A solution to the Füredi–Hajnal conjecture by Marcus & Tardos

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Mittagsseminar

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Combinatorics
Geometry
Computation

Conj. (Füredi & Hajnal, '92)

The number of 1-entries in an $n \times n$ 0/1-matrix avoiding an arbitrary fixed permutation matrix $= O(n)$.

Solved by A. Marcus & G. Tardos (Nov. '03)

Goal of this talk Look at their proof

and unsolved problems

Contents of the talk

(1) Definitions for the conjecture
(2) Proof by Marcus & Tardos
(3) Motivation
(4) Open problems

Setup A an $n \times n$ 0/1-matrix

P a $k \times \ell$ 0/1-matrix $(k, \ell \leq n)$

Def. A contains P if

$\exists$ a $k \times \ell$ submatrix B of A s.t.

$$p_{ij} = 1 \quad \Rightarrow \quad b_{ij} = 1$$

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$$p_{ij} = 1 \quad \Rightarrow \quad b_{ij} = 1$$

Otherwise, A avoids P

Def. A permutation matrix is a 0/1-matrix in which every row and column contains exactly one 1.

$$\begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline\end{array}$$

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Setup P a 0/1-matrix (fixed)

Q. What is the maximum number of 1's in an $n \times n$ 0/1-matrix avoiding P??

Def. $f(n, P)$ = such a maximum

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Trivial Observation

$$f(n, P) \leq n^2$$

Conj. (Füredi & Hajnal '92)

P a permutation matrix

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Easy observation $n \le f(n,P) \le n^2$.

P a $k \times k$ permutation matrix

Lem (Marcus & Tardos)

$$f(n, P) \leq (k-1)^2 f\left(\frac{n}{k^2}, P\right) + 2k^3 \binom{k^2}{k} n.$$

when n divisible by k^2.

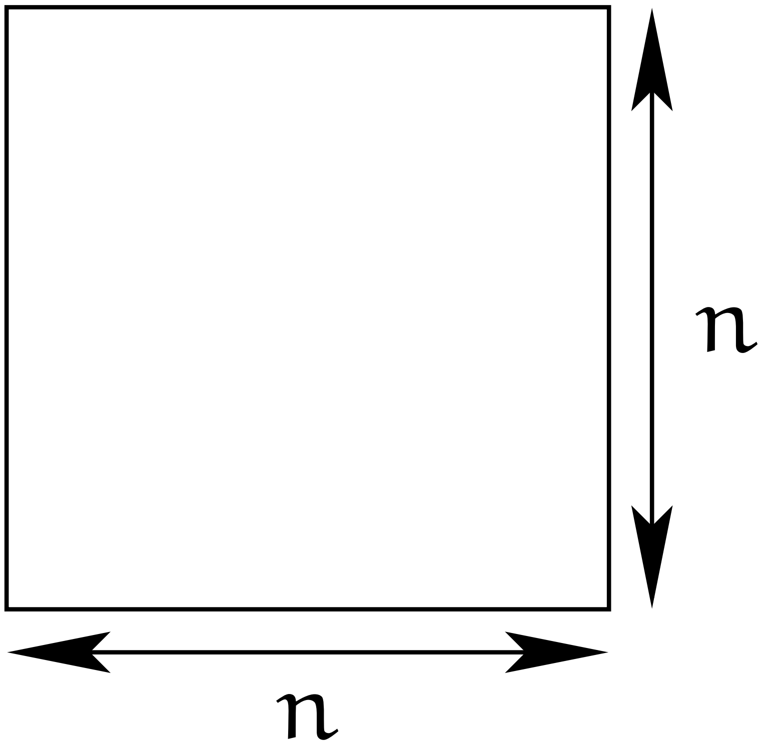
Thm (Marcus & Tardos)

$$f(n, P) \leq 2k^4 \binom{k^2}{k} n.$$

Setup

P a $k{\times}k$ permutation matrix (fixed)

A an $n{\times}n$ matrix attaining $f(n,P)$

$A =$ 

Setup P a $k \times k$ permutation matrix (fixed)

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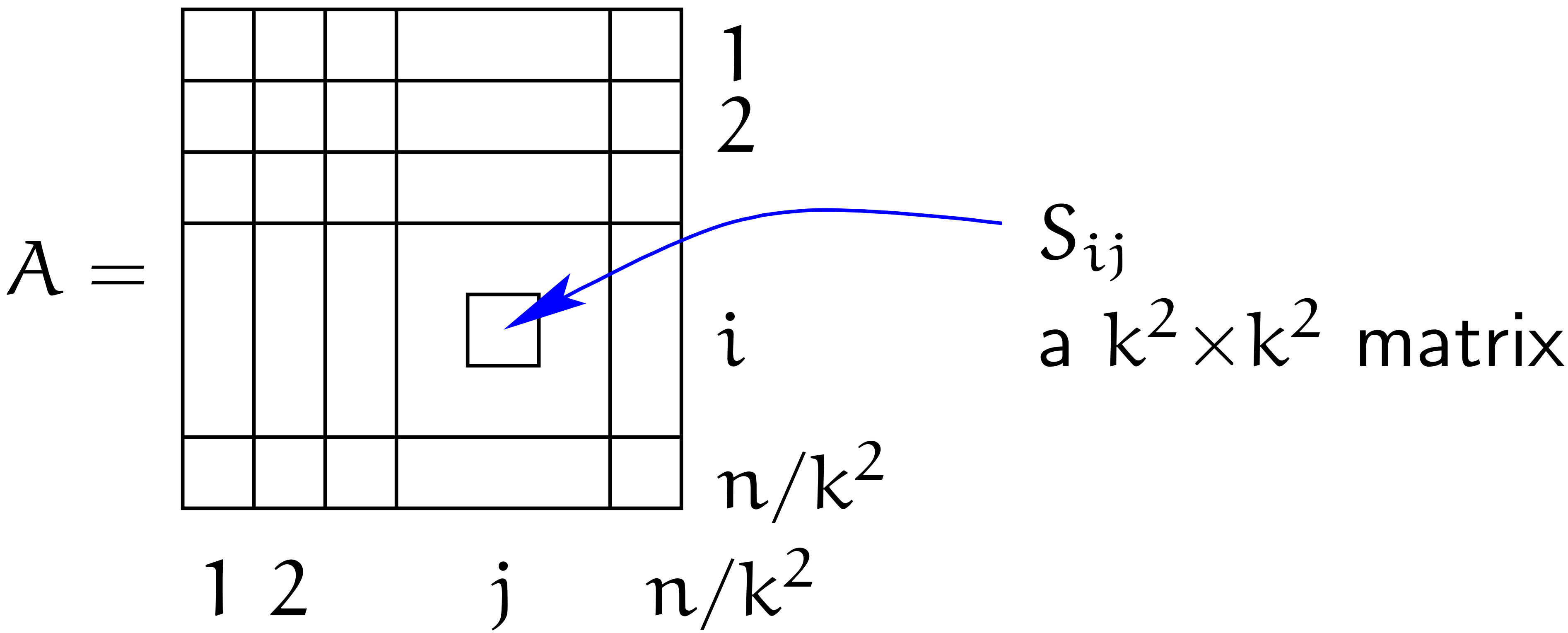
Divide into $(n/k^2) \times (n/k^2)$ blocks



Proof of Lemma (2): A reduced matrix B

Define an $(n/k^2) \times (n/k^2)$ matrix B as

$$B_{ij} = \begin{cases} 0 & \text{if } S_{ij} \text{ contains no 1} \\ 1 & \text{if } S_{ij} \text{ contains a 1} \end{cases}$$

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad B = \left[\begin{array}{c|c|c} 1 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right]$$

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Observation 1: B avoids P.

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So far, we obtained

- # of nonzero blocks $\leq f\left(\frac{n}{k^2}, P\right)$.
- # of 1's in each block $\leq f(k^2, P)$.

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which would give $f(n, P) = O(n)$,
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$\Rightarrow$ Need more observations!!

IDEA: Look at blocks with many 1's separately.

Proof of Lemma (3): Tall blocks

Def. A block S_{ij} is tall if
$\exists$ at least k nonzero rows in it

$$S_{ij} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}}_{k^2} \; k^2$$

Proof of Lemma (3): Tall blocks

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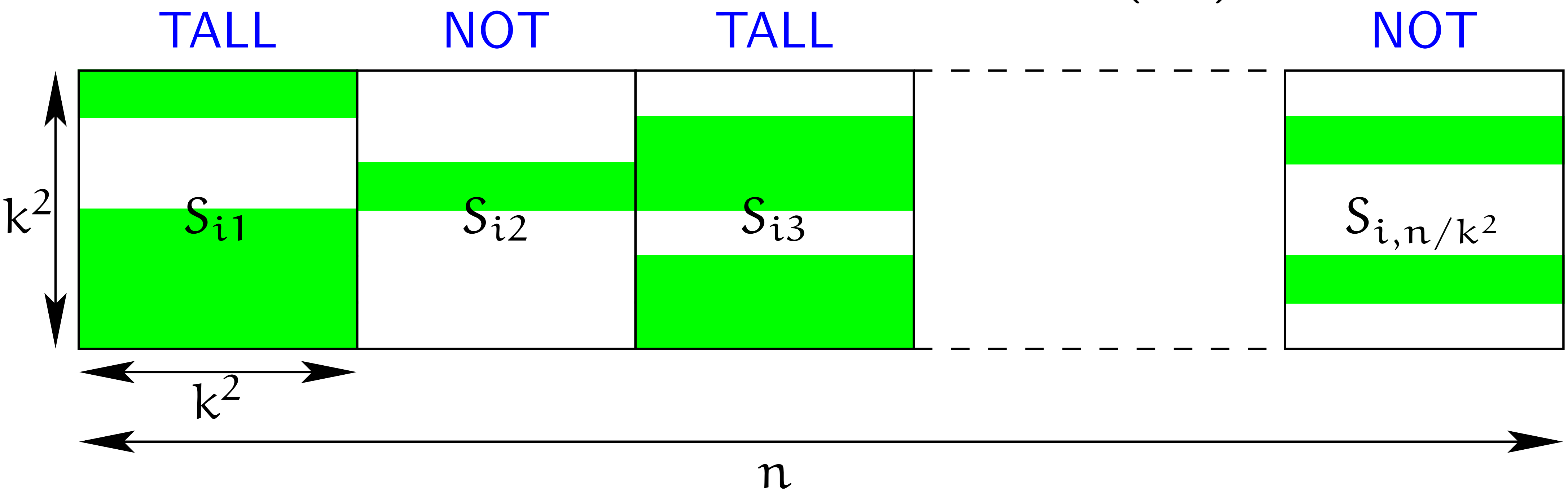
Observation 2: $\forall i$: $|\{j \mid S_{ij} \text{ tall}\}| < k\binom{k^2}{k}$



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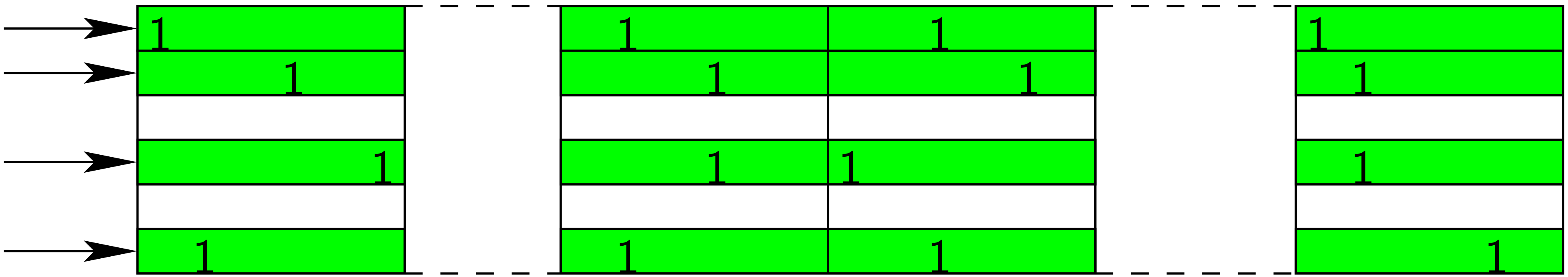
[Proof] Fix k rows out of k^2
$\#$ of S_{ij}'s with 1's in all of these rows $< k$.



$\boxed{\text{Def.}}$ A block S_{ij} is wide if
$\exists$ at least k nonzero columns in it

Observation 3: $\forall j$: $|\{i \mid S_{ij} \text{ wide}\}| < k\binom{k^2}{k}$

[Proof] The same argument as Obs. 2.

Proof of Lemma (5): Classification

(1) Zero blocks

- each contains no 1.

(2) Nonzero tall blocks

- each contains $\leq k^4$ 1's (by triv. Obs.)
- # of such blocks $\leq \frac{n}{k^2} k \binom{k^2}{k}$ (by Obs. 2)

(3) Nonzero wide blocks

- each contains $\leq k^4$ 1's (by triv. Obs.)
- # of such blocks $\leq \frac{n}{k^2} k \binom{k^2}{k}$ (by Obs. 3)

(4) Nonzero blocks neither tall nor wide

- each contains $\leq (k-1)^2$ 1's (by Def.)
- # of such blocks $\leq f(\frac{n}{k^2}, P)$ (by Obs. 1)

Proof of Lemma (6): Putting them together

$$\begin{aligned}
f(n,P) &\leq 0 \\
&+ \quad k^4 \times \frac{n}{k^2} k \binom{k^2}{k} \\
&+ \quad k^4 \times \frac{n}{k^2} k \binom{k^2}{k} \\
&+ \quad (k-1)^2 \times f\left(\frac{n}{k^2}, P\right) \\
&= 2k^3 \binom{k^2}{k} n + (k-1)^2 f\left(\frac{n}{k^2}, P\right).
\end{aligned}$$

[QED]

(1) Definitions for the conjecture

(2) Proof by Marcus & Tardos

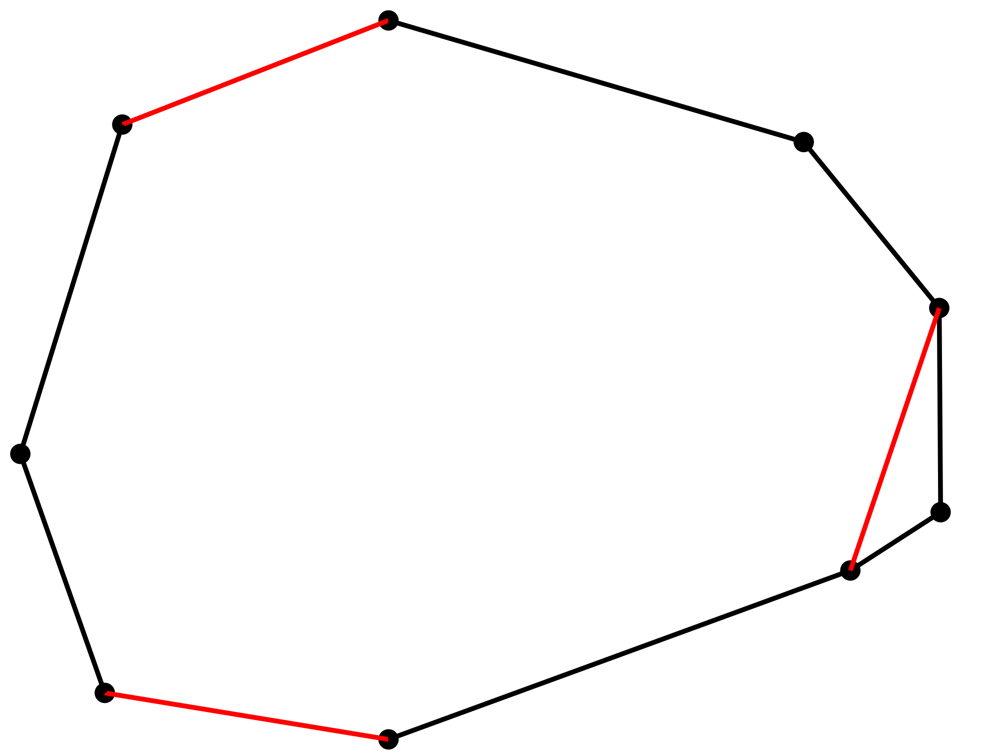
(3) Motivation

(4) Open problems

Question

(Erdős & Moser '59)

What is the maximum possible number of unit distances among n points in convex position on the plane?



$\boxed{\text{Question}}$ (Erdős & Moser '59)

What is the maximum possible number of unit distances among n points in convex position on the plane?

Lower bound

- Erdős & Moser '59 $\to$ $\left\lfloor \frac{5}{3}(n-1) \right\rfloor$.
- Edelsbrunner & Hajnal '91 $\to$ $2n-7$.

Question (Erdős & Moser '59)

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Upper bound

- Füredi '90 $\longrightarrow$ $O(n \log n)$.

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Upper bound

- Füredi '90 $\rightarrow O(n \log n)$.

Key Lemma (Füredi '90)

$$f(n, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}) = \Theta(n \log n).$$

Füredi & Hajnal ('92) investigated the all cases when P contains exactly four 1's. Especially...

$\boxed{\text{Thm}}$ (Füredi & Hajnal '92)

$$f(n, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) = \Theta(n^{3/2})$$

$\boxed{\text{Proof}}$ by Extremal Graph Theory.

Füredi & Hajnal ('92) investigated the all cases when P contains exactly four 1's. Especially...

$\boxed{\text{Thm}}$ (Füredi & Hajnal '92)

$$f(n, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}) = \Theta(n\alpha(n))$$

($\alpha(n)$ = the inverse Ackermann fn)

$\boxed{\text{Proof}}$ by Davenport–Schinzel sequences.

Why is it interesting?

Füredi '90

- Upper bound for the unit distance problem

Füredi & Hajnal '92

- Relation to Extremal Graph Theory
- Generalization of Davenport–Schinzel seq's

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Klazar '00

- Füredi–Hajnal Conj. $\Rightarrow$ Stanley–Wilf Conj.

$\boxed{\text{Conj.}}$ (Stanley & Wilf)

$\forall$ permutation π of $[k]$: $\exists$ a constant c s.t.

"# of perm's of $[n]$ avoiding π" $\le c^n$.

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$\boxed{\text{Cor.}}$

The Stanley–Wilf conjecture is true.

$\boxed{\text{Conj.}}$ (Stanley & Wilf)

$\forall$ permutation π of $[k]$: $\exists$ a constant c s.t.

$$\lim_{n\to\infty} \left(\text{“\# of perm's of } [n] \text{ avoiding } \pi\text{”}\right)^{1/n} = c.$$

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Cor. The conj. is true with $c \le 15^{2k^4\binom{k^2}{k}}$.

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Cor. The conj. is true with $c \leq 15^{2k^4\binom{k^2}{k}}$.

Another Conj. $c \leq (k-1)^2$.

(Arratia '99, still open)

Some more consequences

- Affirmative answer to a conj. of Alon–Friedgut ('00) on the max. length of sparse word avoiding a permutation
- Characterization of $0/1$-matrices P which have at most exponentially many $n \times n$ $0/1$-matrices avoiding P

Some more open problems (1)

$\boxed{\text{Problem}}$ (Füredi & Hajnal '92)

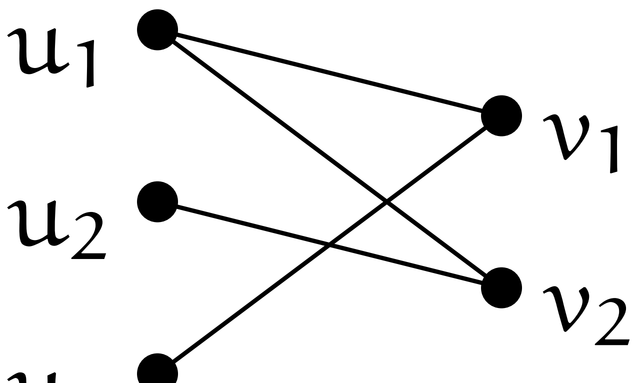
Characterize the matrices P with $f(n, P) = O(n)$.

$\boxed{\text{Problem}}$ (Marcus & Tardos)

Find minimally nonlinear patterns P with more than four 1's.

Problem (Füredi & Hajnal '92)

G a bipartite graph
P_G the 0/1-matrix corresponding to G



G

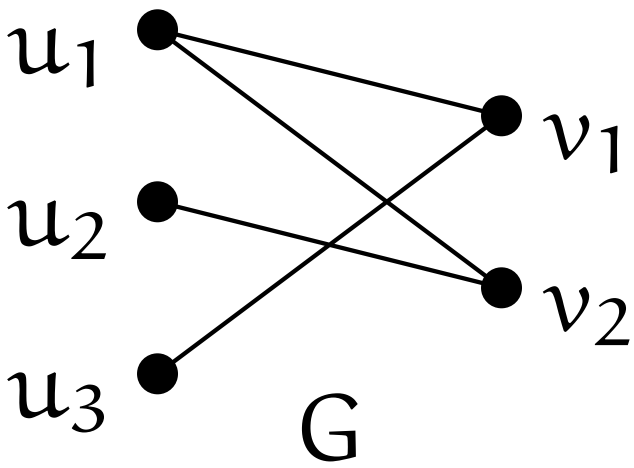
$$P_G = \begin{array}{cc} \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{array}} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \\ \begin{array}{cc} v_1 & v_2 \end{array} & \end{array}$$

$\boxed{\text{Problem}}$ (Füredi & Hajnal '92)

G a bipartite graph
P_G the 0/1-matrix corresponding to G

$$f(n, P) = O(\mathrm{Turan}(n, G) \log n)??$$

$\mathrm{Turan}(n, G) =$ max. # of edges in an n-vert. graph avoiding G



$$\begin{array}{ccc} P_G = & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{matrix}} & \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \\ & \begin{matrix} v_1 & v_2 \end{matrix} & \end{array}$$

Conclusion

- The Füredi–Hajnal Conj. has been solved.
- The Stanley–Wilf Conj. has been solved.
- Still problems are remaining.
- New progress on
  - Extremal 0/1-matrix theory (Füredi & Hajnal '92, Tardos)
  - Extremal ordered graph theory (Klazar '04, Brass, Károlyi & Valtr '03).

[End of the talk]