The affine representation theorem for abstract convex geometries

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Combinatorics
Geometry
Computation

- **Matroids** .... abstraction of dependence

  Application: $\begin{cases} \text{Finite geometry} \\ \text{Coding theory} \\ \text{Combinatorial optimization} \end{cases}$

- **Oriented Matroids** .... abstraction of dependence

  Application: $\begin{cases} \text{Convex polytopes} \\ \text{Computational geometry} \\ \text{Discrete geometry} \\ \text{Optimization} \end{cases}$

- **Convex geometries** .... abstraction of convexity

  Application: $\begin{cases} \text{Discrete geometry} \\ \text{Social choice theory} \\ \text{Mathematical psychology} \end{cases}$

- **Matroids** . . . . abstraction of dependence

  Every matroid can be represented
  as a homotopy-sphere arrangement.
  (Swartz, '03)

- **Oriented Matroids** . . . . abstraction of dependence

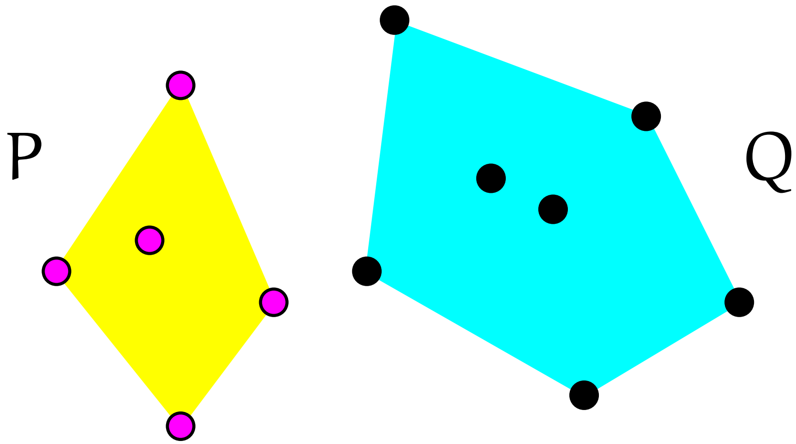
  Every oriented matroid can be represented
  as a pseudohyperplane arrangement.
  (Forkman–Lawrence, '78)

- **Convex geometries** . . . . abstraction of convexity

  ??????????

Our Theorem:

Every convex geometry is isomorphic to some generalized convex shelling,



determined by two point sets P and Q satisfying that $\mathrm{conv}(\mathrm{P}) \cap \mathrm{conv}(\mathrm{Q}) = \emptyset$.

This gives an affine representation of a convex geometry.

Our Theorem:

Every convex geometry is isomorphic to some generalized convex shelling.

In the rest of my talk

- Definition of a convex geometry
- Examples of a convex geometry
- Definition of a generalized convex shelling
- Our theorem
- Outline of the proof

(Edelman–Jamison '85)

E a nonempty finite set

$\mathcal{L}$ a nonempty family of subsets of E

Def. $\mathcal{L} \subseteq 2^E$ is called a convex geometry on E if $\mathcal{L}$ satisfies the following three conditions.

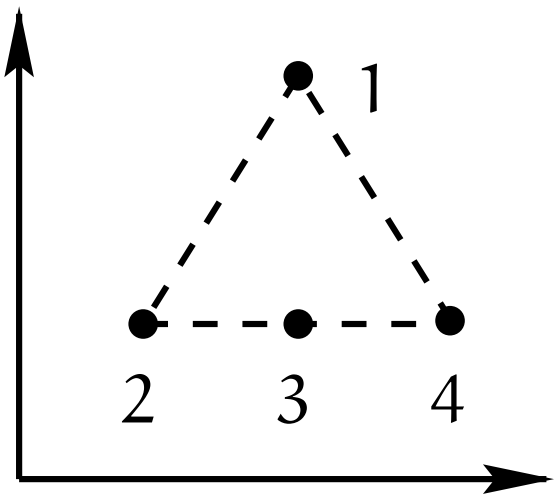
(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$.

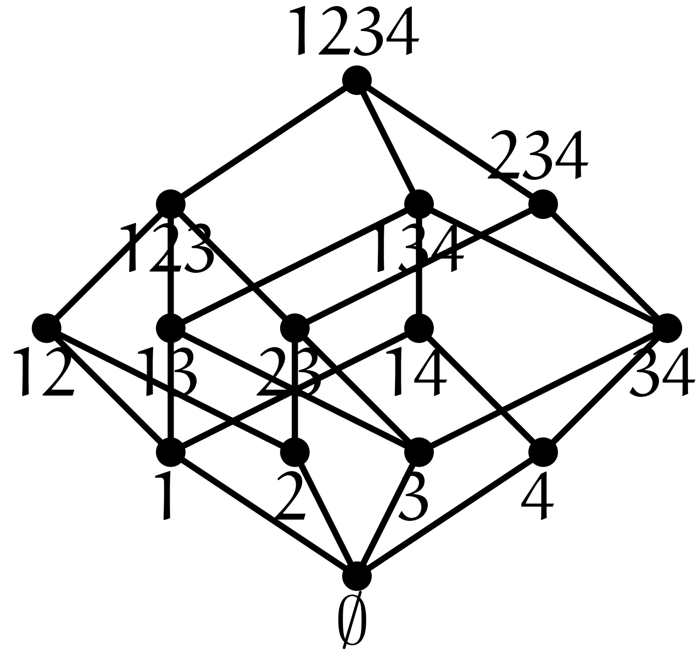
(2) $X, Y \in \mathcal{L} \Longrightarrow X \cap Y \in \mathcal{L}$.

(3) $X \in \mathcal{L} \setminus \{E\} \Longrightarrow \exists\ e \in E \setminus X$ s.t. $X \cup \{e\} \in \mathcal{L}$.

Q a finite point set in $\mathbb{R}^d$

Define: $\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X) \cap (Q \setminus X) = \emptyset\}$.

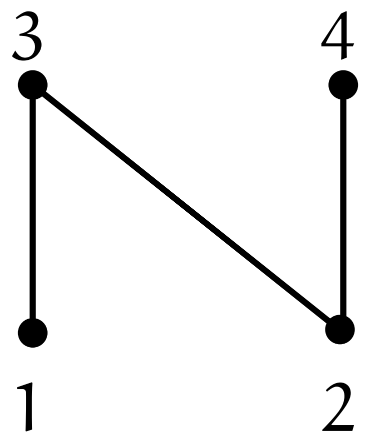


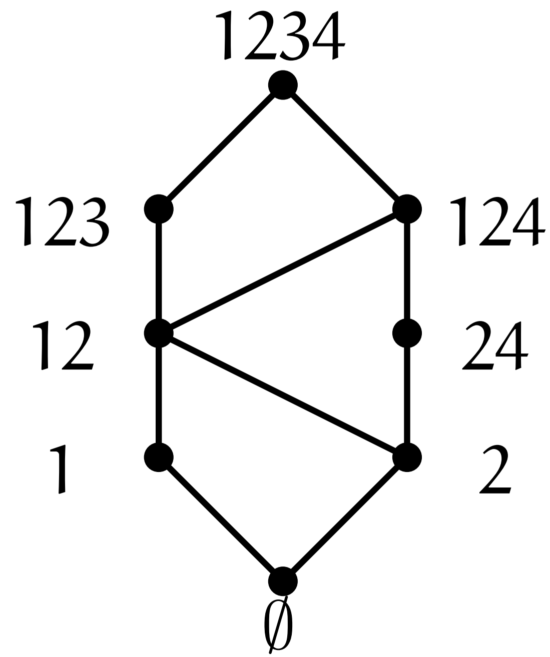


$\mathcal{L}$ is a convex geometry and called the convex shelling on Q.

$\mathcal{P} = (E, \leq)$ a partially ordered set

Define: $\mathcal{L} = \{X \subseteq E : e \in X, f \leq e \Rightarrow f \in X\}$.



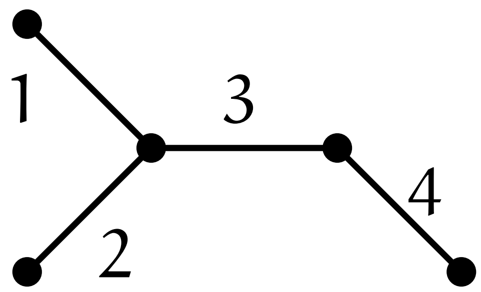


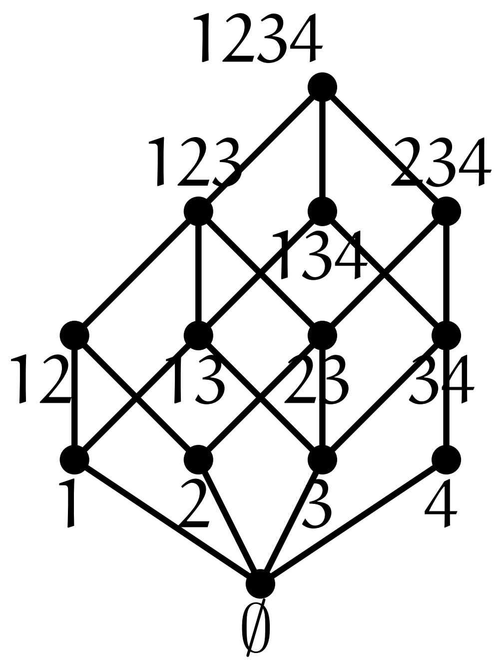
$\mathcal{L}$ is a convex geometry on E and called the poset shelling of $\mathcal{P}$.

$T = (V, E)$ a tree

Define:

$$\mathcal{L} = \{X \subseteq E : X \text{ forms a subtree of } T\}.$$

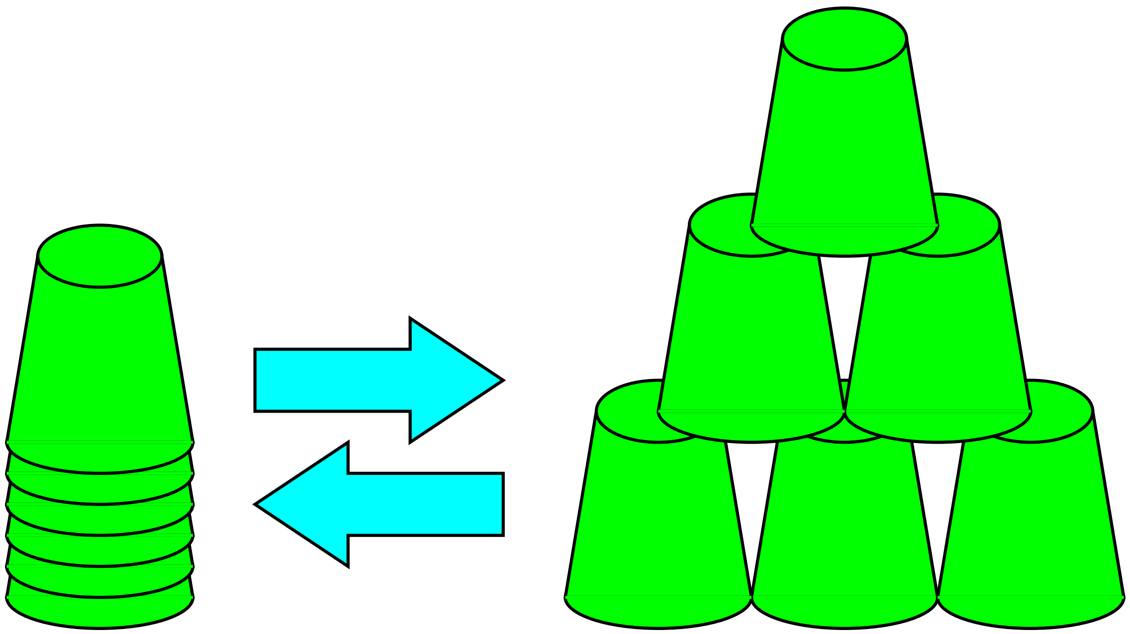




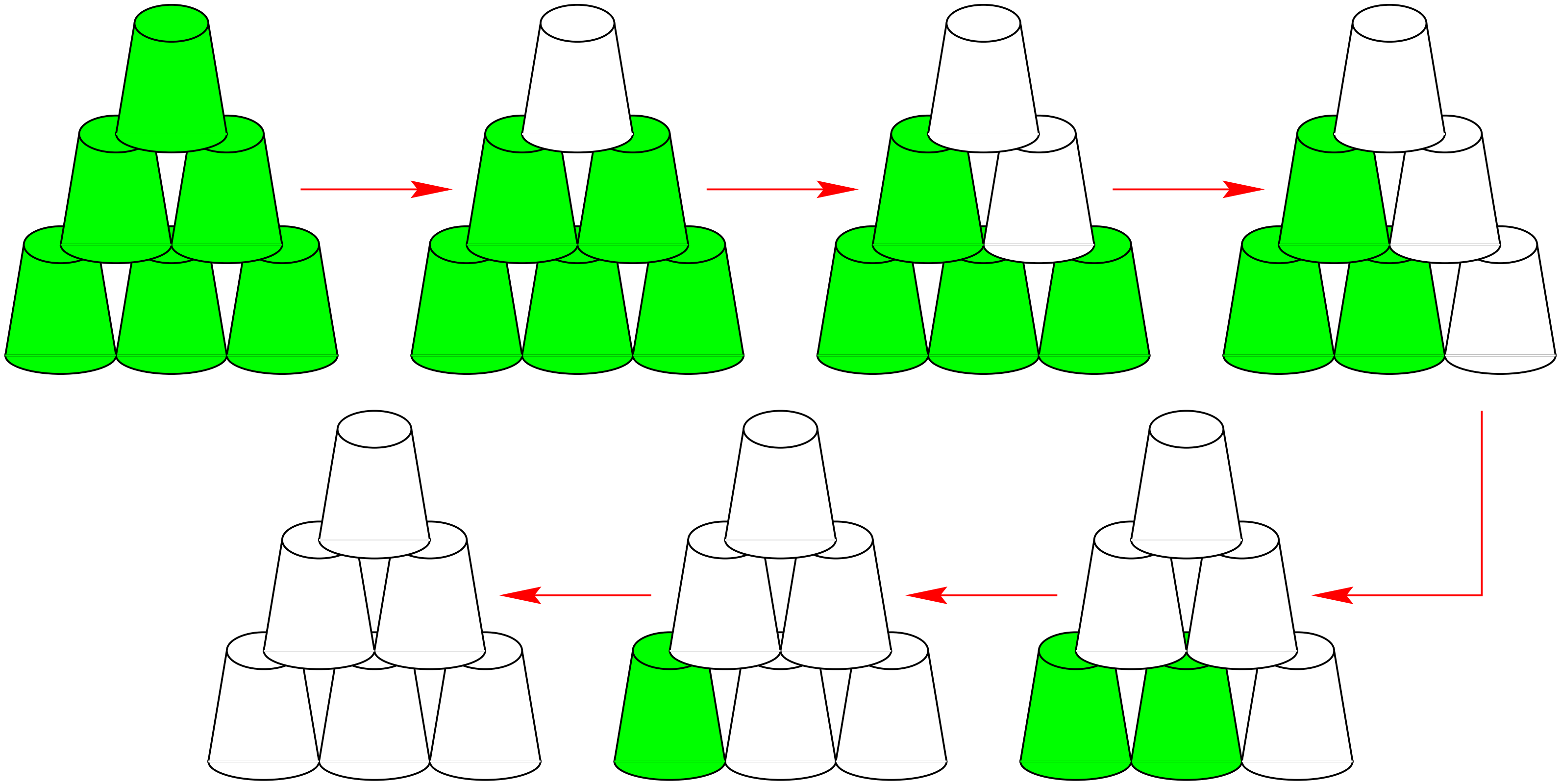
$\mathcal{L}$ is a convex geometry on E and called the tree shelling of T

What is "cupstacks"?

Construct the tower from the pile and get it back as quickly as possible.

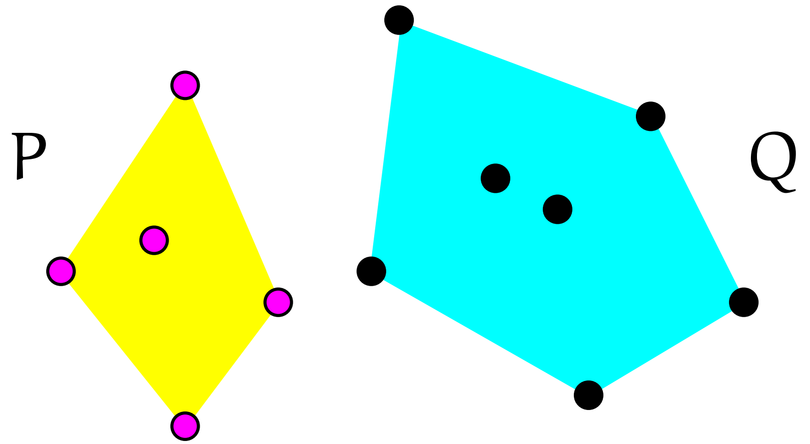


A sequence in collapsing



Our Theorem:

Every convex geometry is isomorphic to some generalized convex shelling,

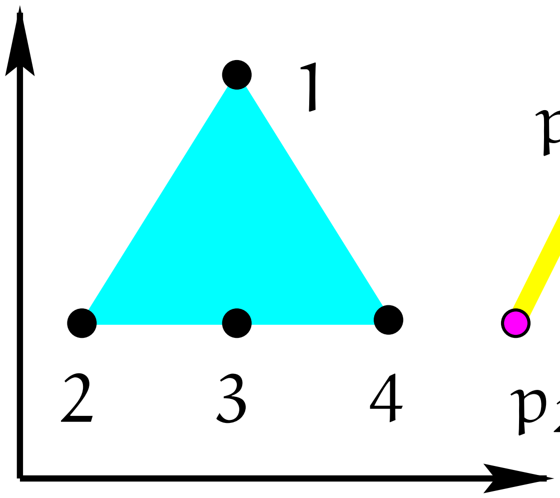


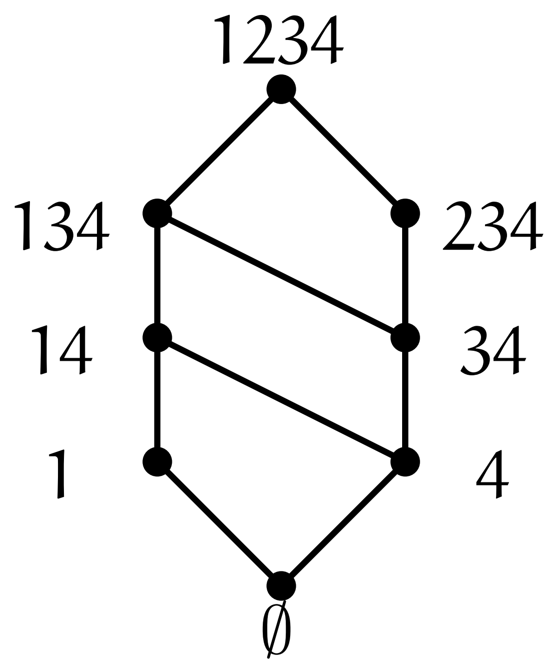
determined by two point sets P and Q satisfying that $\mathrm{conv}(\mathrm{P}) \cap \mathrm{conv}(\mathrm{Q}) = \emptyset$.

This gives an affine representation of a convex geometry.

P,Q finite point sets in $\mathbb{R}^d$ satisfying $\mathrm{conv}(P) \cap Q = \emptyset$

Define: $\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X \cup P) \cap (Q \setminus X) = \emptyset\}$.





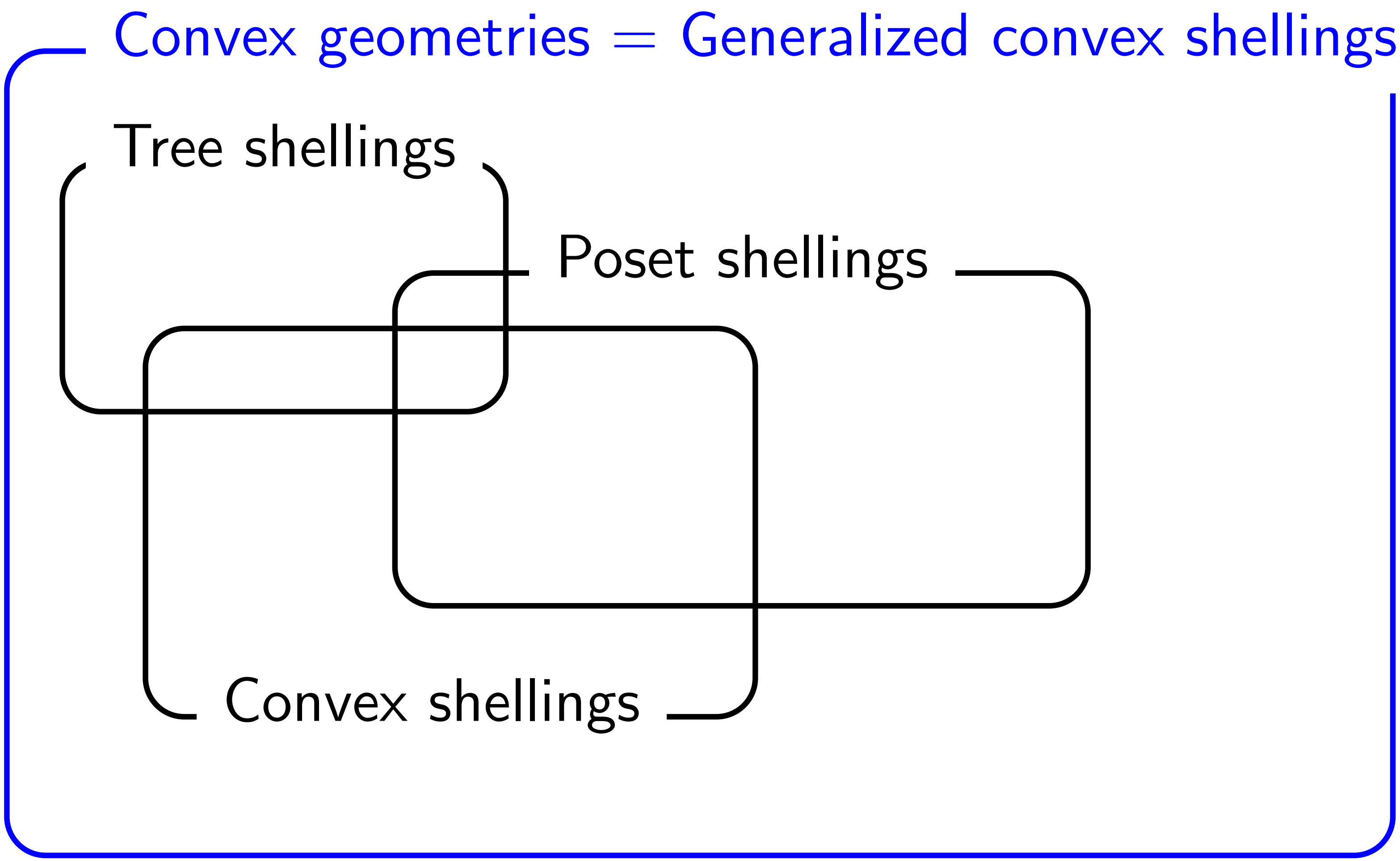
$\mathcal{L}$ is a convex geometry on Q and called
the generalized convex shelling on Q with respect to P.

$\boxed{\text{Our Theorem}}$

Every convex geometry is isomorphic to
some generalized convex shelling.

In other words,

For any convex geometry $\mathcal{L}$,
there exist finite point sets P and Q such that
$\mathcal{L}$ is isomorphic to
the generalized convex shelling on Q w.r.t. P.


Convex geometries = Generalized convex shellings
Tree shellings
Poset shellings
Convex shellings

What does the theorem mean? (2)

- For oriented matroids and matroids, we have

  Topological representation theorems.

- For convex geometries, we have

  Affine representation theorem.

$\Longrightarrow$ An intrinsic simplicity of convex geometries

The proof goes along the following line.

We are given a convex geometry $\mathcal{L}$.

(1) Construct:
point sets P and Q from $\mathcal{L}$.

(2) Show:
$\mathcal{L} \cong$ the generalized convex shelling on Q w.r.t. P.

To illustrate the proof, we will show a much weaker version.

What we will show

For any poset shelling $\mathcal{L}$
there exist point sets P and Q such that
$\mathcal{L}$ is isomorphic to
the generalized convex shelling on Q w.r.t. P.

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

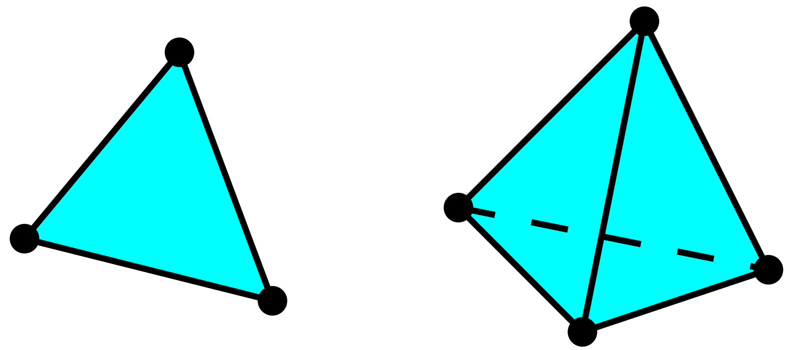
$\boxed{\text{Construction of } Q}$

We use the $(n-1)$-dimensional space $\mathbb{R}^{n-1}$.

For each $e \in E$, put a point $\mathbf{q}(e)$ such that

$$\{\mathbf{q}(e) : e \in E\} \text{ is affinely independent,}$$

$$(\mathrm{conv}(\{\mathbf{q}(e) : e \in E\}) \text{ is an } (n-1)\text{-simplex}).$$



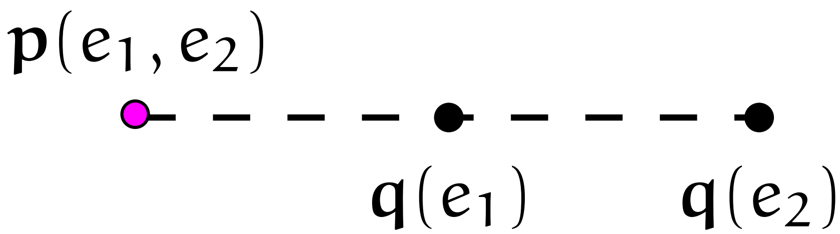
Let $Q = \{\mathbf{q}(e) : e \in E\}$.

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

$\boxed{\text{Construction of } P}$

For each $e_1, e_2 \in E$ such that $e_1 < e_2$,

Put a point $\mathbf{p}(e_1, e_2)$ such that $\mathbf{q}(e_1) = \dfrac{\mathbf{p}(e_1, e_2) + \mathbf{q}(e_2)}{2}$.



Let $P = \{\mathbf{p}(e_1, e_2) : e_1, e_2 \in E, e_1 < e_2\}$.

Claim the poset shelling of $\mathcal{P} = (\mathrm{E}, \leq)$

$\simeq$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in \mathrm{E}$ allowed to be removed?

e is allowed to be removed $\iff$ all f's such that $\boxed{e < \mathrm{f}}$ have been already removed.

(2) When is $\mathbf{q}(e) \in \mathrm{Q}$ allowed to be removed?

$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(\mathrm{f})$'s s.t. $\boxed{\mathbf{p}(\mathrm{e},\mathrm{f}) \in \mathrm{P}}$ have been already removed.

(3) $\boxed{\mathrm{e} < \mathrm{f}} \iff \boxed{\mathbf{p}(\mathrm{e},\mathrm{f}) \in \mathrm{P}}$.

What was our theorem??

$\boxed{\text{Our Theorem}}$

Every convex geometry is isomorphic to
some generalized convex shelling.

This theorem is expected to be useful
for a lot of problems in convex geometries.

$\Longrightarrow$ Opens a new research direction!

Based on our theorem...

- Hachimori & Nakamura
  - Consider a certain clutter associated with a convex geometry
  - Characterized the 2-dim. generalized convex shellings with MFMC clutters.
- Okamoto
  - Study the local topology of a certain simplicial complex associated with a convex geometry (conjectured by Edelman & Reiner '00)
  - Solved the conjecture for 2-dim. generalized convex shellings.