Counting the Independent Sets of a Chordal Graph in Linear Time

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Rényi Institute, Budapest, Hungary

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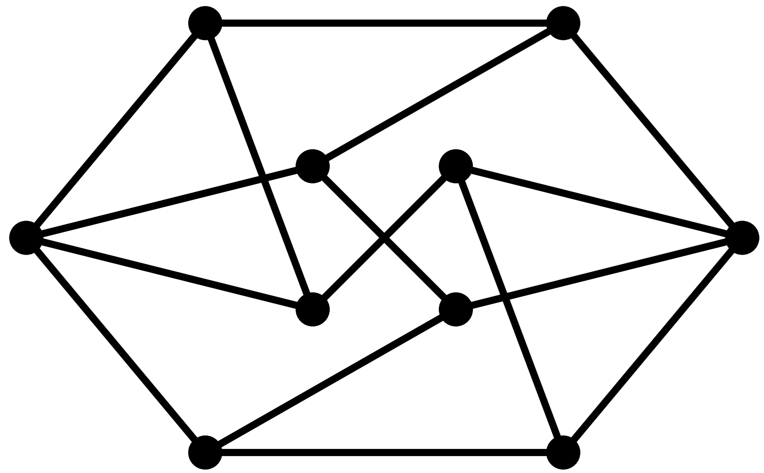
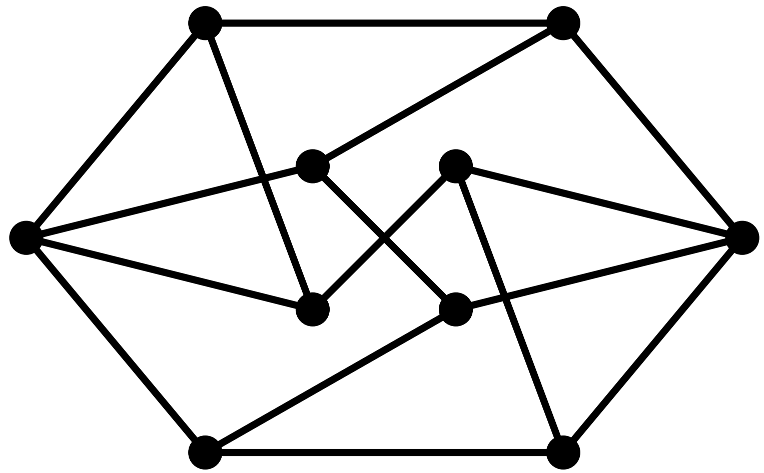


Setup $G = (V, E)$ a graph (undirected, finite, simple)

Def. A set $I \subseteq V$ is an **independent set** of G if no two vertices in I are adjacent.



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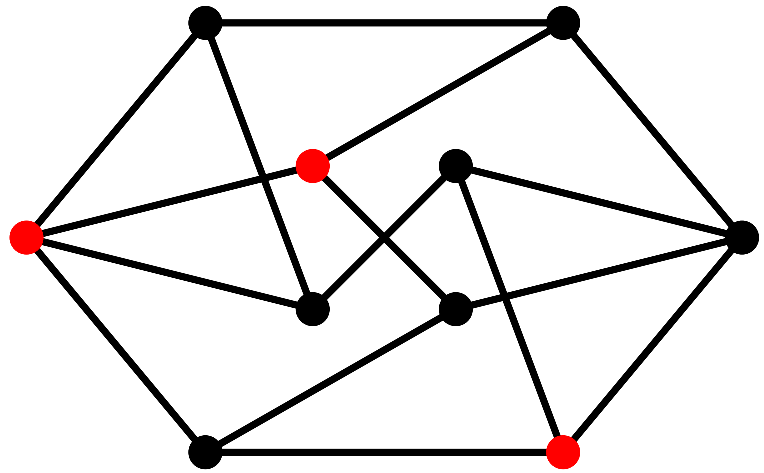

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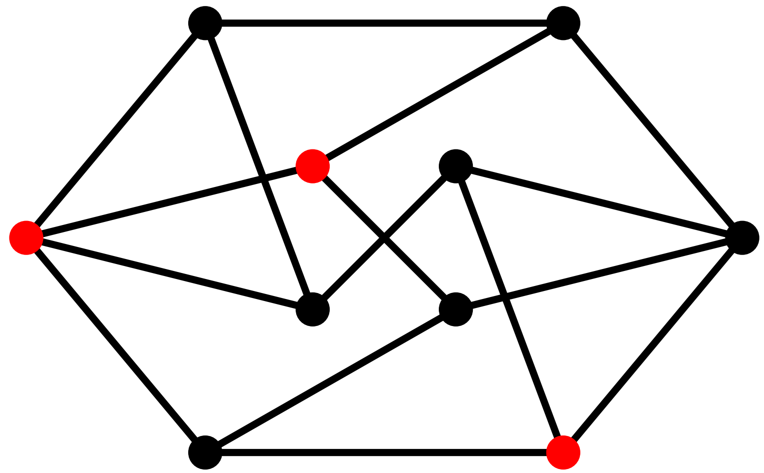

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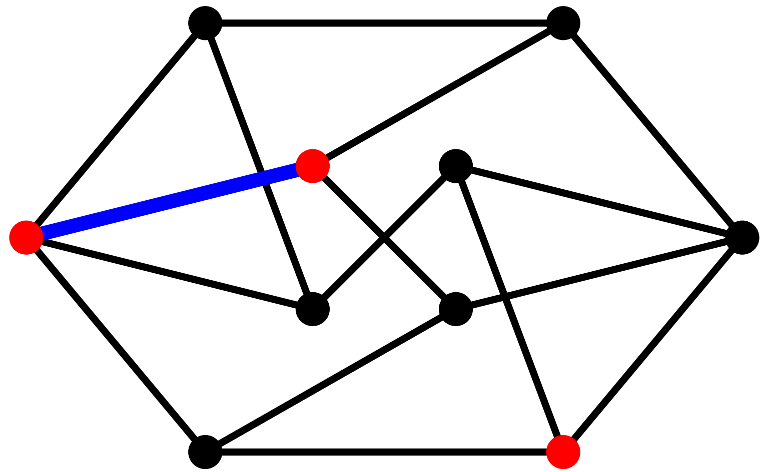

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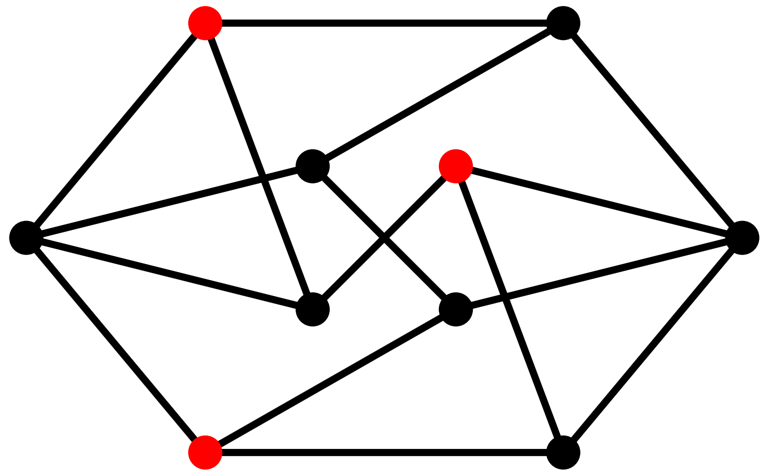

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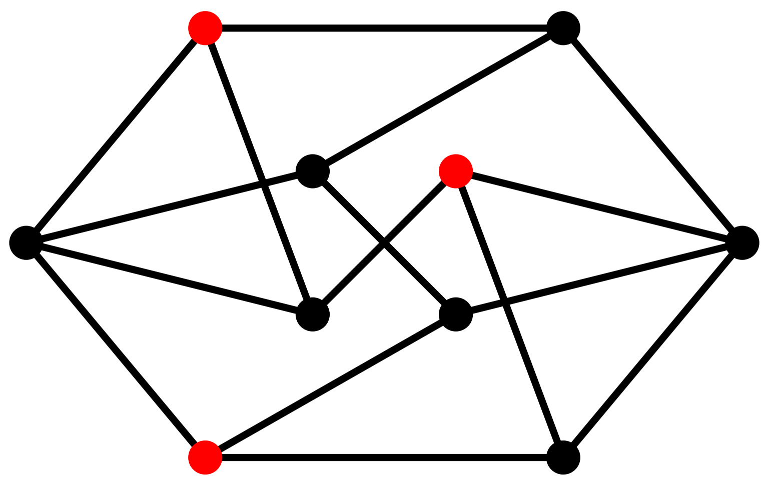

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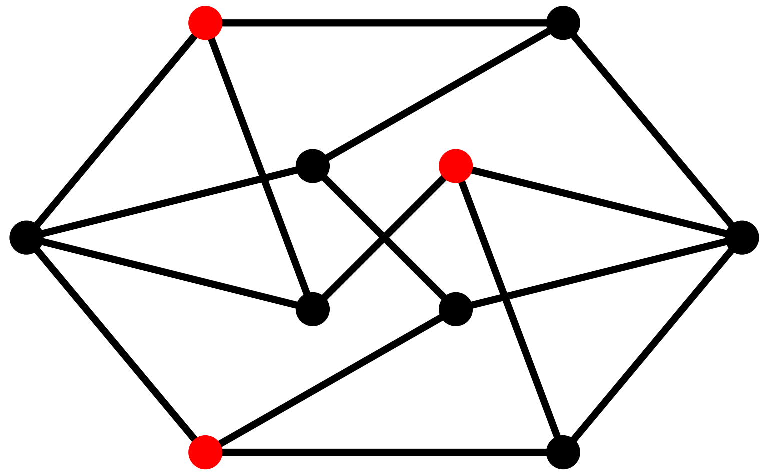
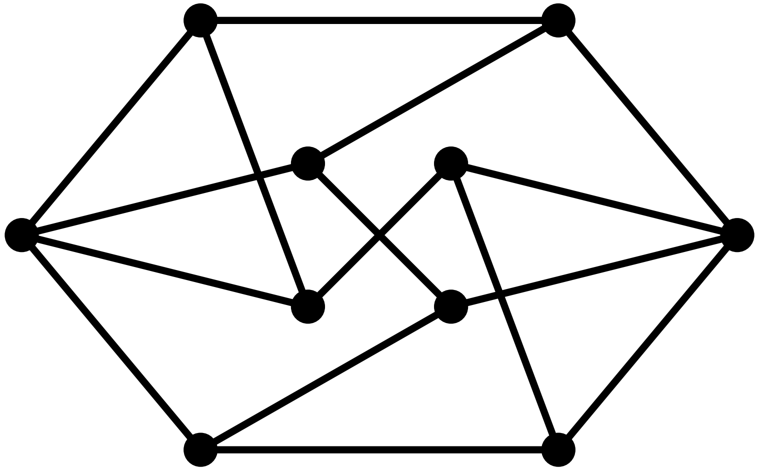


An independent set

Also called a stable set of G

Input $G = (V, E)$ a graph

Output All independent sets of G



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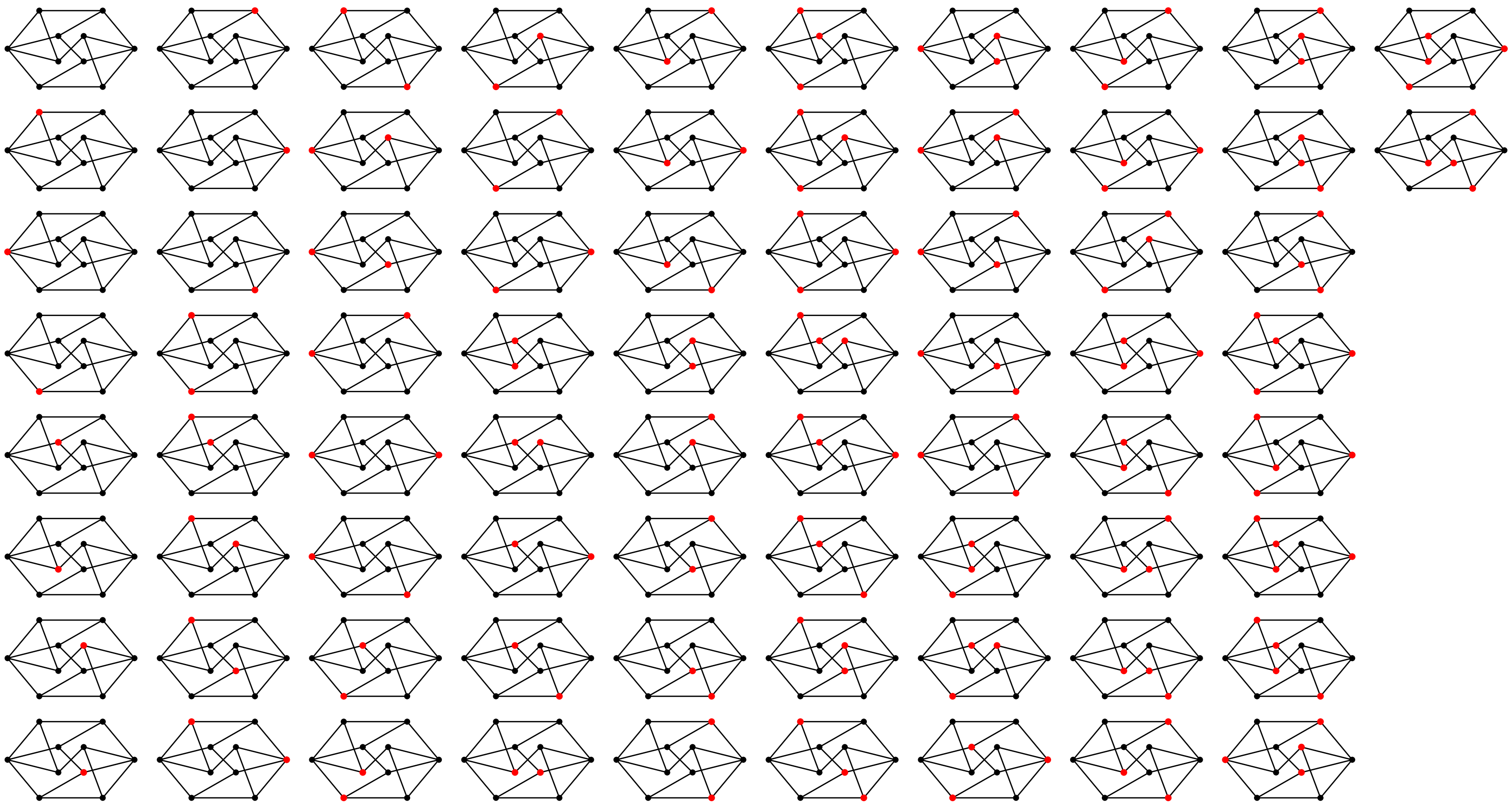
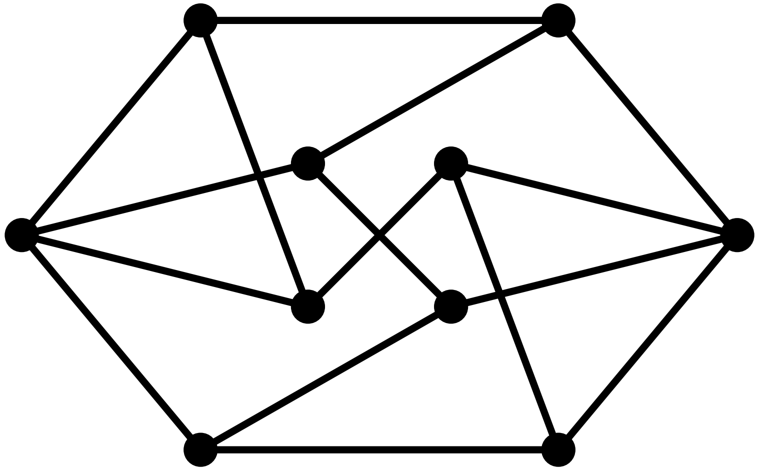


74 independent sets in total

Input $G = (V, E)$ a graph

Output # independent sets of G



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Input: $G = (V, E)$ a graph

Output:

(1) # independent sets of G

(2) # maximum independent sets of G

(3) # independent sets of G of fixed size

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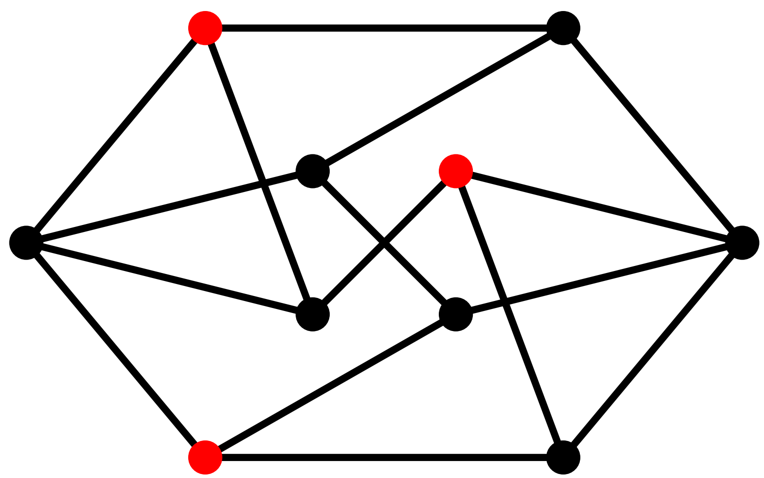

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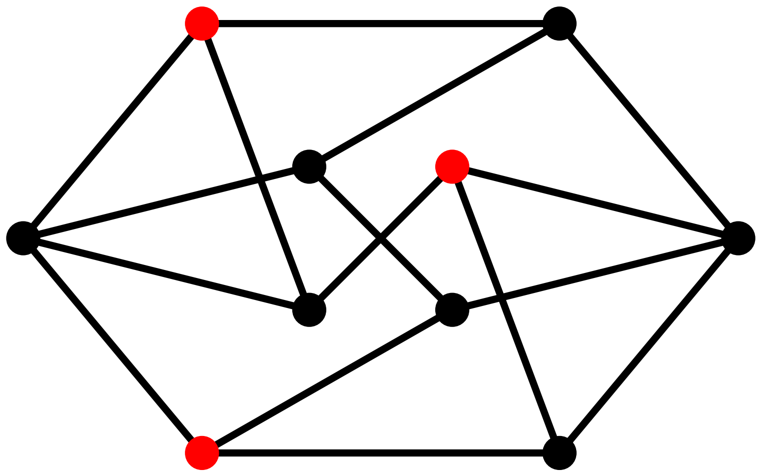

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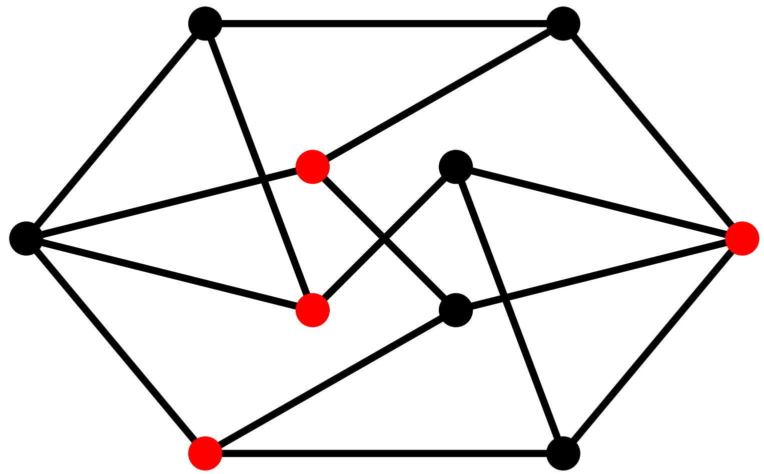

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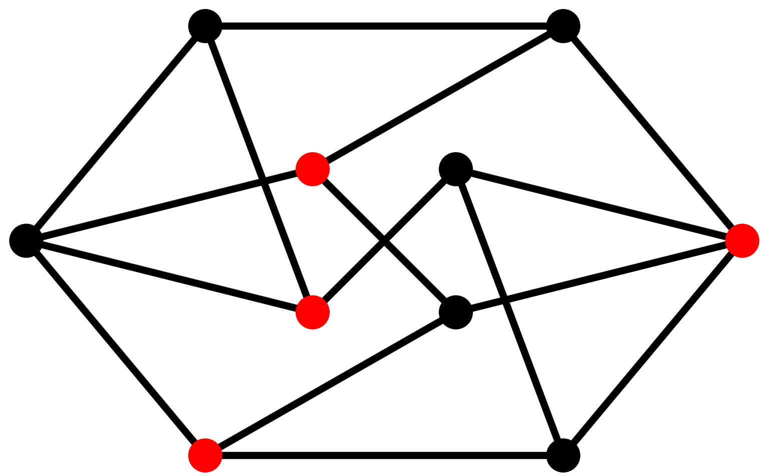
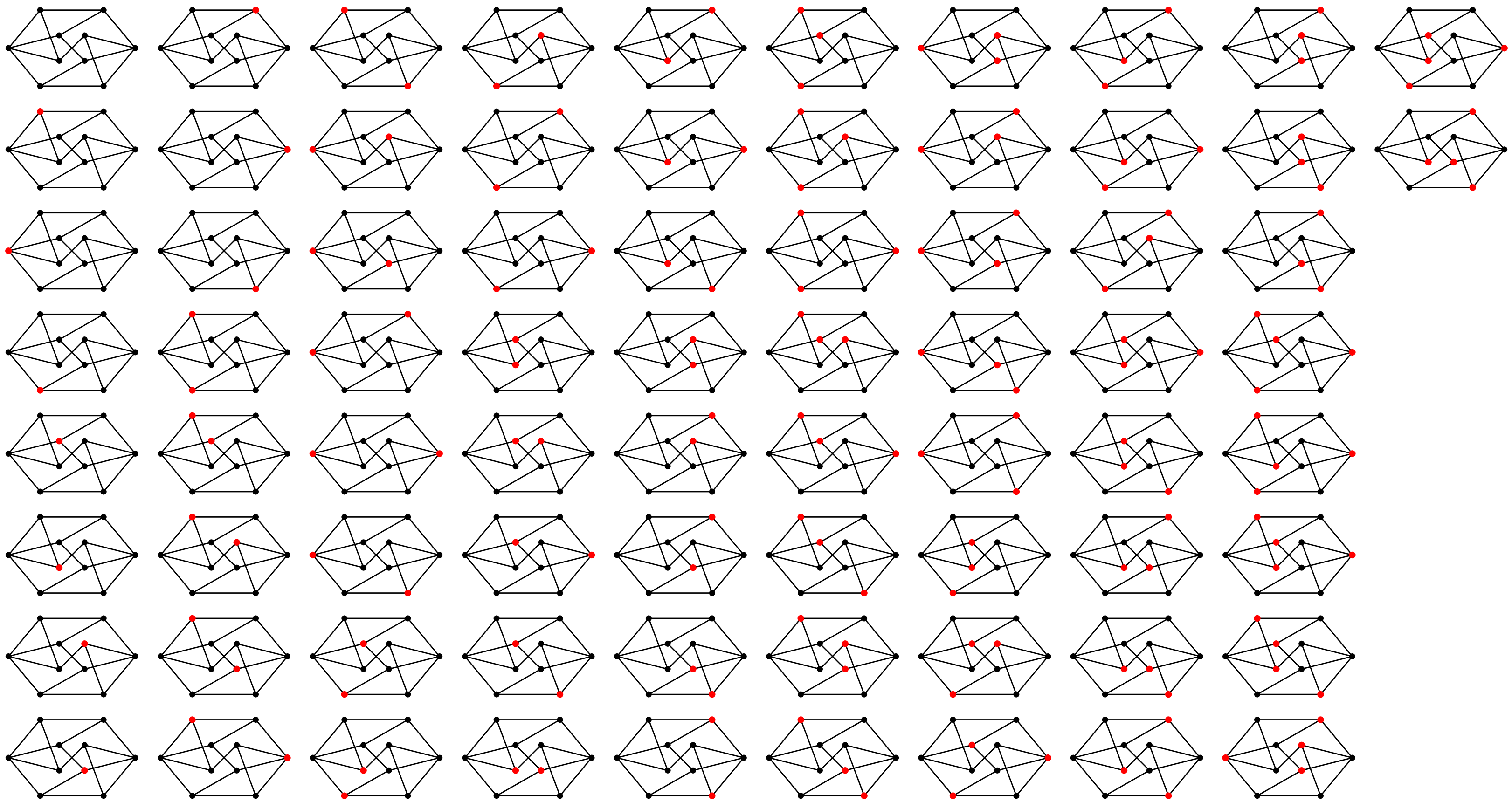


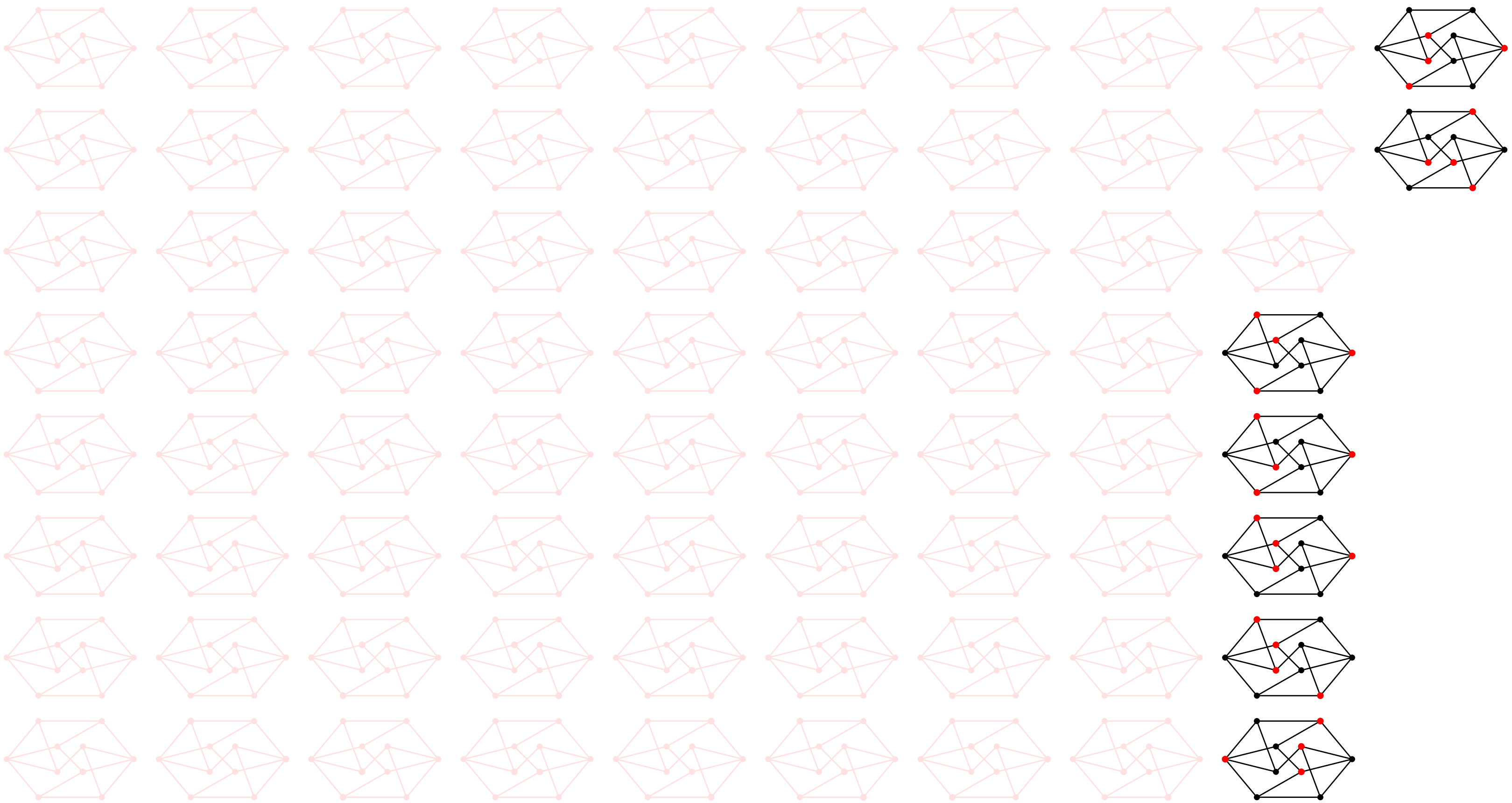
Maximum

Maximum independent sets



74 independent sets

Maximum independent sets

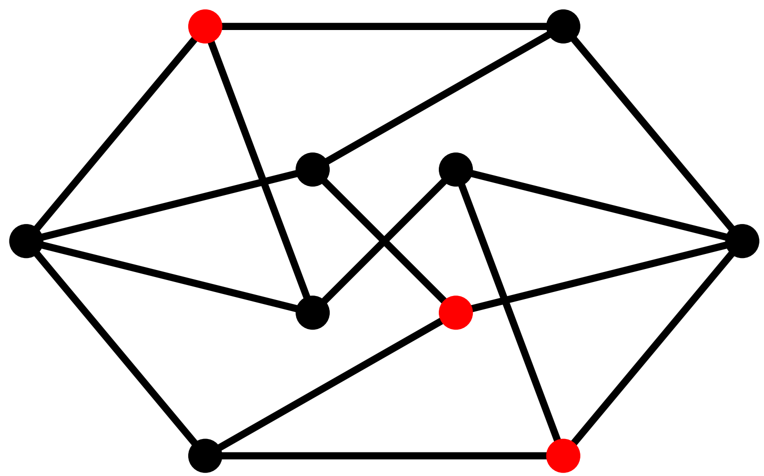


74 independent sets

7 maximum independent sets

Setup $G = (V, E)$ a graph

Def. An independent set I of G is **maximal** if no proper superset of I is independent.



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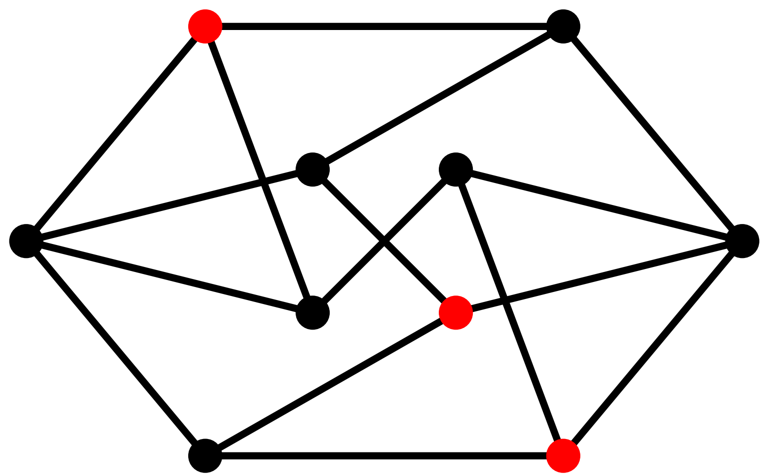

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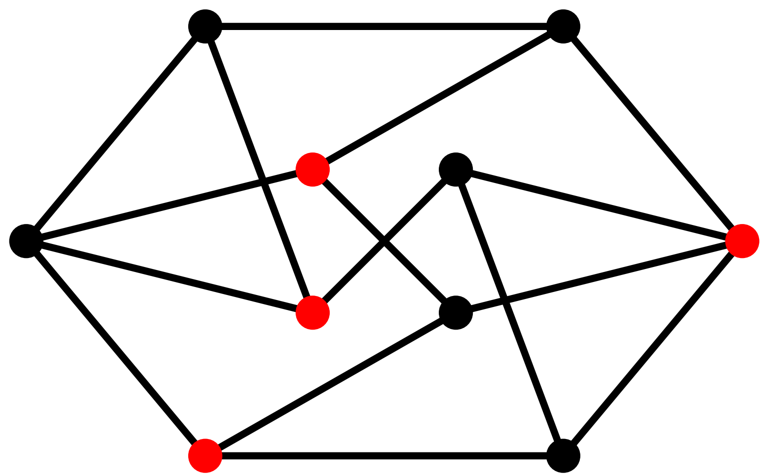

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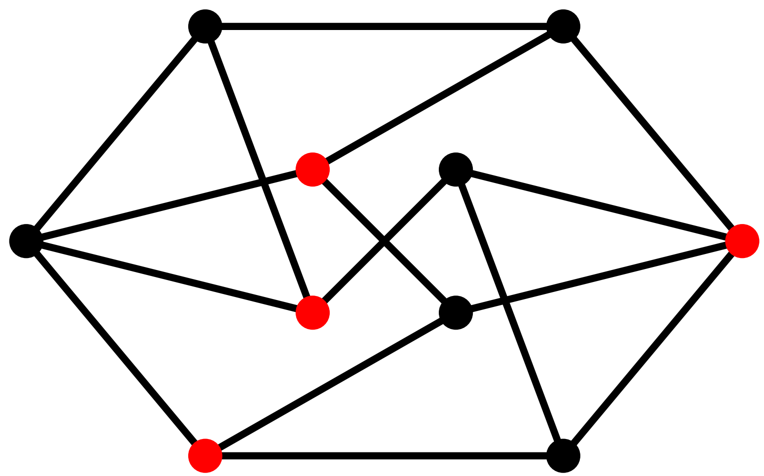

Maximal

A maximal independent set

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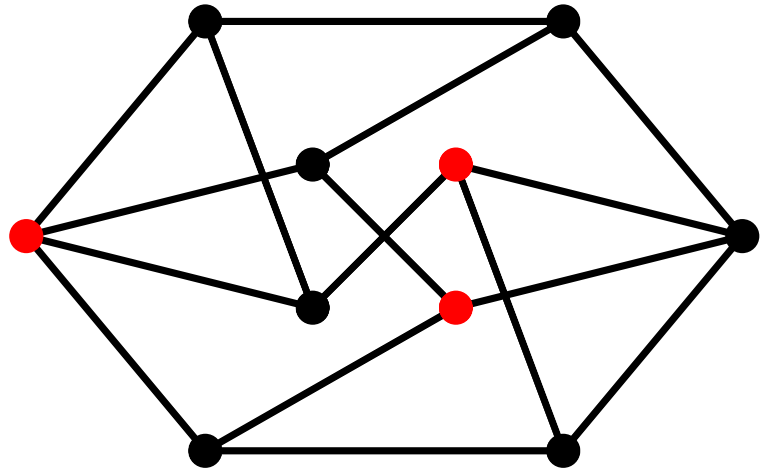

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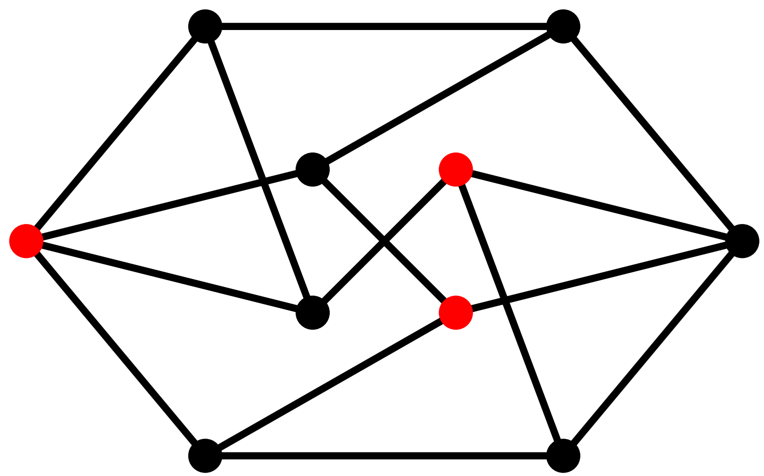

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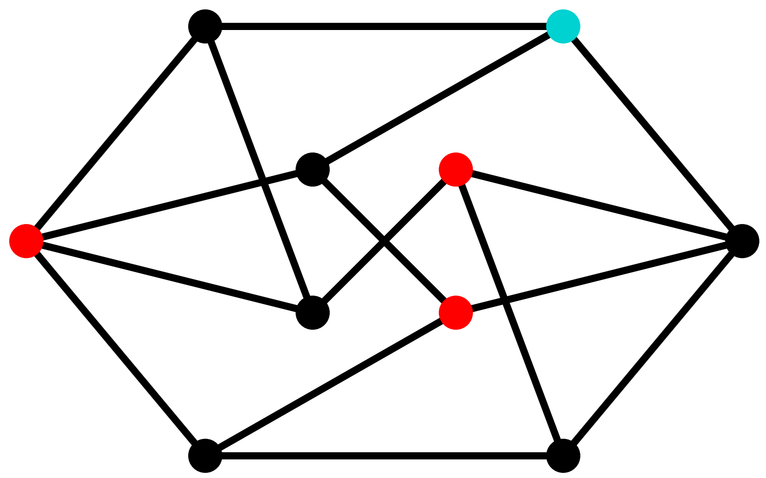
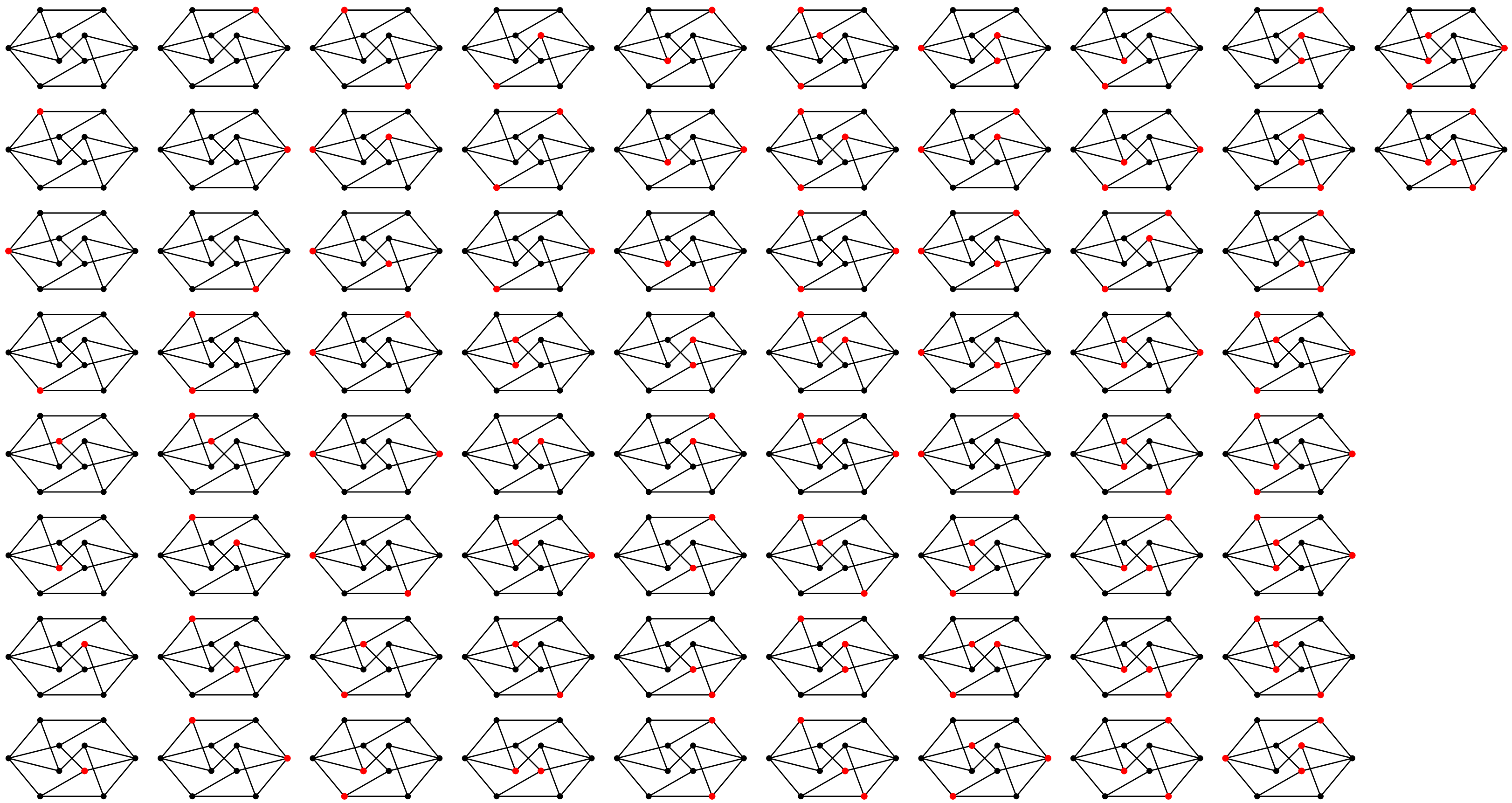


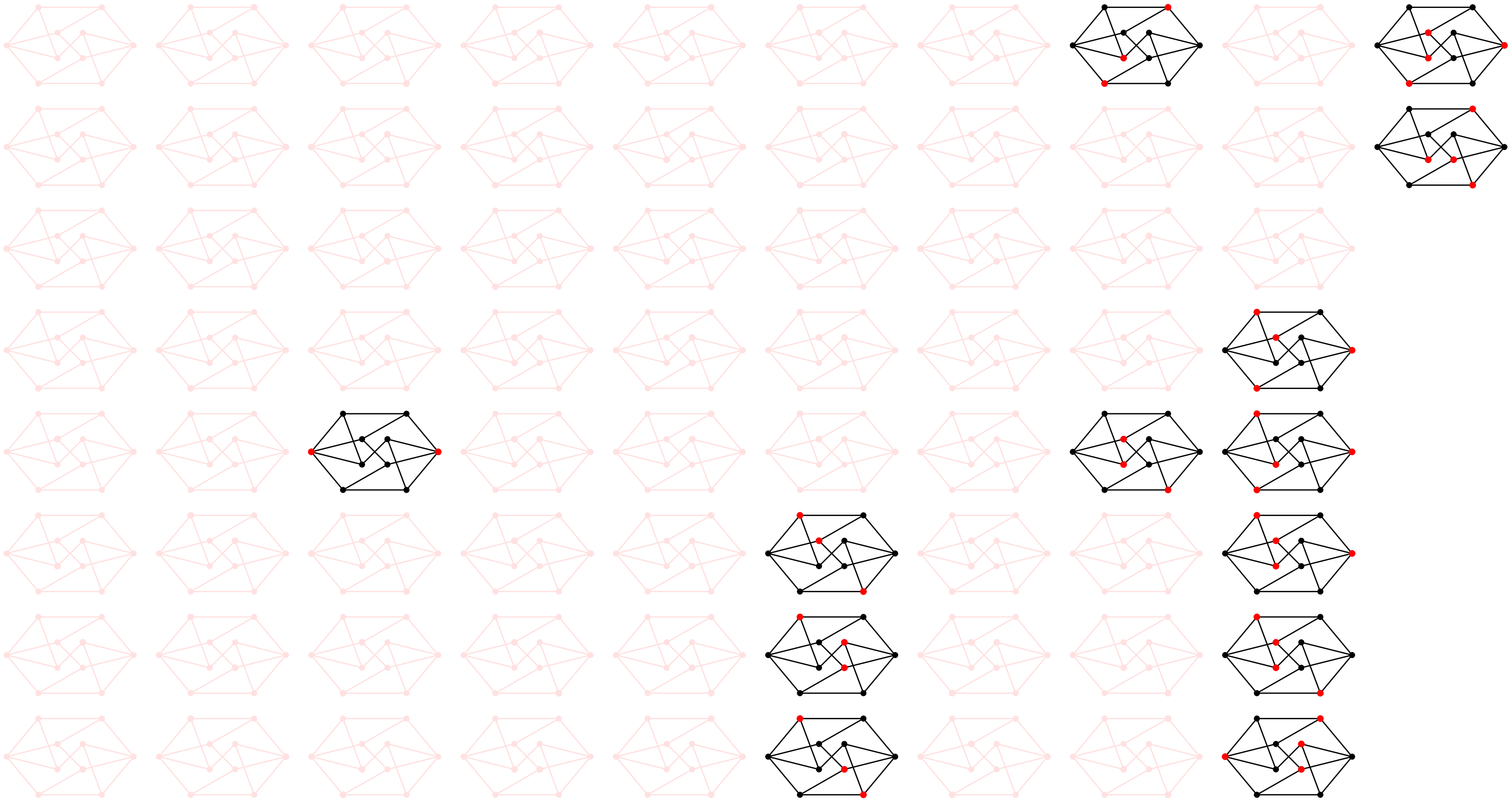
Not maximal

Maximal independent sets



74 independent sets

Maximal independent sets



74 independent sets

13 maximal independent sets

Setup $G = (V, E)$ a graph

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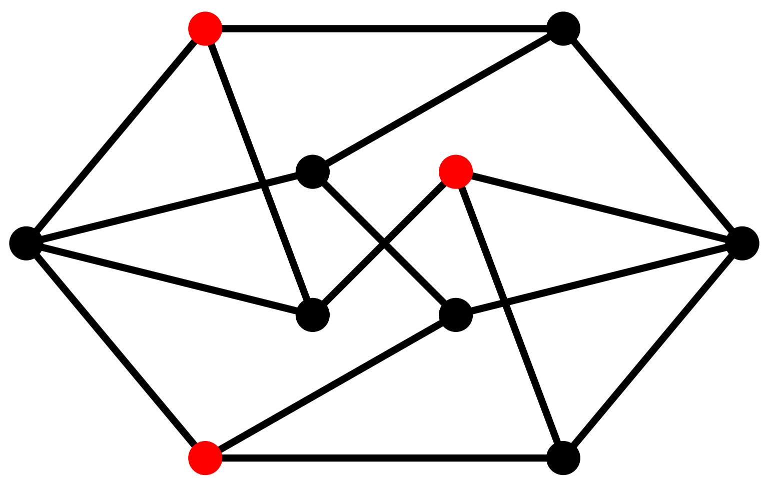

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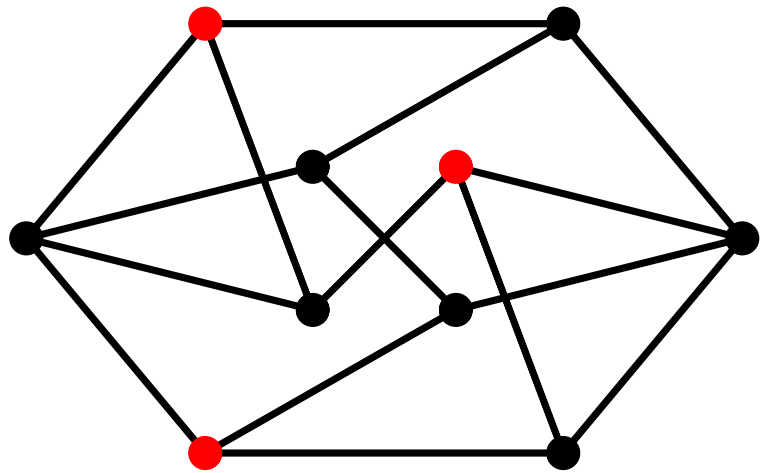

Not minimum maximal

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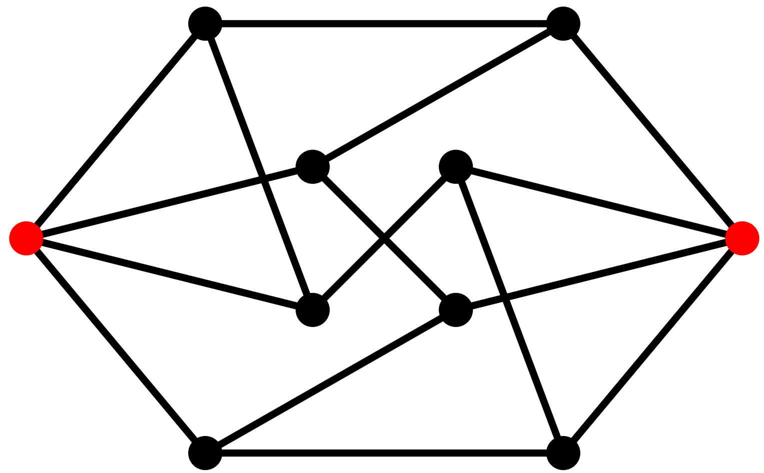

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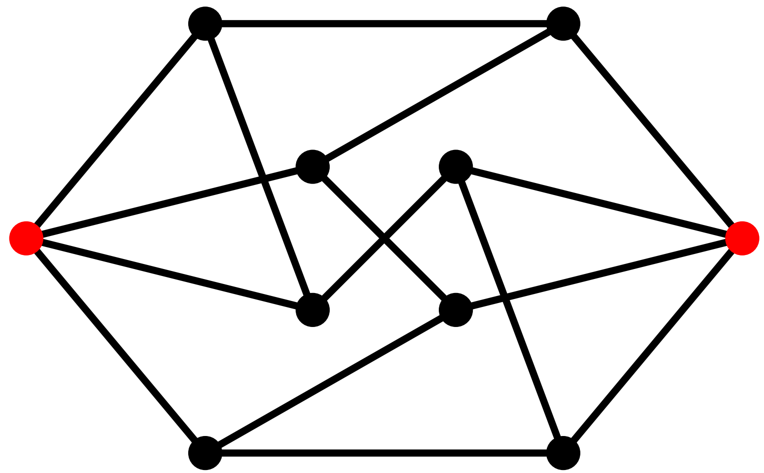

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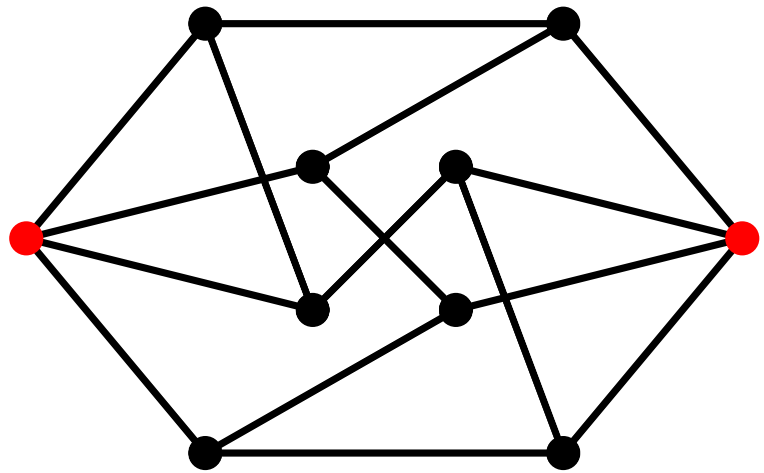
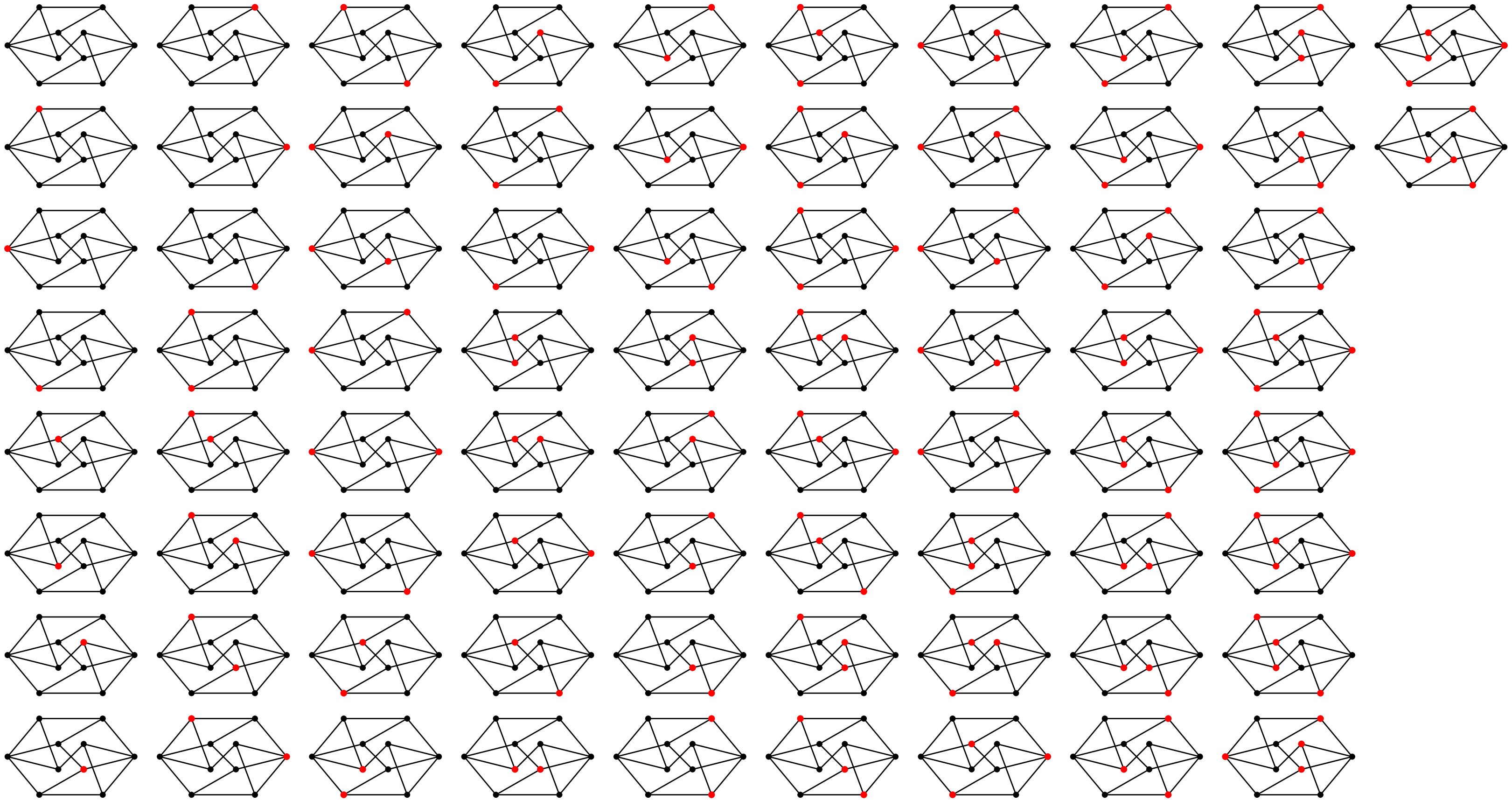


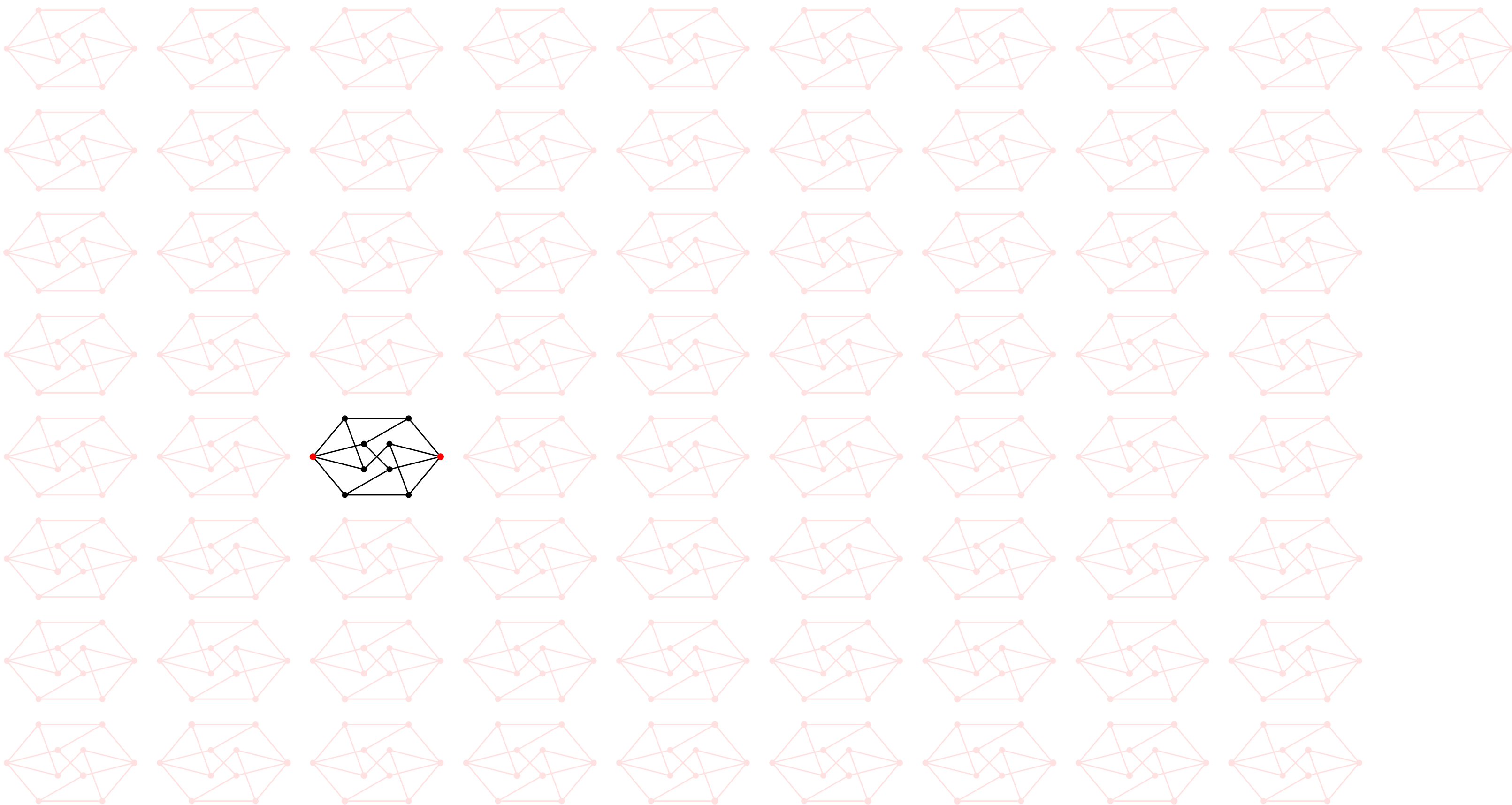
Minimum maximal

Also called a minimum independent dominating set of G



74 independent sets

Minimum maximal independent sets



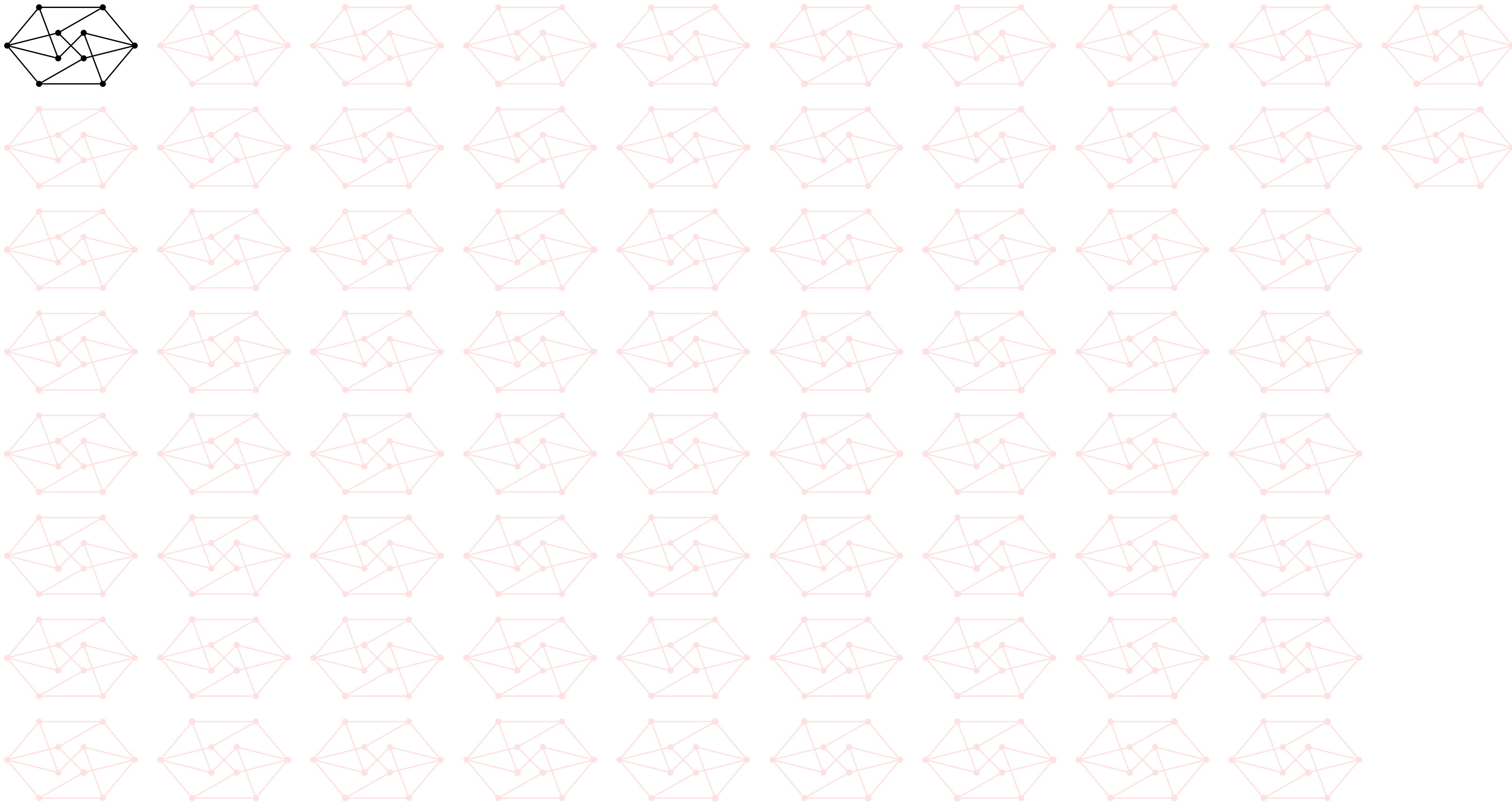
74 independent sets

1 minimum maximal independent set

We study the following **counting problems**:

Input	$G = (V, E)$ a graph
Output	(1) # independent sets of G **74**
	(2) # maximum independent sets of G **7**
	(3) # independent sets of G of fixed size
	(4) # maximal independent sets of G **13**
	(5) # minimum maximal independent sets of G **1**

Independent sets of fixed size



74 independent sets

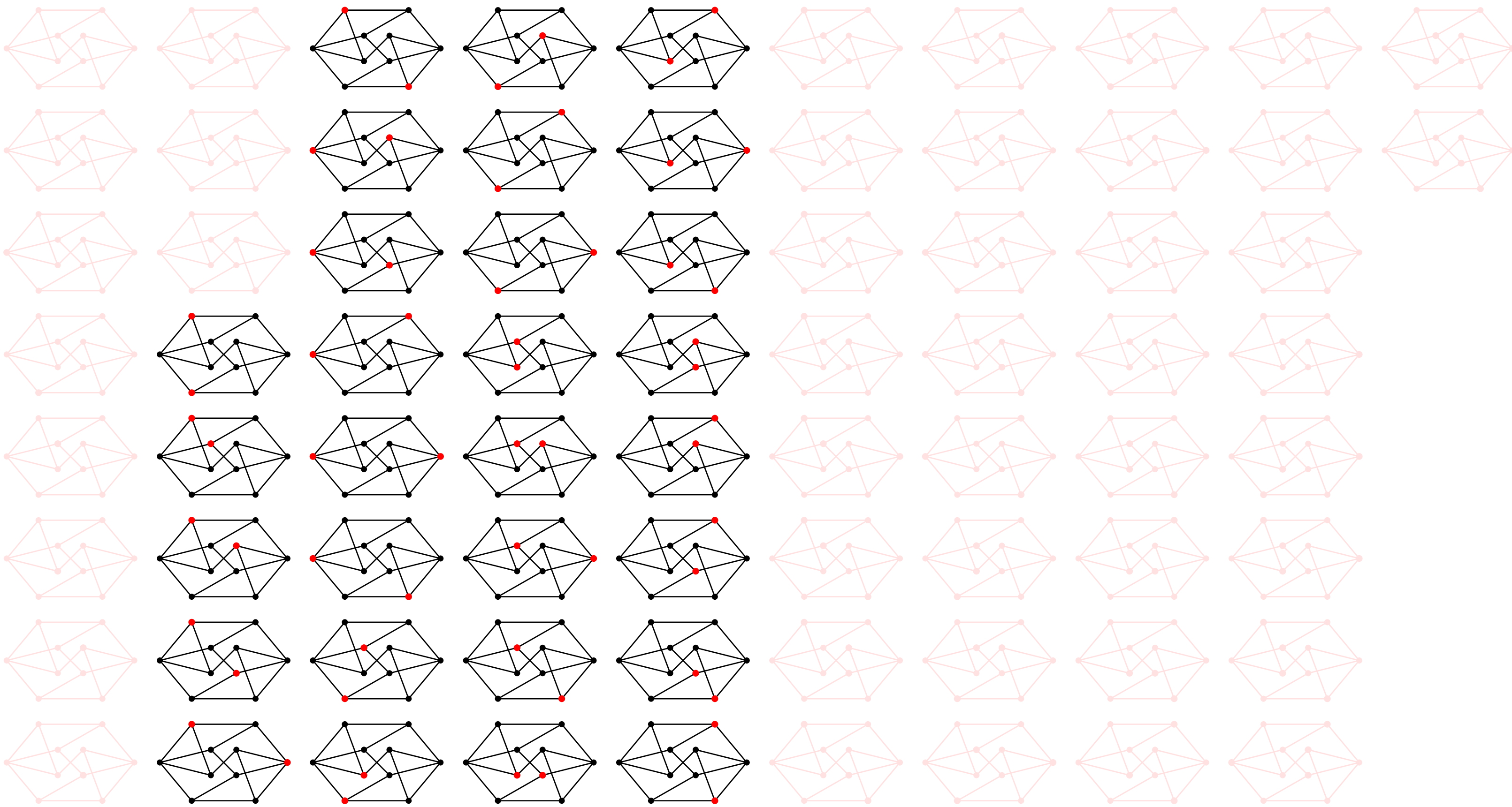
1 independent set of size 0

Independent sets of fixed size

74 independent sets

10 independent sets of size 1

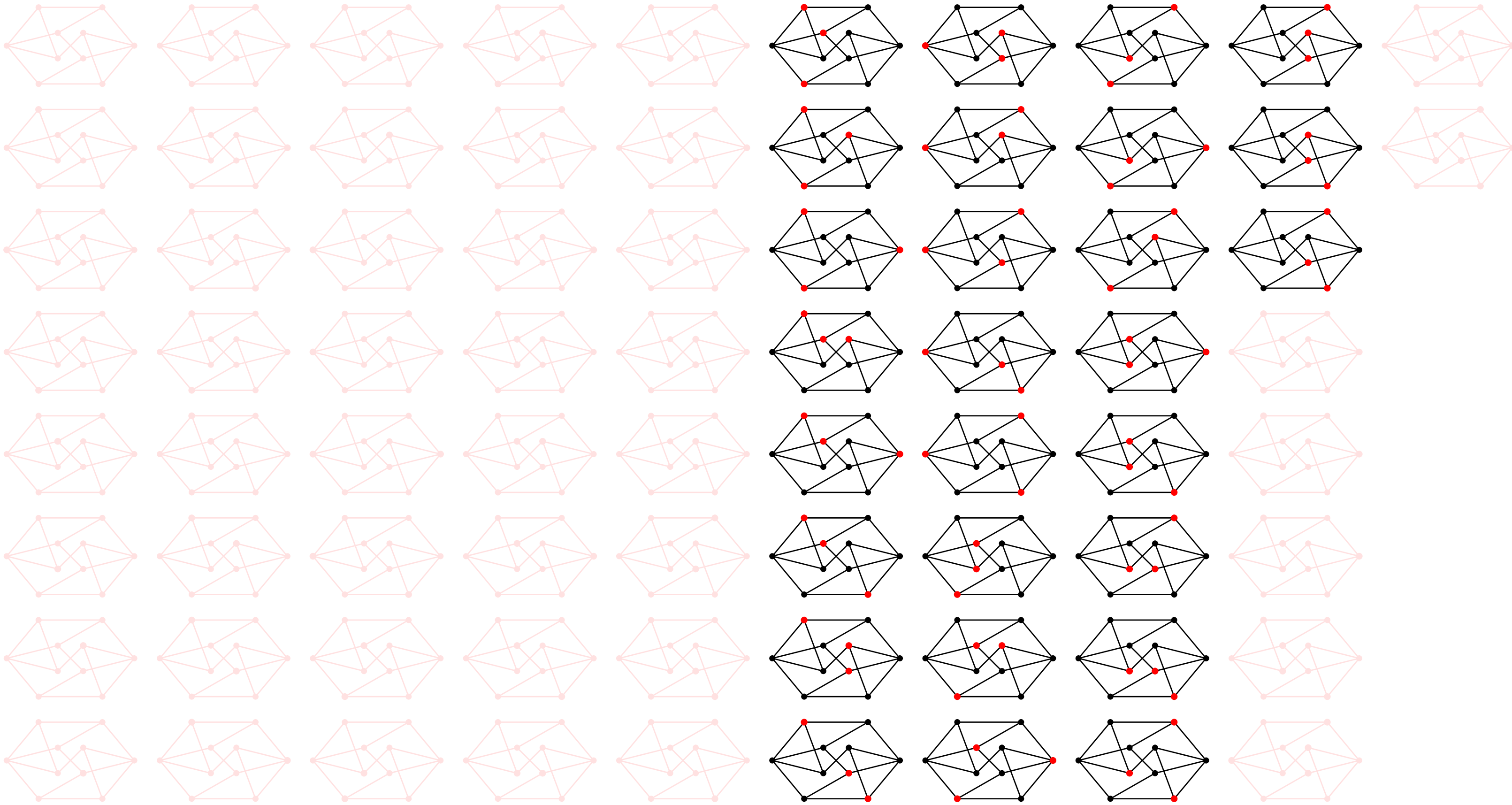
Independent sets of fixed size



74 independent sets

29 independent sets of size 2

Independent sets of fixed size



74 independent sets

27 independent sets of size 3

Independent sets of fixed size

74 independent sets

7 independent sets of size 4

$\boxed{\text{Fact}}$ These counting problems are #P-complete (analoguous to NP-completeness).

$\Longrightarrow$ Cannot hope for a poly-time algorithms.

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- bipartite graphs (Provan & Ball)
- planar bipartite graphs of max deg 4 (Vadhan).

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$\Longrightarrow$ Cannot hope for a poly-time algorithms even for

- the line graphs of bipartite graphs (Valiant)
- bipartite graphs (Provan & Ball)
- planar bipartite graphs of max deg 4 (Vadhan).

$\Longrightarrow$ Focus on another class of perfect graphs!!

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if every induced cycle is of length three.

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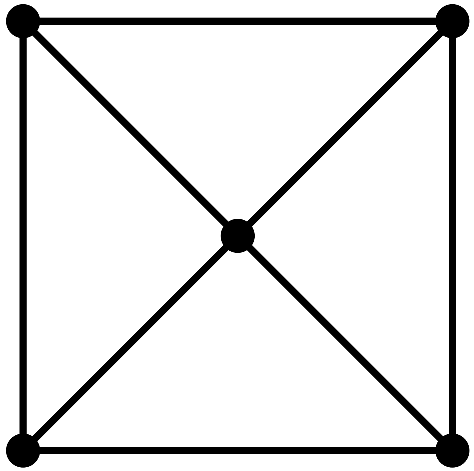

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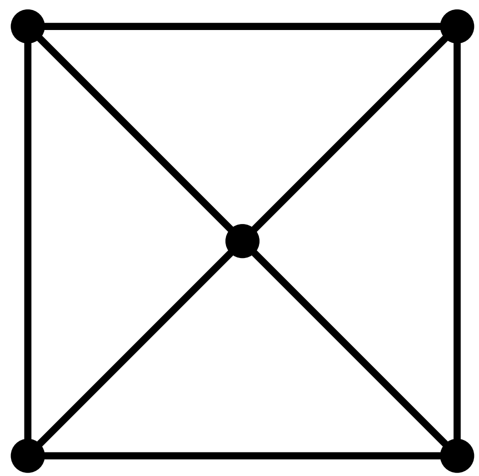

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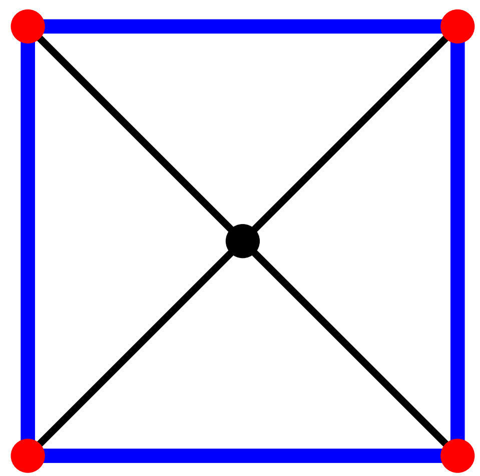

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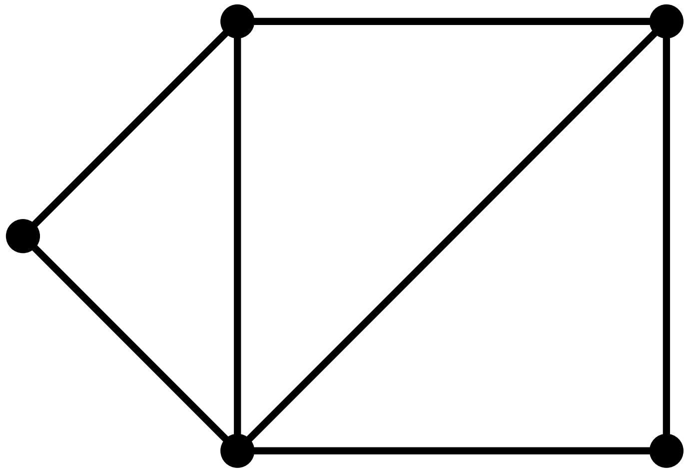

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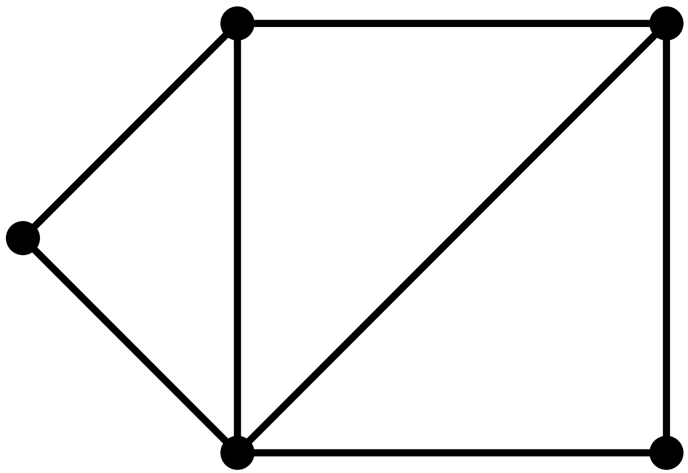

Chordal

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Input: $G = (V, E)$ a chordal graph

Output:

(1) # independent sets of G

(2) # maximum independent sets of G

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We obtain the following results for chordal graphs.

(1) # independent sets of G
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$O(|V| + |E|)$ alg.

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#P-complete

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#P-complete

Given $G = (V, E)$ a (connected) chordal graph

Def A **clique tree** of G is a tree T st

(1) the nodes of $T =$ the maximal cliques of G,

(2) $\forall\ v \in V$,

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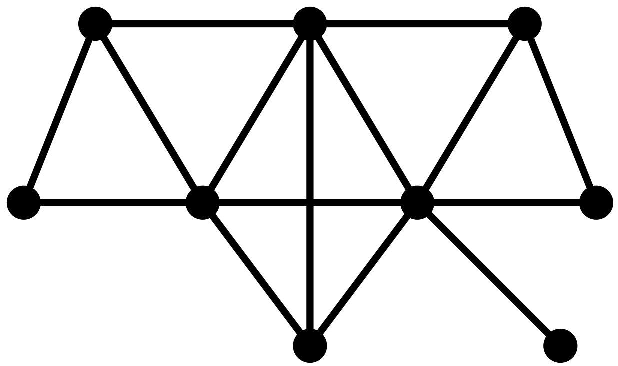

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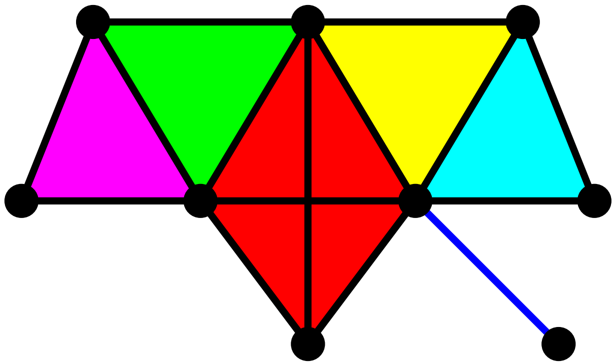

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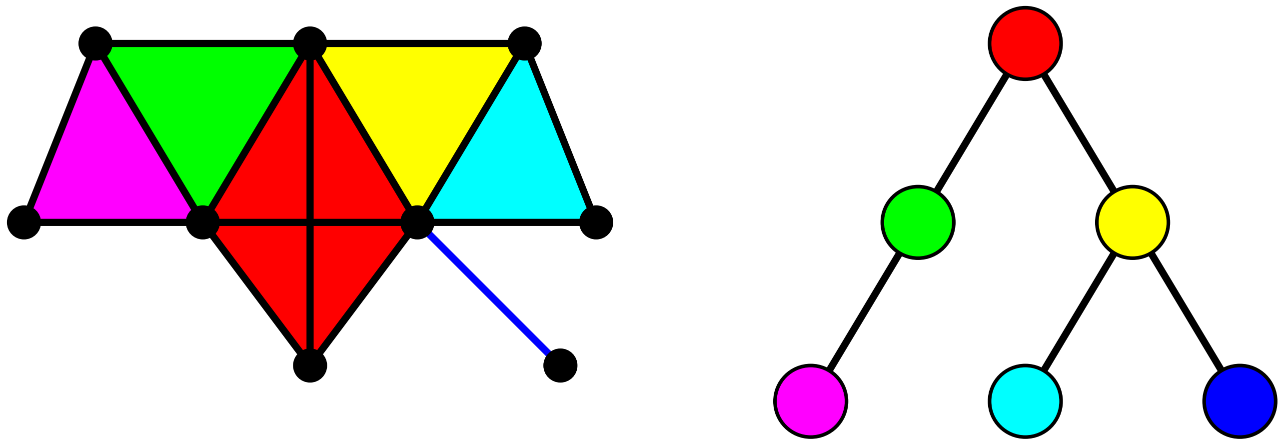

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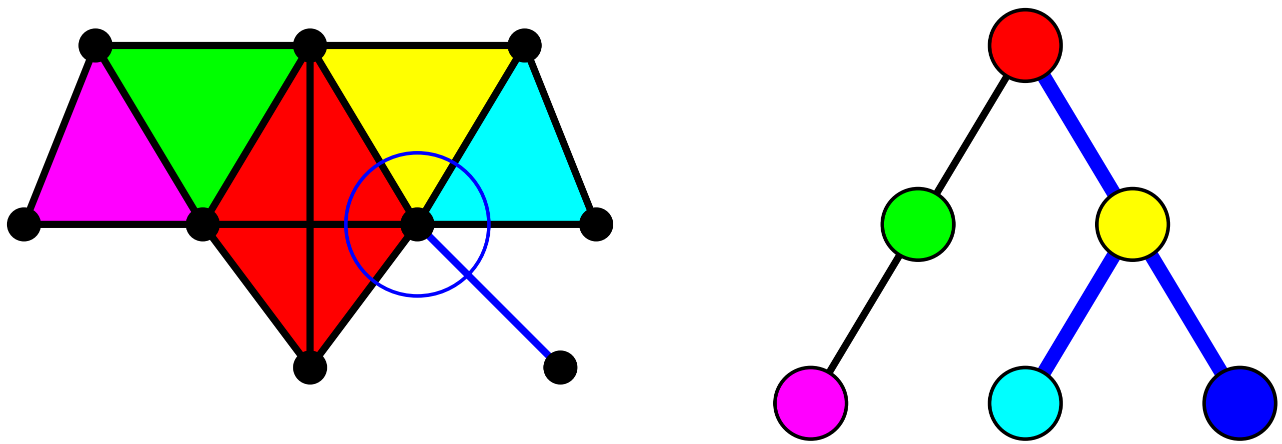

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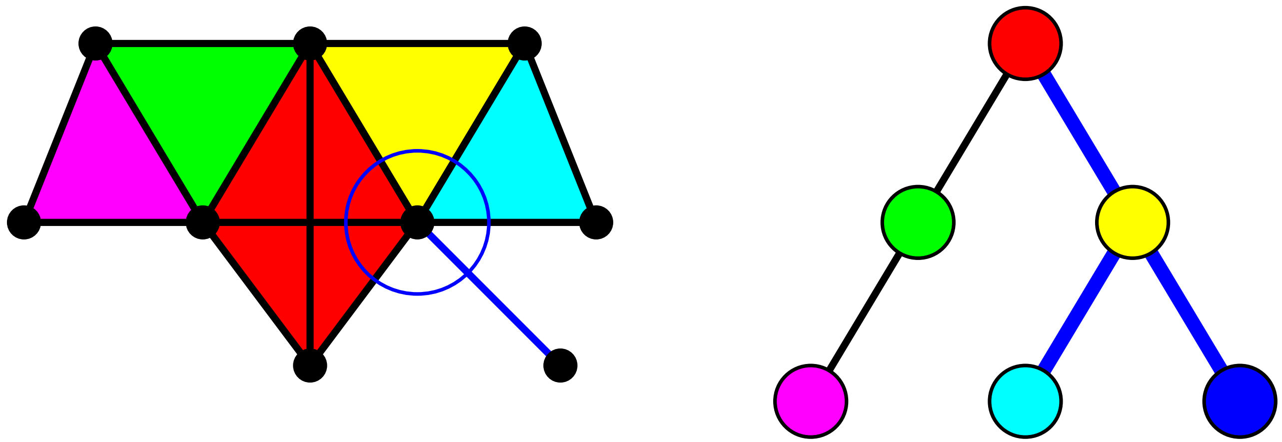

A clique tree can be computed in linear time.

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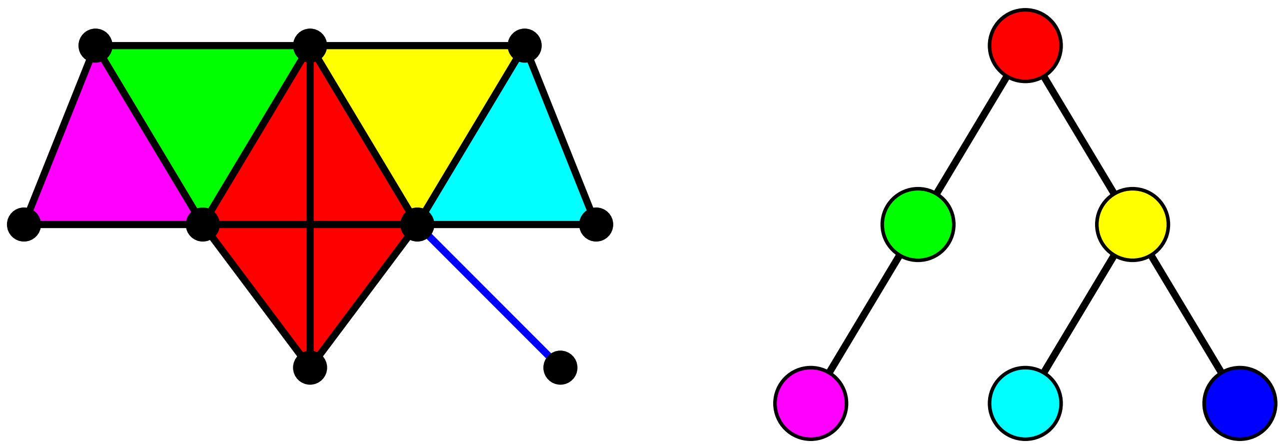

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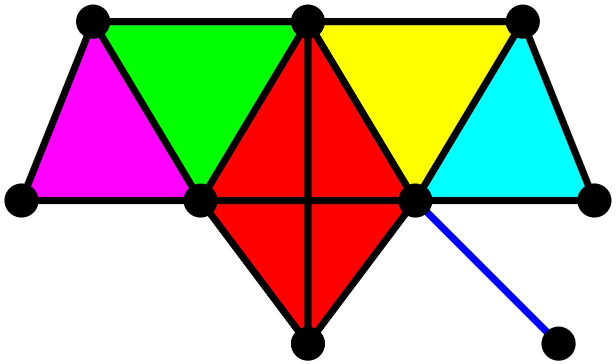
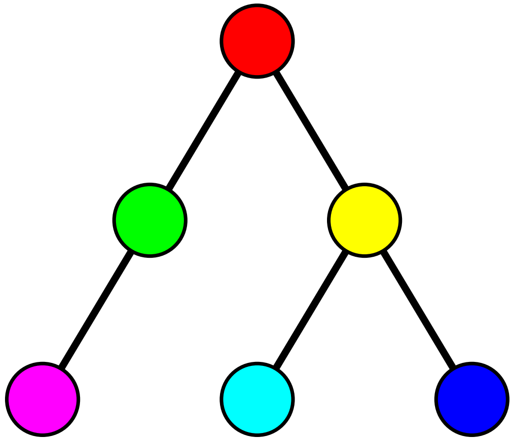
Property Every independent set of G either contains exactly one vertex in K or not.





Given $G = (V, E)$ a (connected) chordal graph, T a clique tree of G, with root K

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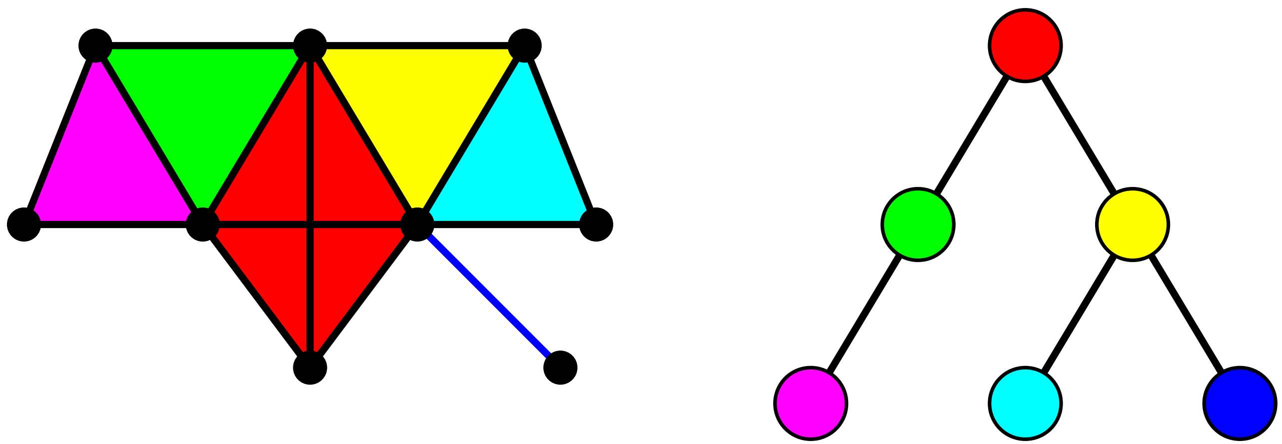

Leads to a recursive formula...

Recursive formula

Let G be a chordal graph and T be a rooted clique tree of G. For a maximal clique K of G which is not a leaf of the clique tree, let $K_1, \dots, K_\ell$ be the children of K in T. Furthermore, let $v \in K$. Then, the following identities hold.

$$\mathcal{IS}(G(K)) = \overline{\mathcal{IS}}(G(K), K) \,\dot\cup \dot{\bigcup_{v \in K}} \mathcal{IS}(G(K), v);$$

$$\mathcal{IS}(G(K), v) = \{S \cup \{v\} \mid S = \bigcup_{i=1}^{\ell} S_i, S_i \in \left\{ \begin{array}{ll} \mathcal{IS}(G(K_i), v) & \text{if } v \in K_i \\ \overline{\mathcal{IS}}(G(K_i), K \cap K_i) & \text{otherwise} \end{array} \right\}\};$$

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where $\mathcal{IS}(G)$ denotes the family of independent sets in G, $\mathcal{IS}(G, v)$ denotes for a vertex v the family of independent sets in G including v, $\overline{\mathcal{IS}}(G, U)$ denotes the family of independent sets in G including no vertex of U for a vertex set U.

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A detailed analysis yields a linear-time algorithm to count the independent sets in a chordal graph!

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A detailed analysis yields a linear-time algorithm to count the independent sets in a chordal graph!

$\boxed{\text{Thm}}$ The independent sets in a chordal graph $G = (V, E)$ can be counted in $O(|V| + |E|)$ time.

We obtain the following results for chordal graphs.

(1) # independent sets of G
(2) # maximum independent sets of G
(3) # independent sets of G of fixed size

$O(|V| + |E|)$ alg.

(4) # maximal independent sets of G
(5) # minimum maximal independent sets of G

#P-complete

We use the following counting problem
(a counting version of the set cover problem).

Given	X a finite set, $\mathcal{S} \subseteq 2^{X}$ a family
Output	The number of subfamilies $\mathcal{S}' \subseteq \mathcal{S}$ st $\bigcup_{Y \in \mathcal{S}'} Y = X$.

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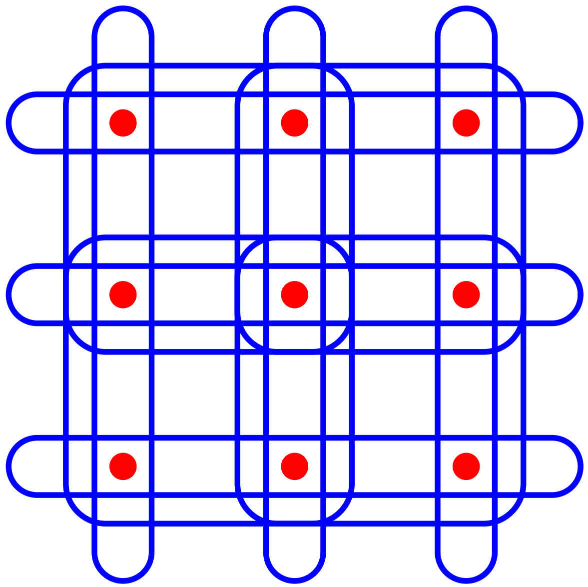

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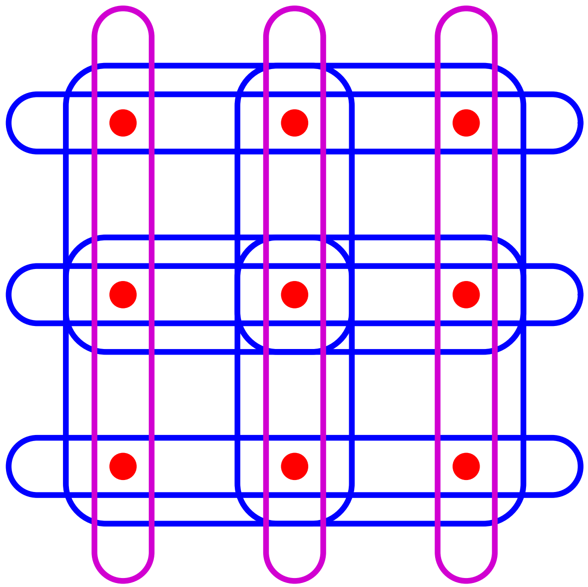

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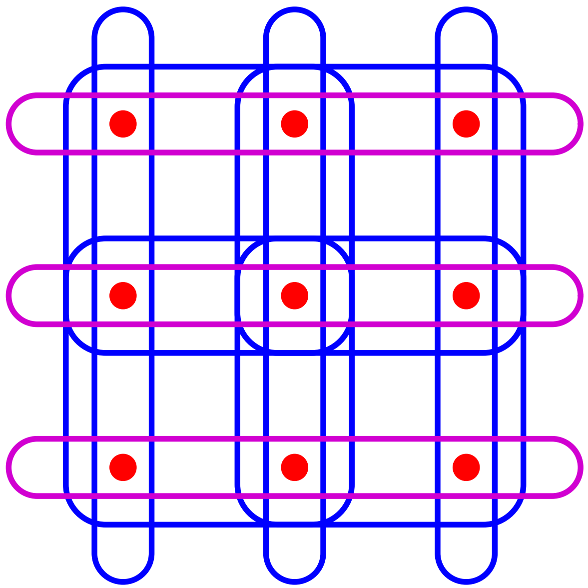

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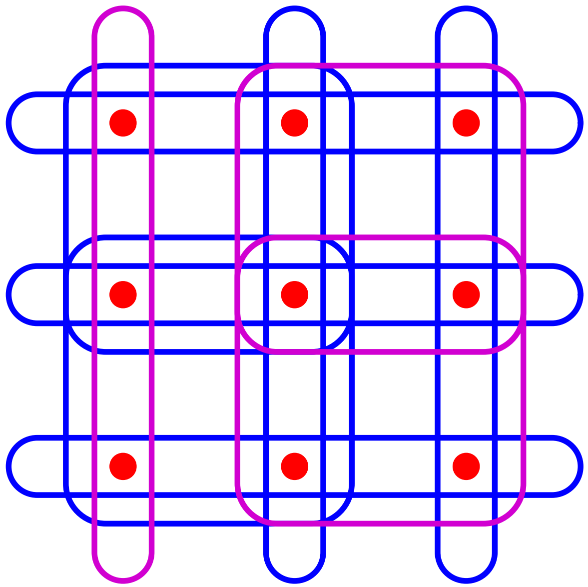

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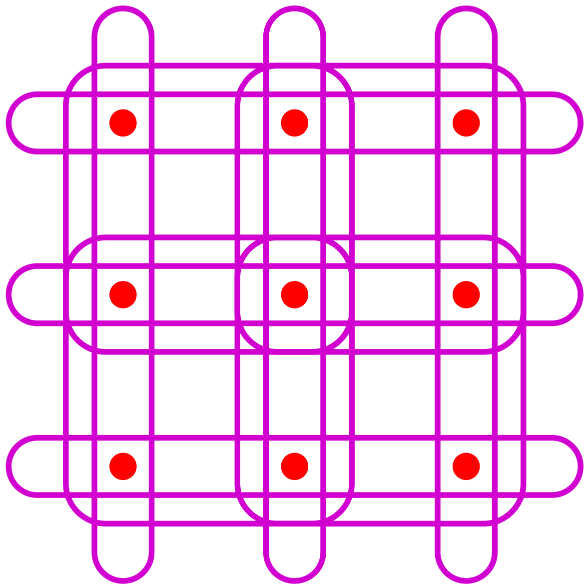

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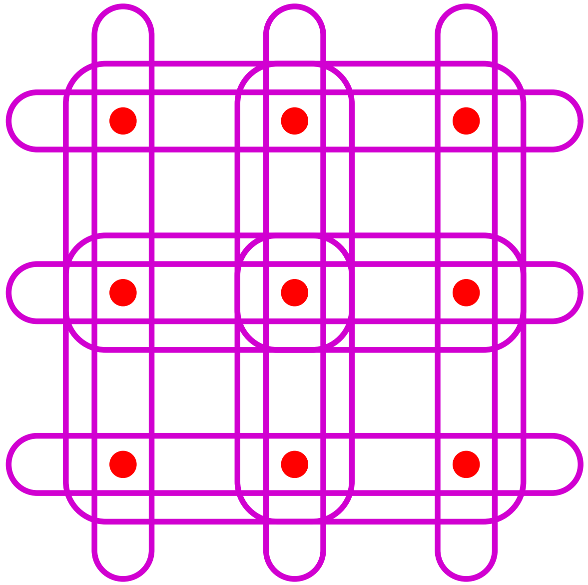
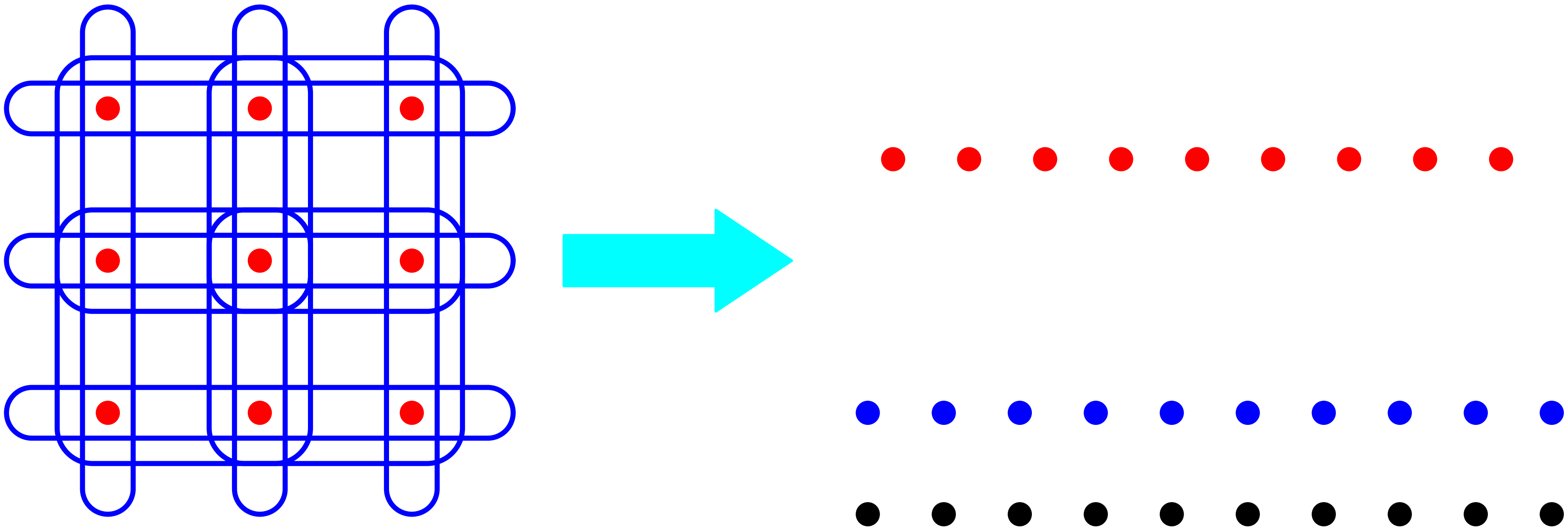
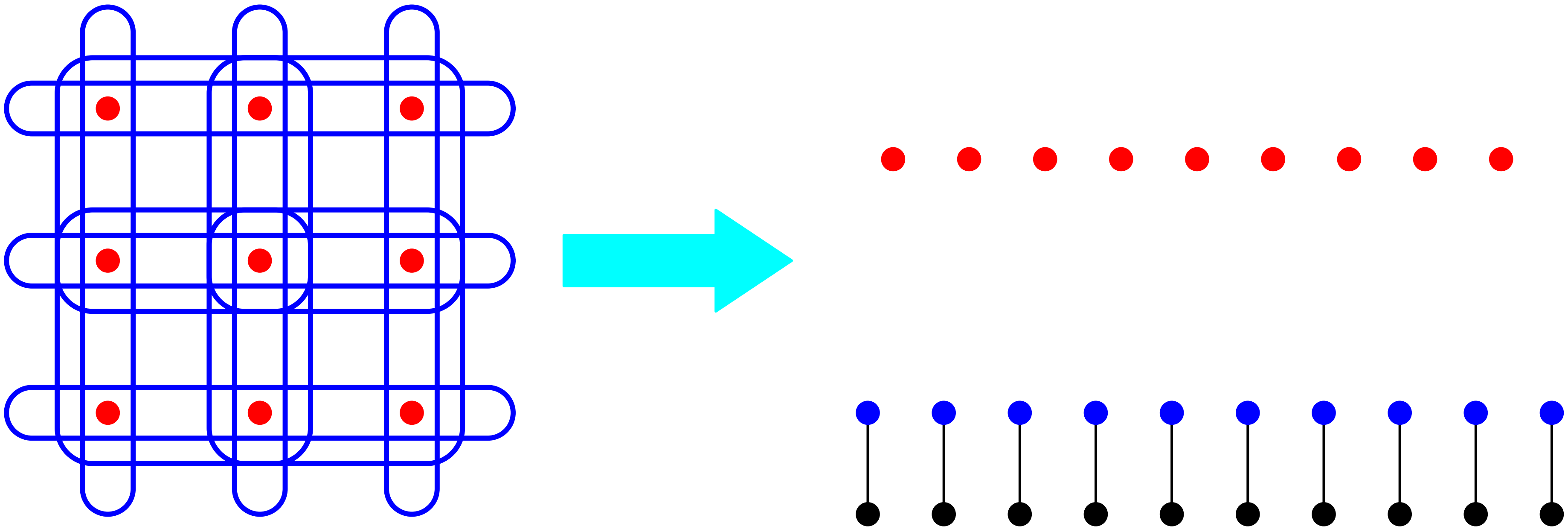


Actual reduction





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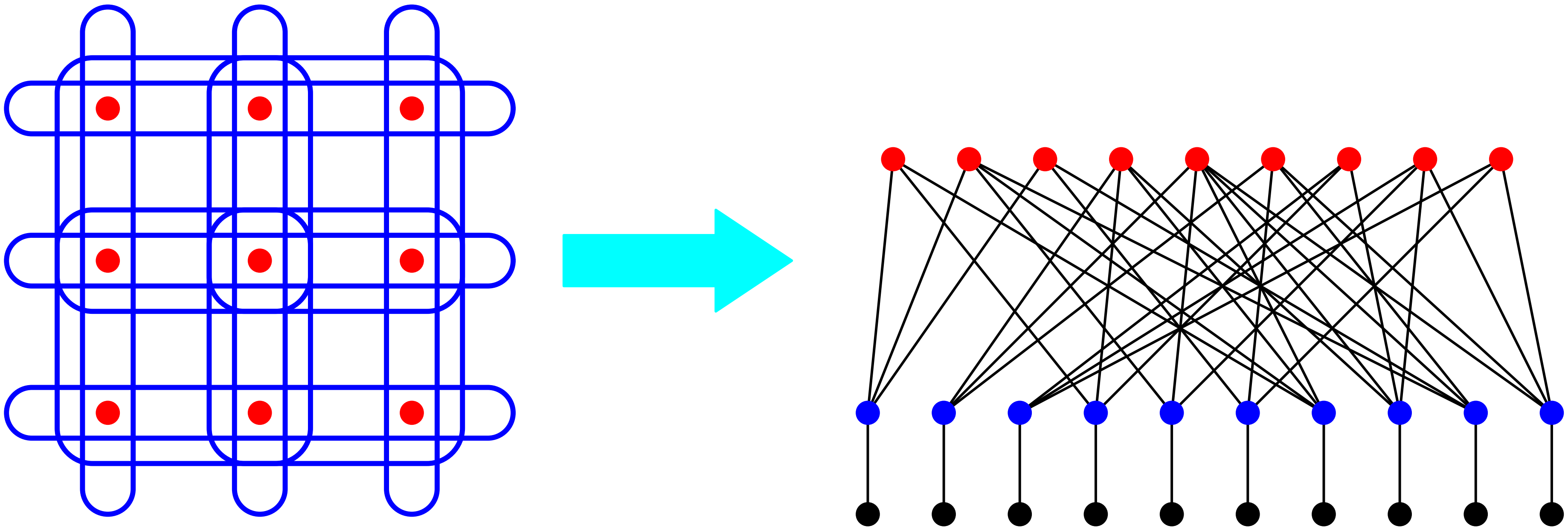

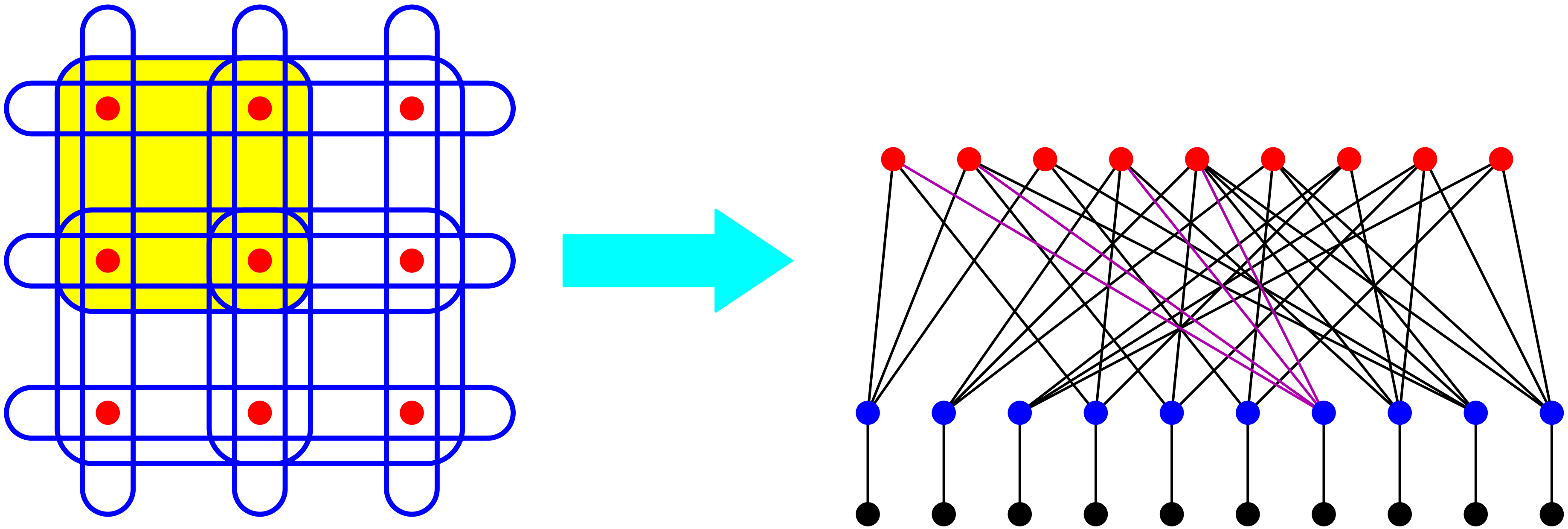

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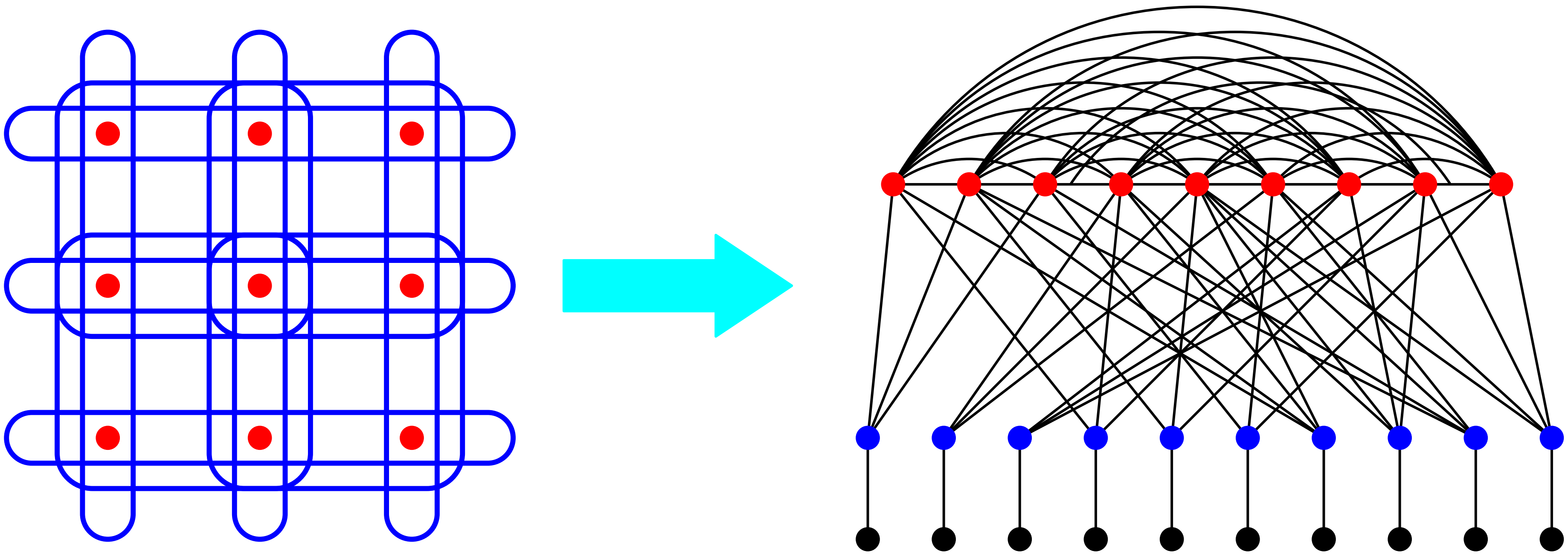

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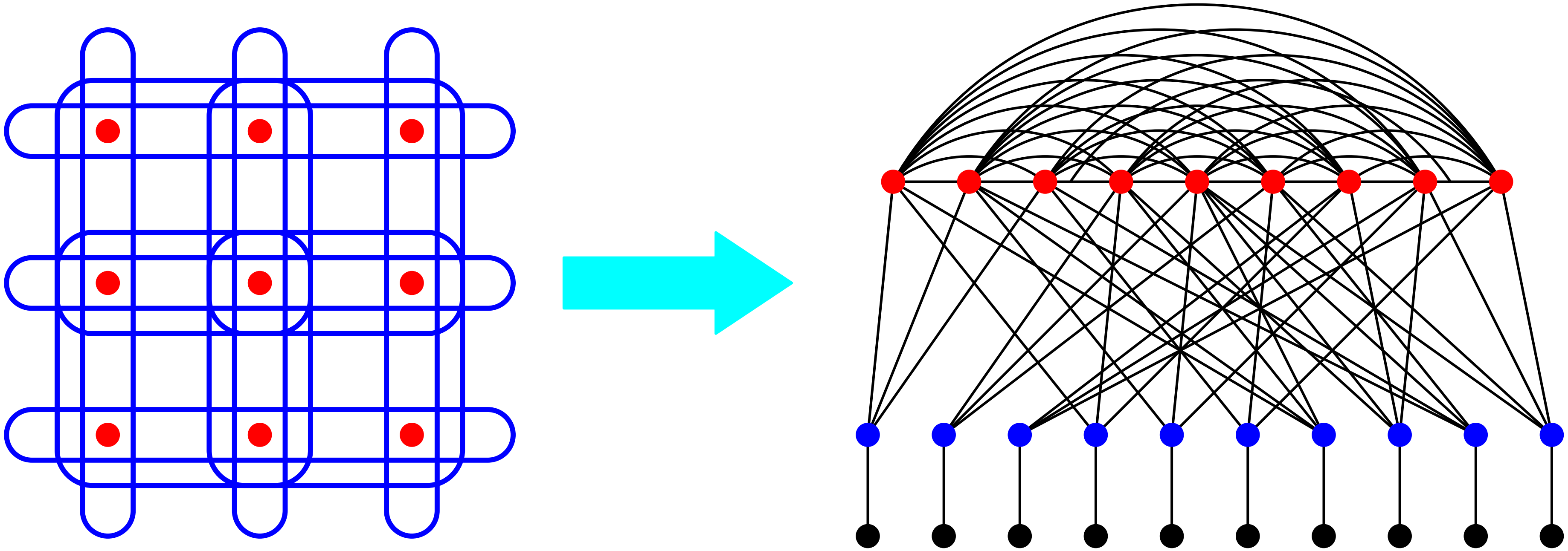

$$\#\ \text{set covers} + \sum_{x \in X} 2^{\#\ \text{sets not containing } x}$$

$$\|$$

$$\#\ \text{maximal independent sets}$$

Plan

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[End of my talk]

Köszönöm szépen!