamoto [20] studied an open problem posed by Edelman and Reiner [8] from the viewpoint of our theorem. Especially, he solved the question affirmatively for 2-dimensional generalized convex shellings.

We hope that our theorem will give a fruitful tool in the theory of convex geometries and related field.

Acknowledgements

Authors are grateful to Masahiro Hachimori and Tadashi Sakuma for discussion and useful comments. Furthermore, they thank an anonymous referee very much for the comments which improved the presentation of the paper.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G.M. Ziegler: Oriented Matroids (2nd Edition). Cambridge University Press, Cambridge, 1999.

[2] A. Björner and G.M. Ziegler: Introduction to Greedoids. In: N. White ed., Matroid Applications, Cambridge University Press, Cambridge, 1992.

[3] E.A. Boyd and U. Faigle: An algorithmic characterization of antimatroids. Discrete Applied Mathematics **28**, 1990, 197–205.

[4] B.L. Dietrich: A circuit set characterization of antimatroids. Journal of Combinatorial Theory Series B **43**, 1987, 314–321.
[5] B.L. Dietrich: Matroids and antimatroids — a survey. Discrete Mathematics **78**, 1989, 223–237.
[6] J.-P. Doignon and J.-C. Falmagne: Knowledge spaces. Springer Verlag, Berlin, 1999.
[7] P.H. Edelman and R.E. Jamison: The theory of convex geometries. Geometriae Dedicata **19**, 1985, 247–270.
[8] P. Edelman and V. Reiner: Counting the interior points of a point configuration. Discrete & Computational Geometry **23**, 2000, 1–13.
[9] J. Folkman and J. Lawrence: Oriented matroids. Journal of Combinatorial Theory Series B **25**, 1978, 199–235.
[10] S. Fujishige: Dual greedy polyhedra, choice functions, and abstract convex geometries. Discrete Optimization **1**, 2004, 41–49.
[11] P. Glasserman and D.D. Yao: Monotone Structure in Discrete-Event Systems. John Wiley & Sons, New York, 1994.
[12] M. Hachimori and M. Nakamura: Rooted circuits of closed-set systems and the max-flow min-cut property of stem clutters of affine convex geometries. Submitted.
[13] M.R. Johnson and R.A. Dean: Locally complete path independent choice functions and their lattices. Mathematical Social Sciences **42**, 2001, 53–87.
[14] B. Korte and L. Lovász: Shelling structures, convexity, and a happy end. In: B. Bollóbas, ed., Graph Theory and Combinatorics: Proceedings of the Cambridge Combinatorial Conference in Honour of Paul Erdős, Academic Press, London New York San Francisco, 1984, 219–232.
[15] B. Korte, L. Lovász and R. Schrader: Greedoids. Springer-Verlag, Berlin Heidelberg, 1991.
[16] K. Kashiwabara and Y. Okamoto: A greedy algorithm for convex geometries. Discrete Applied Mathematics **131**, 2003, 449–465.
[17] G.A. Koshevoy: Choice functions and abstract convex geometries. Mathematical Social Sciences **38**, 1999, 35–44.
[18] B. Monjardet and V. Raderanirina: The duality between the anti-exchange closure operators and the path independent choice operators on a finite set. Mathematical Social Sciences **41**, 2001, 131–150.
[19] K. Murota: Matrices and Matroids for Systems Analysis. Springer Verlag, Berlin, 2000.
[20] Y. Okamoto: Local topology of the free complex of a two-dimensional generalized convex shelling. In preparation.
[21] J. Oxley: Matroid Theory. Oxford University Press, New York, 1992.
[22] A. Schrijver: Combinatorial Optimization – Polyhedra and Efficiency. Springer Verlag, Berlin, 2003.
[23] M. Stern: Semimodular Lattices. Cambridge University Press, Cambridge, 1999.
[24] E. Swartz: Topological representations of matroids. Journal of the American Mathematical Society **16**, 2003, 427–442.