Traveling salesman games with the Monge property*

Yoshio Okamoto†‡

Abstract

Several works have indicated the relationships between polynomially solvable combinatorial optimization problems and the core non-emptiness of cooperative games associated with them, and between intractable combinatorial optimization problems and the hardness of the problem to decide the core non-emptiness of the associated games. In this paper, we study the core of a traveling salesman game, which is associated with the traveling salesman problem. First, we show that in general the problem to test the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard. This corresponds to the $\mathcal{NP}$-hardness of the traveling salesman problem. Second, we show that the core of a traveling salesman game is non-empty if the distance matrix is a symmetric Monge matrix, and also that a traveling salesman game is submodular (or concave) if the distance matrix is a Kalmanson matrix. These correspond to the fact that the Monge property and the Kalmanson property are polynomially solvable special cases of the traveling salesman problem.

Key words. Cooperative game; Core; Traveling salesman; Polynomially solvable case

1 Introduction

1.1 Background — combinatorial optimization games

Several works have indicated the relationship between polynomially solvable combinatorial optimization problems and the core non-emptiness of cooperative games associated with them. Even the first example in the history of combinatorial optimization games, due to Shapley–Shubik [62], fitted into this framework. They introduced the assignment games [62], which are derived from the assignment problem (or the maximum weighted matching problem on bipartite graphs), and showed that the core of an assignment game is always non-empty. Corresponding to this, the assignment problem can be solved in polynomial time. The first polynomial time algorithm is due to Kuhn [44] (which is called the Hungarian method) and other algorithms can be found in a textbook of combinatorial optimization (like Korte–Vygen [42]). Another early example is a minimum cost spanning tree game by Bird [3], which is based on the minimum cost spanning tree problem. It was shown that every minimum cost spanning tree game has a non-empty core, by constructing an explicit vector

*A preliminary extended abstract version of this paper has appeared in Proceedings of the 6th Korea-Japan Workshop on Algorithms and Computation, 2001, 75–82. This work was mainly done while the author was at Department of Systems Science, Graduate School of Arts and Sciences, The University of Tokyo.

†Institute of Theoretical Computer Science, Department of Computer Science, ETH Zürich, CH-8092 Zürich, Switzerland

‡Supported by the Berlin/Zürich Joint Graduate Program “Combinatorics, Geometry, and Computation” (CGC), financed by ETH Zürich and the German Science Foundation (DFG). *E-mail address:* `okamotoy@inf.ethz.ch`

belonging to the core [3, 29]. Again, corresponding to this, the minimum cost spanning tree problem can be solved in polynomial time by the algorithms, for example, due to Borůska [51], Kruskal [43] and Prim [59]. (A textbook of combinatorial optimization like Korte–Vygen [42] provides a further account.)

In the proof of the core non-emptiness of an assignment game by Shapley–Shubik [62], it was a key observation that the linear programming relaxation of the ordinary integer programming formulation of the assignment problem always has an integral optimal solution. In fact, the core of an assignment game is characterized as the set of optimal solutions of the dual of the linear programming relaxation. Extending this result, Deng–Ibaraki–Nagamochi [12] gave a necessary and sufficient condition for maximum packing games and minimum covering games to have non-empty cores. It says that the linear programming relaxation of a maximum packing problem (and a minimum covering problem) has an integral optimal solution if and only if the associated game has a non-empty core, and if so the core is characterized by the set of optimal solutions of the dual of the linear programming relaxation. It gives rise to good characterizations for the core non-emptiness of some combinatorial optimization games such as maximum matching games, minimum vertex cover games, maximum independent set games, etc. Similar results based on linear programming duality were shown for other kinds of games as well. For facility location games, Kolen [40] showed the core of an uncapacitated facility location game is non-empty if and only if the linear programming relaxation has an integral optimal solution (see also [41]). Chardaire [6] generalized Kolen's results to some sorts of capacitated facility location games, and Goemans–Skutella [26] characterized the more generalized facility location games with non-empty cores, including the results by Kolen [40] and Chardaire [6]. (Note that a facility location game studied by these papers [6, 26, 40] is different from a location game of Tamir [66] and of Curiel [8].) For partition games, Faigle–Kern [18] proved that the core is non-empty if and only if the linear programming relaxation of the corresponding partition problem has an integral optimal solution. Moreover, Granot–Hamers–Tijs [28] investigated the core non-emptiness of delivery games in relation with the structures of the underlying graphs. In the books by Bilbao [2] and by Curiel [8], we can find a lot of results and properties of cooperative games associated with combinatorial optimization problems, which we may call combinatorial optimization games.

On the other hand, some papers have indicated the relationship between intractable combinatorial optimization problems and the hardness of the problem to test the core non-emptiness of cooperative games associated with them. For example, Deng–Ibaraki–Nagamochi [12] showed that testing the core non-emptiness of a minimum coloring game is $\mathcal{NP}$-complete; Matsui [48] showed that testing the core non-emptiness of a bin packing game is $\mathcal{NP}$-complete; Goemans–Skutella [26] showed that testing the core non-emptiness of a facility location game is $\mathcal{NP}$-complete. Notice that Deng–Ibaraki–Nagamochi [12] showed that for a minimum vertex cover game and a maximum independent set game, we can test the core non-emptiness in polynomial time, while the minimum vertex cover problem and the maximum independent set problem are known to be $\mathcal{NP}$-hard. Therefore, it is not always the case that, for a class of cooperative games arising from an $\mathcal{NP}$-hard optimization problem, testing the core non-emptiness is hard. Note that Deng–Papadimitriou [13] also discussed cooperative games from the computational (or algorithmic) point of view, not only for the cores but also for other kinds of solution concepts.

1.2 Traveling salesman games

In this paper, we will study traveling salesman games, introduced by Potters–Curiel–Tijs [58] from the viewpoint of Section 1.1. In the literature, some of the properties of traveling salesman games were discussed. Tamir [65] showed that a metric traveling salesman game with at most four players always has a non-empty core, and that there exists a metric traveling salesman game with six players such that the core is empty. Furthermore, Faigle–Fekete–Hochstättler–Kern [16] provided an instance of a traveling salesman game in the 2-dimensional Euclidean space with six players such that the core is empty. On the other hand, Kuipers [45] showed that a metric traveling salesman game with five players always has a non-empty core. Also, Potters–Curiel–Tijs [58] gave an example of an asymmetric traveling game with four players which has an empty core, and provided some conditions for an asymmetric traveling salesman game to have a non-empty core. In other papers [8, 65], we can find other conditions for a traveling salesman game to have a non-empty core. On the other hand, approximation of the core of a traveling salesman game was discussed by Faigle–Fekete–Hochstättler–Kern [16] and Faigle–Kern [17].

Stimulated by a sort of the vehicle routing problems, Herer [31] initiated the study of the underlying graph structure which always yields submodular traveling salesman games. Such graphs are called naturally submodular. (The more precise definition will be given in Section 6.5.5.) Herer–Penn [32] characterized undirected graphs which are naturally submodular, and Granot–Granot–Zhu [27] characterized directed graphs and bidirected graphs which are naturally submodular. In fact, submodularity, which is also called concavity, is an important concept in cooperative game theory. First of all, submodularity implies core non-emptiness [61]. In addition, submodularity has other good properties: for example, the Shapley value is the barycenter of the core [61]; the core is a unique von Neumann–Morgenstern solution [61]; the bargaining set coincides with the core and the kernel coincides with the nucleolus [47]; the τ-value can be calculated in polynomial time [67]; the nucleolus can be calculated in polynomial time [19, 46]. Besides, submodularity plays an important role in the fields of network flows and combinatorial optimization. Fujishige [23] provides a survey of submodular-type optimization problems, and Murota [50] gives a further account on this topic.

1.3 Contributions of this paper

In this paper, we will show that in general the problem to test the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard. This corresponds to the $\mathcal{NP}$-hardness of the traveling salesman problem. Next, we will provide some conditions for a symmetric traveling salesman game to have a non-empty core, which are related to polynomially solvable cases of the traveling salesman problems. First, we will show that the core of a traveling salesman game is always non-empty if the distance matrix is a symmetric Monge matrix. It is known that if the distance matrix is an (asymmetric) Monge matrix then the traveling salesman problem can be solved in linear time in the number of cities [55] while in general the traveling salesman problem is $\mathcal{NP}$-hard already for the 2-dimensional Euclidean case [54]. Second, we will show a traveling salesman game is always submodular if the distance matrix is a Kalmanson matrix. The Kalmanson property also yields a polynomially solvable special class of the traveling salesman problem [38].

It is known that Monge matrices are related with polynomially solvable cases for combinatorial optimization problems other than the traveling salesman problem, which was surveyed by Burkard–Klinz–Rudolf [5]. On the other hand, there are many other polynomially solvable cases of traveling salesman problems. They were surveyed in [4, 25], and some recent results appeared in [1, 14, 35, 36, 37, 52], etc.

1.4 Organization

This paper is organized as follows. The next section is devoted to some definitions and basic properties, and our results are described again in a more formal manner. In Sections 3–5, we will provide the proofs of our results. Concluding remarks are provided in the final section.

2 Definitions and results

Let $N_0 = \{0, 1, \ldots, n\}$ and an $N_0 \times N_0$-matrix D be given. We treat N_0 as a set of cities and D as the distance matrix. We assume that the diagonal components of D are all zero and the non-diagonal components of D are all positive, and in this paper we will call a matrix D a *distance matrix* if D satisfies these assumptions. The (i, j)-component of D is denoted by $d[i, j]$. Note that possibly a distance matrix D does not satisfy the triangle inequality: $d[i, j] + d[j, k] \geq d[i, k]$ for all $i, j, k \in N_0$. The *traveling salesman problem* (TSP, for short) is the problem to find a shortest tour around the cities in N_0 with respect to a given distance matrix D; more formally, to find a tour τ on N_0 to

$$\text{minimize} \sum_{i=0}^{n} d[i, \tau(i)], \tag{1}$$

where $\tau(i)$ denotes the successor of i. We denote the k-th successor and the k-th predecessor of $i \in N_0$ in the tour τ by $\tau^k(i)$ and $\tau^{-k}(i)$, respectively. A tour τ is sometimes denoted by $\tau = \langle 0, \tau(0), \tau^2(0), \ldots, \tau^n(0) \rangle$.

Let $N = N_0 \setminus \{0\} = \{1, 2, \ldots, n\}$ and D be an $N_0 \times N_0$ distance matrix. We define a function $c_D : 2^N \to \mathbb{R}$ as follows. For $S \subseteq N$, $c_D(S)$ is the total distance of a shortest tour around the cities in $S \cup \{0\}$ with respect to D. Notice that $c_D(\emptyset) = 0$. We call the pair (N, c_D) a *traveling salesman game*. In terminology of cooperative game theory, N is called the set of *players* and c_D is the *characteristic function*. In this paper, when we consider traveling salesman games, we always assume that the distance matrix D is symmetric, i.e., $d[i, j] = d[j, i]$ for all $i, j \in N_0$, unless stated otherwise. A traveling salesman game arises from the following cost allocation problem, originated in Fishburn–Pollak [21]. A professor has visited, starting from his home institute 0, the universities $1, 2, \ldots, n$ which invited him and after the visits he has returned to his home. The total cost of the trip should be paid by the inviting universities. The problem is to find a "fair" rule for the allocation of the total cost among these universities. Stimulated by this example, we sometimes say that the city $0 \in N_0$ (which is not a player) is the *home*.

In cooperative game theory, a core is frequently used as a fair allocation rule. For a traveling salesman game (N, c_D), the *core* is defined as

$$\mathrm{Core}(N, c_D) = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} x(N) = c_D(N) \text{ and} \\ x(S) \leq c_D(S) \text{ for all } S \subseteq N \end{array} \right\}, \tag{2}$$

where we use a convention that $x(S) = \sum_{i \in S} x[i]$ (and $x(\emptyset) = 0$) for a vector $x \in \mathbb{R}^N$. See the chapters [39, 56] for some properties of the core of a cooperative game.

As we can see from the definition, cores can be empty for some cases. This means that in such cases we cannot find a fair allocation by this rule. Hence, it is important to test the core non-emptiness of a given traveling salesman game. Our first theorem is a solution to this algorithmic problem. Here is the theorem.

Theorem 1. *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. Then, the problem to test the core non-emptiness of the traveling salesman game (N, c_D) is $\mathcal{NP}$-hard.*

This theorem implies that it is almost impossible to have a good characterization of the core non-emptiness of a traveling salesman game in terms of distance matrices. So we may only hope to determine some classes of matrices which give rise to non-empty cores. The class of Monge matrices is one of such classes.

The Monge property is known as a polynomially solvable special case of TSP, while TSP is $\mathcal{NP}$-hard even if N_0 is the set of points on the 2-dimensional plane and each entry

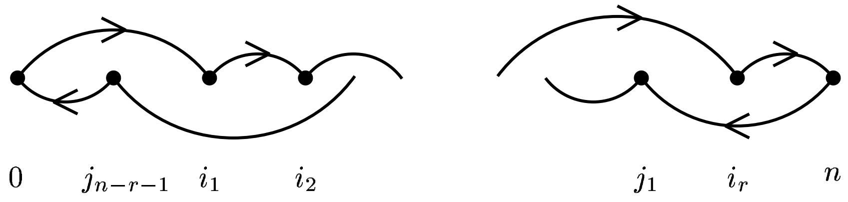

Figure 1: A pyramidal tour.

of D is defined as the Euclidean distance between the corresponding two points [54]. An $N_0 \times N_0$ matrix D is a *Monge matrix* if D satisfies

$$d[i,k] + d[j,l] \leq d[i,l] + d[j,k] \tag{3}$$

for all $i < j$ and $k < l$. If a matrix D is a Monge matrix, then it is also said to have the *Monge property*. Note that a Monge matrix does not need to satisfy the triangle inequality. As we can observe from the definition of a Monge matrix, the Monge property is dependent on the order of the indices of a given matrix. To resolve this dependency, we will use a permuted Monge matrix. An $N_0 \times N_0$ matrix D is called a *permuted Monge matrix* if there exists a permutation σ on the indices N_0 such that the matrix whose (i,j)-component is $d[\sigma(i), \sigma(j)]$ has the Monge property. Note that we can determine that a given matrix is a permuted Monge matrix and if so we can find such a permutation which results in a Monge matrix in $\mathrm{O}(n^2)$ [9]. See also [5].

It is well-known that Monge matrices are related to some polynomially solvable combinatorial optimization problems [5]. The next proposition is a basic result on Monge matrices and TSP. A tour $\langle 0, i_1, i_2, \ldots, i_r, n, j_1, j_2, \ldots, j_{n-r-1} \rangle$ on N_0 is called a *pyramidal* tour if $i_1 < i_2 < \cdots < i_r$ and $j_1 > j_2 > \cdots > j_{n-r-1}$. Fig. 1 is an illustration of a pyramidal tour. As we can see, in a pyramidal tour the cities $i_1, \ldots, i_r$ are visited on the way from the home 0 to n in a monotone manner and the cities $j_1, \ldots, j_{n-r-1}$ are visited on the way back from n to 0 also in a monotone manner.

Proposition 2 (Gilmore et al. [25]). *Consider a traveling salesman problem on the cities N_0 with a distance matrix D. If D is a Monge matrix, then there exists a shortest tour which is pyramidal.*

Generally, a shortest pyramidal tour can be found in $\mathrm{O}(n^2)$ by the dynamic programming technique [25]. Moreover, Park [55] showed that a shortest pyramidal tour for TSP with a Monge distance matrix can be found in $\mathrm{O}(n)$ time invoking the structure of a Monge matrix. Therefore, TSP with a permuted Monge distance matrix can be solved in polynomial time.[1]

Furthermore, for symmetric Monge matrices, Supnick [64] showed the following proposition. For this case, even the concrete "shape" of a shortest tour can be determined. See also [4].

Proposition 3 (Supnick [64]). *Consider a traveling salesman problem on the cities N_0 with a distance matrix D. If D is a symmetric Monge matrix, then the tour $\langle 0, 2, 4, \ldots, n, \ldots, 5, 3, 1 \rangle$ is a shortest tour.*

Proposition 3 will be useful for proving the next theorem. This theorem relates the Monge property with the core non-emptiness of a traveling salesman game.

[1] Note that some past works like [4] say that TSP with a permuted Monge distance matrix is $\mathcal{NP}$-hard. However, our definition of permuted Monge distance matrix is different from that in Section 4 of [4]. So there is no contradiction with what we have just described. Actually, in Section 4 of [4] a permuted Monge matrix can have different permutations on rows and columns, while our permuted Monge matrix must have the same permutation on rows and columns. You should not be misled; in Section 2 of [4] a permuted Monge matrix is defined as the same as ours.

Theorem 4. *Consider a traveling salesman problem on the cities $N_0 = \{0, 1, \ldots, n\}$ with a distance matrix D. If D is a symmetric permuted Monge matrix, then the core of the traveling salesman game (N, c_D) is non-empty. Furthermore, an element in the core can be found in $\mathrm{O}(n^2)$.*

For a traveling salesman game (N, c_D) and $T \subseteq N$, we define the *subgame* $(T, c_D^{(T)})$ as $c_D^{(T)}(S) = c_D(S)$ for all $S \subseteq T$. Observe that if the distance matrix is a symmetric Monge matrix, then every subgame of a traveling salesman game also has a non-empty core. That is because every submatrix of a Monge matrix is also a Monge matrix and because of Theorem 4. In cooperative game theory, a game with the property that every subgame has a non-empty core is called *totally balanced*. Hence, the discussion above immediately leads to the following corollary.

Corollary 5. *Consider a traveling salesman problem on the cities N_0 with a distance matrix D. If D is a symmetric permuted Monge matrix, then the traveling salesman game (N, c_D) is totally balanced.*

We have another property which yields a polynomially solvable case of TSP. That is the Kalmanson property. Let D be an $N_0 \times N_0$ matrix. We call D a *Kalmanson matrix* if D is symmetric and fulfills the following: for every $i < j < k < l$

$$d[i, j] + d[k, l] \leq d[i, k] + d[j, l], \tag{4}$$

$$d[i, l] + d[j, k] \leq d[i, k] + d[j, l]. \tag{5}$$

We also say that D has the *Kalmanson property* if D is a Kalmanson matrix. Note that the class of Kalmanson matrices and that of symmetric Monge matrices have no inclusion-relationship. The following proposition relates Kalmanson matrices to TSP.

Proposition 6 (Kalmanson [38]). *Consider a traveling salesman problem on the cities N_0 with a distance matrix D. If D is a Kalmanson matrix, then the tour $\langle 0, 1, 2, 3, \ldots, n\rangle$ is a shortest tour.*

We can define permuted Kalmanson matrices similarly to permuted Monge matrices. Namely, a matrix D is called a *permuted Kalmanson matrix* if there exists a permutation σ on the indices N_0 such that the matrix whose (i, j)-component is $d[\sigma(i), \sigma(j)]$ has the Kalmanson property. Note that we can determine that a given matrix is a permuted Kalmanson matrix and if so we can find a permutation which results in a Kalmanson matrix in $\mathrm{O}(n^2 \log n)$ [7, 10].

There is another important relation between TSP and the Kalmanson property: the master tour problem. A tour τ on N_0 is a *master tour* if, for any $T \subseteq N_0$, a shortest tour on T is obtained by removal of the cities not in T from τ. Note that, in particular, a master tour on N_0 itself is a shortest tour on N_0. For example, if the cities lie in a convex position on the plane and the distance is measured by the Euclidean metric, the tour along the boundary of the convex hull of the cities is a master tour. The *master tour problem* is the problem of deciding whether the cities have a master tour with respect to a given distance matrix. The next proposition states that a master tour exists if and only if the distance matrix is a permuted Kalmanson matrix.

Proposition 7 (Burkard et al. [4] Deĭneko et al. [10]). *Consider a traveling salesman problem on the cities N_0 with a symmetric distance matrix D. The tour $\langle 0, 1, 2, \ldots, n-1, n\rangle$ is a master tour if and only if D is a Kalmanson matrix.*

We will prove the next theorem by using Proposition 6. A traveling salesman game (N, c_D) is *submodular* or *concave* if $c_D(S) + c_D(T) \geq c_D(S \cup T) + c_D(S \cap T)$ for any $S, T \subseteq N$. Submodular games have a bunch of important properties (see Section 1.2).

Theorem 8. *Consider a traveling salesman problem on the cities N_0 with a distance matrix D. Then the traveling salesman game (N, c_D) is submodular if D is a permuted Kalmanson matrix.*

Note that submodularity implies total balancedness [61]. Potters [57] shows that if the cities have a master tour, the (asymmetric) traveling salesman game has a non-empty core and that for a special subcase the game is submodular. When we concentrate on the symmetric case, we can find that Theorem 8 is a stronger statement than the above argument by Potters [57] with help of Proposition 7.

In the subsequent sections, we will prove Theorems 1, 4 and 8. Some additional remarks will be provided in the final section.

3 Proof of Theorem 1

First we will state the problem more formally.

Problem. Core Non-emptiness of Traveling Salesman Games.

Instance. Cities N_0 and an $N_0 \times N_0$ symmetric distance matrix D.

Question. Is the core of the traveling salesman game (N, c_D) non-empty?

Theorem 1 states the $\mathcal{NP}$-hardness of Core Non-emptiness of Traveling Salesman Games. To prove that, we use Hamiltonian Path, a famous $\mathcal{NP}$-complete problem.

Problem. Hamiltonian Path.

Instance. A graph $G = (V, E)$.

Question. Does G have a Hamiltonian path, i.e., a path which visits each vertex exactly once (i.e., a simple spanning path)?

Proof of Theorem 1. We reduce Hamiltonian Path to Core Non-emptiness of Traveling Salesman Games. For a given graph $G = (V, E)$, put $N = V = \{1, \ldots, n\}$ and $N_0 = N \cup \{0\}$. We define an $N_0 \times N_0$ symmetric distance matrix D as

$$d[i,j] = \begin{cases} 0 & (i = j), \\ n/(n+1) & (\{i,j\} \in E, \text{ or exactly one of } i \text{ and } j \text{ is } 0), \\ 2 & (\text{otherwise}). \end{cases}$$

Let (N, c_D) be a traveling salesman game derived from the cities N_0 and the distance matrix D. We now show that G has a Hamiltonian path if and only if the game (N, c_D) has a non-empty core, which completes the reduction.

First, assume that G has a Hamiltonian path. Then we have $c_D(N) = n$, $c_D(\emptyset) = 0$, and $c_D(S) \geq n(|S|+1)/(n+1)$ for all $S \in 2^N \setminus \{\emptyset, N\}$. Define $x = (x[1], \ldots, x[n])$ as $x[i] = 1$ for all $i \in N$. Since $x(N) = n = c_D(N)$ and $x(S) = |S| \leq n(|S|+1)/(n+1) \leq c_D(S)$ for all $S \in N \setminus \{\emptyset\}$, We can see that x belongs to the core.

Next, assume that G has no Hamiltonian path. Let

$$\tau = \langle 0, \tau(0), \tau^2(0), \ldots, \tau^n(0) \rangle$$

be a shortest tour on N_0. For the convenience, let us assume that $\tau^i(0) = i$ without loss of generality. We define a subgraph $\tilde{G} = (N, \tilde{E})$ of G as $\tilde{E} = E \cap \{\{i, i+1\} : i \in V\}$. Namely $\tilde{E}$ is the set of edges of G which appear in the tour τ. Let P be the vertices of a longest path in $\tilde{G}$ and l and m be the endpoints of P, where $l < m$. Since G has no Hamiltonian path and $\tilde{G}$ is a subgraph of G, $\tilde{G}$ has no Hamiltonian path either. This implies that $P \neq N$, namely $l \neq 1$ or $m \neq n$ holds. Also we have

$$c_D(N) = d[0,1] + \sum_{i=1}^{n-1} d[i,i+1] + d[n,0]$$

(clearly from the assumption that $\tau^i(0) = i$) and

$$c_D(P) = d[0,l] + \sum_{i=l}^{m-1} d[i,i+1] + d[m,0]$$

(since the distances involved in this expression are all $n/(n+1)$ by the choice of P).

Now, we claim the following.

Claim 9. *It holds that $c_D(P) < c_D(N) - c_D(N \setminus P)$.*

Proof of Claim 9. We distinguish two cases.

Case 1: Both $l \neq 1$ and $m \neq n$ hold. Fig. 2 illustrates the situation.

Since P is a longest path in $\tilde{G}$, we have $d[l-1,l] = d[m,m+1] = 2$. Consider a tour $\tau' = \langle 0, 1, \ldots, l-1, m+1, \ldots, n \rangle$ on $(N \setminus P) \cup \{0\}$. Then we have

$$\begin{aligned} c_D(N \setminus P) &\leq \text{the total distance of the tour } \tau' \\ &= d[0,1] + \sum_{i=1}^{l-2} d[i,i+1] \\ &\quad + d[l-1,m+1] + \sum_{i=m+1}^{n-1} d[i,i+1] + d[n,0] \\ &= \left(d[0,1] + \sum_{i=1}^{n-1} d[i,i+1] + d[n,0] \right) \\ &\quad - \left(d[0,l] + \sum_{i=l}^{m-1} d[i,i+1] + d[m,0] \right) \\ &\quad + d[0,l] + d[m,0] - d[l-1,l] - d[m,m+1] \\ &\quad + d[l-1,m+1] \\ &= c_D(N) - c_D(P) + 2\frac{n}{n+1} - 2 - 2 + d[l-1,m+1] \\ &\leq c_D(N) - c_D(P) + 2\frac{n}{n+1} - 2 - 2 + 2 \\ &< c_D(N) - c_D(P). \end{aligned}$$

Thus, we have $c_D(P) < c_D(N) - c_D(N \setminus P)$ as desired.

Case 2: Either $l = 1$ or $m = n$ holds, but not both.

Without loss of generality, we assume that $l = 1$ holds. (If not, we turn τ in the reverse order to make $l = 1$ hold.) Fig. 3 depicts the situation.

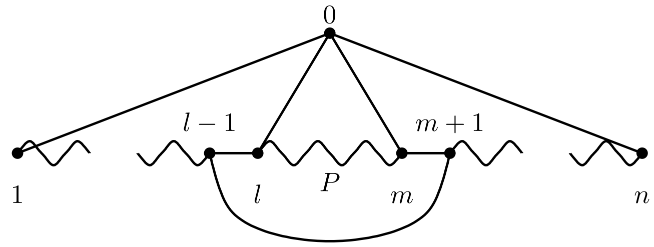

Figure 2: Case 1 in the proof of Claim 9

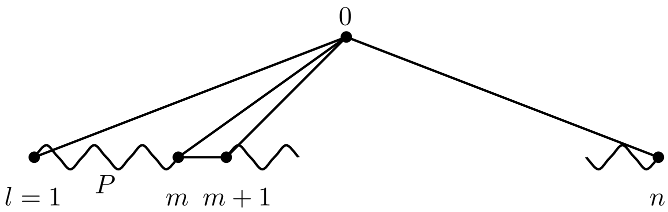

Figure 3: Case 2 in the proof of Claim 9

Since P is a longest path in $\tilde{G}$, we have $d[m, m+1] = 2$. Consider a tour $\tau' = \langle 0, m+1, m+2, \ldots, n\rangle$ on $(N \setminus P) \cup \{0\}$. Then we have

$$\begin{aligned}
c_D(N \setminus P) &\leq \text{the total distance of the tour } \tau' \\
&= d[0, m+1] + \sum_{i=m+1}^{n-1} d[i, i+1] + d[n, 0] \\
&= \left(d[0,1] + \sum_{i=1}^{n-1} d[i, i+1] + d[n, 0]\right) \\
&\quad - \left(d[0, l] + \sum_{i=l}^{m-1} d[i, i+1] + d[m, 0]\right) \\
&\quad + d[m, 0] + d[0, m+1] - d[m, m+1] \\
&= c_D(N) - c_D(P) + 2\frac{n}{n+1} - 2 \\
&< c_D(N) - c_D(P).
\end{aligned}$$

Thus, we have $c_D(P) < c_D(N) - c_D(N \setminus P)$ as expected.

In both cases, it holds that $c_D(P) < c_D(N) - c_D(N \setminus P)$. In this way, the claim has been proved. □

Let us go back to the proof of Theorem 1. Now, suppose that the core is non-empty, i.e., there exists $x \in \mathbb{R}^N$ such that $x(N) = c_D(N)$ and $x(S) \leq c_D(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Particularly, we have $x(P) \leq c_D(P)$ and $x(N \setminus P) \leq c_D(N \setminus P)$. Then we obtain

$$x(N) = x(P) + x(N \setminus P) \leq c_D(P) + c_D(N \setminus P) < c_D(N),$$

using Claim 9 at the last inequality. This contradicts the assumption that $x(N) = c_D(N)$. Hence the core is empty. Thus we have finished the reduction. □

4 Proof of Theorem 4.

Now we will give a proof of Theorem 4. In the proof, we will explicitly construct a vector belonging to the core, and this construction can be done in $\mathrm{O}(n^2)$.

Proof of Theorem 4. Let D be an $N_0 \times N_0$ permuted Monge matrix and assume that N_0 is renumbered by a permutation σ so that D is a Monge matrix. Here, let $h = \sigma^{-1}(0) \in N_0$ be the home. So the set of players is $N_0 \setminus \{h\}$. Set $N_0^h = N_0 \setminus \{h\}$ and consider the traveling salesman game (N_0^h, c_D).

We use the induction in terms of the number of players, i.e., the size of N_0^h.

Fix a linear order $\preceq$ on N_0^h. Define the *marginal contribution* $m_{\preceq}^{c_D}[i]$ of a player $i \in N_0^h$ with respect to a linear order $\preceq$ as

$$m_{\preceq}^{c_D}[i] = c_D(X_{\preceq}(i)) - c_D(X_{\preceq}(i) \setminus \{i\}),$$

where $X_{\preceq}(i) = \{j \in N_0^h : j \preceq i\}$. We treat $m_{\preceq}^{c_D}$ as a vector whose i-th component is $m_{\preceq}^{c_D}[i]$, and call it the *marginal contribution vector* with respect to $\preceq$. Note that $i \in X_{\preceq}(i)$ and $m_{\preceq}^{c_D}(N_0^h) = c_D(N_0^h)$ for any linear order $\preceq$. (Remember our convention that $m_{\preceq}^{c_D}(S) = \sum_{i \in S} m_{\preceq}^{c_D}[i]$ for $S \subseteq N_0^h$.)

Now we will construct the marginal contribution vector with respect to a suitable linear order and will show that this vector belongs to the core. For a Monge matrix, we can compute $c_D(S)$ for any $S \subseteq N_0^h$ in polynomial time, hence this gives rise to a polynomial-time algorithm to find a vector in the core. (The detail will be given at the end of the proof.)

Take a linear order $\preceq$ on N_0^h determined as $h-1 \preceq h-2 \preceq \cdots \preceq 1 \preceq 0 \preceq h+1 \preceq h+2 \preceq \cdots \preceq n$, and consider the marginal contribution vector $m_{\preceq}^{c_D}$ with respect to this order.

As the base case of our induction, we can easily verify that $m_{\preceq}^{c_D}$ belongs to the core when $n = 1$.

As the induction hypothesis, we assume that the marginal contribution vector $m_{\preceq}^{c_D}$ with respect to this order belongs to the core when $|N_0^h| < n$. Now we show that this vector belongs to the core when $|N_0^h| = n$.

Since $m_{\preceq}^{c_D}$ is a marginal contribution vector, we have $m_{\preceq}^{c_D}(N_0^h) = c_D(N_0^h)$. Hence it suffices to show that $m_{\preceq}^{c_D}(S) \le c_D(S)$ for every $S \subseteq N_0^h$.

We distinguish two cases.

Case 1: $h = n$.

From the induction hypothesis, we have $m_{\preceq}^{c_D}(S) \le c_D(S)$ for $S \subseteq N_0^h \setminus \{0\}$. So it suffices to show that $m_{\preceq}^{c_D}(S \cup \{0\}) \le c_D(S \cup \{0\})$ for $S \subseteq N_0^h \setminus \{0\}$.

Here, from the induction hypothesis and the definition of the marginal contribution, we have $m_{\preceq}^{c_D}(S \cup \{0\}) = m_{\preceq}^{c_D}(S) + m_{\preceq}^{c_D}[0] \le c_D(S) + m_{\preceq}^{c_D}[0] = c_D(S) + c_D(N_0^h) - c_D(N_0^h \setminus \{0\})$. Moreover, by Proposition 3 and the appropriate choice of pyramidal tours, we have $c_D(N_0^h) - c_D(N_0^h \setminus \{0\}) = d[0,1] + d[0,2] - d[1,2]$ and $c_D(S \cup \{0\}) - c_D(S) = d[0,i_1] + d[0,i_2] - d[i_1,i_2]$, where S is represented as $\{i_1, i_2, \ldots, i_s\}$ with $i_1 < i_2 < \cdots < i_s$. Therefore, what we want to show is now replaced to $d[0,i_1] + d[0,i_2] - d[i_1,i_2] - d[0,1] - d[0,2] + d[1,2] \ge 0$.

We consider the following four subcases.

Case 1-1: $i_1 = 1$ and $i_2 = 2$.

We have $d[0,i_1] + d[0,i_2] - d[i_1,i_2] - d[0,1] - d[0,2] + d[1,2] = d[0,1] + d[0,2] - d[1,2] - d[0,1] - d[0,2] + d[1,2] = 0$.

Case 1-2: $i_1 = 1$ and $i_2 > 2$.

We have $d[0, i_1] + d[0, i_2] - d[i_1, i_2] - d[0, 1] - d[0, 2] + d[1, 2] = d[0, 1] + d[0, i_2] - d[1, i_2] - d[0, 1] - d[0, 2] + d[1, 2] = d[0, i_2] - d[1, i_2] - d[0, 2] + d[1, 2] \geq 0$, using the Monge property of D.

Case 1-3: $i_1 = 2$ and $i_2 > 2$.

We have $d[0, i_1] + d[0, i_2] - d[i_1, i_2] - d[0, 1] - d[0, 2] + d[1, 2] = d[0, 2] + d[0, i_2] - d[2, i_2] - d[0, 1] - d[0, 2] + d[1, 2] = d[0, i_2] - d[2, i_2] - d[0, 1] + d[1, 2] = d[0, i_2] - d[2, i_2] - d[0, 1] + d[2, 1] \geq 0$, using the Monge property and the symmetry of D.

Case 1-4: $i_1 > 2$ and $i_2 > i_1$.

We have $d[0, i_1] + d[0, i_2] - d[i_1, i_2] - d[0, 1] - d[0, 2] + d[1, 2] = (d[0, i_1] + d[1, 2] - d[0, 2] - d[1, i_1]) + (d[1, i_1] + d[0, i_2] - d[0, 1] - d[i_1, i_2]) \geq 0 + (d[i_1, 1] + d[0, i_2] - d[0, 1] - d[i_1, i_2]) \geq 0$, using the Monge property and the symmetry of D.

Thus, we are done for Case 1.

Case 2: $h < n$.

From the induction hypothesis, we have $m^{c_D}_{\preceq}(S) \leq c_D(S)$ for $S \subseteq N_0^h \setminus \{n\}$. So it suffices to show that $m^{c_D}_{\preceq}(S \cup \{n\}) \leq c_D(S \cup \{n\})$ for $S \subseteq N_0^h \setminus \{n\}$.

Here, from the induction hypothesis and the definition of the marginal contribution, we have $m^{c_D}_{\preceq}(S \cup \{n\}) = m^{c_D}_{\preceq}(S) + m^{c_D}_{\preceq}[n] \leq c_D(S) + m^{c_D}_{\preceq}[n] = c_D(S) + c_D(N_0^h) - c_D(N_0^h \setminus \{n\})$. Moreover, by Proposition 3 and the appropriate choice of pyramidal tours, we have $c_D(N_0^h) - c_D(N_0^h \setminus \{n\}) = d[n-2, n] + d[n-1, n] - d[n-2, n-1]$ and $c_D(S \cup \{n\}) - c_D(S) = d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s]$, where S is represented as $\{i_1, i_2, \ldots, i_s\}$ with $i_1 < i_2 < \cdots < i_s$. Therefore, what we want to show is now replaced to $d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] \geq 0$.

We consider the following four subcases.

Case 2-1: $i_s = n - 1$ and $i_{s-1} = n - 2$.

We have $d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = d[n-2, n] + d[n-1, n] - d[n-2, n-1] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = 0$.

Case 2-2: $i_s = n - 1$ and $i_{s-1} < n - 2$.

We have $d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = d[i_{s-1}, n] + d[n-1, n] - d[i_{s-1}, n-1] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = d[i_{s-1}, n] - d[i_{s-1}, n-1] - d[n-2, n] + d[n-2, n-1] \geq 0$, using the Monge property of D.

Case 2-3: $i_s = n - 2$ and $i_{s-1} < n - 2$.

We have $d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = d[i_{s-1}, n] + d[n-2, n] - d[i_{s-1}, n-2] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = d[i_{s-1}, n] - d[i_{s-1}, n-2] - d[n-1, n] + d[n-1, n-2] \geq 0$, using the Monge property and the symmetry of D.

Case 2-4: $i_s < n - 2$ and $i_{s-1} < i_s$.

We have $d[i_{s-1}, n] + d[i_s, n] - d[i_{s-1}, i_s] - d[n-2, n] - d[n-1, n] + d[n-2, n-1] = (d[i_{s-1}, n] + d[n-1, i_s] - d[i_{s-1}, i_s] - d[n-1, n]) + (d[i_s, n] + d[n-2, n-1] - d[i_s, n-1] - d[n-2, n]) \geq 0$, using the Monge property and the symmetry of D.

Thus, we are done for Case 2 as well, and we can conclude that $m^{c_D}_{\preceq}$ belongs to the core. Now we will show that $m^{c_D}_{\preceq}$ can be computed in $O(n^2)$. An algorithm will look as follows.

1. First, find a permutation σ such that the matrix resulting from σ is a permuted Monge matrix. This can be done in $O(n^2)$ [5, 9].

2. Next, compute $c_D(X_{\preceq}(i))$ for each $i \in N_0^h$. Each of these values is the optimal value for an instance of TSP with the symmetric Monge property. In each instance we just need to look up at most $n+1$ entries of D due to Proposition 3. In total, it requires $\mathrm{O}(n^2)$.

Therefore, in total, this algorithm runs in $\mathrm{O}(n^2) + \mathrm{O}(n^2) = \mathrm{O}(n^2)$. □

5 Proof of Theorem 8.

In this section we prove Theorem 8. The proof is similar to that of Theorem 4. First, notice that the submodularity of c_D is equivalent to the following condition: for all $S \subseteq N$ such that $|S| \geq 2$ and distinct $i, j \in S$

$$c_D(S \setminus \{i\}) + c_D(S \setminus \{j\}) \geq c_D(S) + c_D(S \setminus \{i, j\}).$$

We use this fact in order to shorten the proof.

Proof of Theorem 8. Similarly to the proof of Theorem 4, we assume that N_0 is renumbered by a permutation σ so that D is a Kalmanson matrix. Here, let $h = \sigma^{-1}(0) \in N_0$ be the home and regard $N_0^h = N_0 \setminus \{h\}$ as the set of the players. Choose arbitrarily $S = \{i_1, \ldots, i_s\} \subseteq N_0^h$ where $i_1 < i_2 < \cdots < i_s$ and distinct $i, j \in S$. Assume that $i < j$ without loss of generality. For $k \in S$, define $\mathrm{Pred}(k) = \max\{l \in S \cup \{h\} : l < k\}$ and $\mathrm{Succ}(k) = \min\{l \in S \cup \{h\} : l > k\}$.

We consider the following three cases. Note that every submatrix of a Kalmanson matrix is also a Kalmanson matrix.

Case 1: $h < i_1$.

It has three subcases.

Case 1-1: $h < i_1 \leq i < \mathrm{Succ}(i) < j \leq i_s$.

By Proposition 6, we can easily see that $c_D(S \setminus \{i\}) + c_D(S \setminus \{j\}) = c_D(S) + c_D(S \setminus \{i, j\})$.

Case 1-2: $h < i_1 \leq i < \mathrm{Succ}(i) = j < i_s$.

By Proposition 6, we have $(c_D(S) + c_D(S \setminus \{i, j\})) - (c_D(S \setminus \{i\}) + c_D(S \setminus \{j\})) = d[i, j] + d[\mathrm{Pred}(i), \mathrm{Succ}(j)] - d[\mathrm{Pred}(i), j] - d[i, \mathrm{Succ}(j)] \leq 0$. Here, we used the Kalmanson property (the inequality (5)).

Case 1-3: $h < i_1 \leq i < \mathrm{Succ}(i) = j = i_s$.

By Proposition 6, we have $(c_D(S) + c_D(S \setminus \{i, j\})) - (c_D(S \setminus \{i\}) + c_D(S \setminus \{j\})) = d[i, j] + d[h, \mathrm{Pred}(i)] - d[\mathrm{Pred}(i), j] - d[h, i] \leq 0$. Here, we used the Kalmanson property (the inequality (4)).

Case 2: $i_1 < h < i_s$.

It has two subcases.

Case 2-1: $i_1 = i < \mathrm{Succ}(i) \leq h \leq \mathrm{Pred}(j) < j = i_s$.

By Proposition 6, we have $(c_D(S) + c_D(S \setminus \{i, j\})) - (c_D(S \setminus \{i\}) + c_D(S \setminus \{j\})) = d[i, j] + d[\mathrm{Succ}(i), \mathrm{Pred}(j)] - d[\mathrm{Succ}(i), j] - d[i, \mathrm{Pred}(j)] \leq 0$. Here, we used the Kalmanson property (the inequality (5)).

Case 2-2: Other situations from Case 2-1.

By Proposition 6, we can see that $c_D(S \setminus \{i\}) + c_D(S \setminus \{j\}) = c_D(S) + c_D(S \setminus \{i, j\})$.

Case 3: $i_s < h$.

It has three subcases.

Case 3-1: $i_1 \leq i < \mathrm{Succ}(i) < j \leq i_s < h$.

By Proposition 6, we can see that $c_D(S\setminus\{i\})+c_D(S\setminus\{j\}) = c_D(S)+c_D(S\setminus\{i,j\})$.

Case 3-2: $i_1 < i < \mathrm{Succ}(i) = j \leq i_s < h$.

By Proposition 6, we have $(c_D(S)+c_D(S\setminus\{i,j\}))-(c_D(S\setminus\{i\})+c_D(S\setminus\{j\})) = d[i,j] + d[\mathrm{Pred}(i), \mathrm{Succ}(j)] - d[\mathrm{Pred}(i), j] - d[i, \mathrm{Succ}(j)] \leq 0$. Here, we used the Kalmanson property (inequality (5)).

Case 3-3: $i_1 = i < \mathrm{Succ}(i) = j \leq i_s < h$.

By Proposition 6, we have $(c_D(S)+c_D(S\setminus\{i,j\}))-(c_D(S\setminus\{i\})+c_D(S\setminus\{j\})) = d[i,j] + d[\mathrm{Succ}(j), h] - d[j,h] - d[i, \mathrm{Succ}(j)] \leq 0$. Here, we used the Kalmanson property (inequality (4)).

This completes the proof. $\square$

6 Summary and concluding remarks

In this paper, we have considered the core of a symmetric traveling salesman game. We have proved that the problem to test the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard. Moreover, we have proved that a traveling salesman game is totally balanced if the distance matrix is a permuted symmetric Monge matrix, and that a traveling salesman game is submodular if the distance matrix is a permuted Kalmanson matrix.

Now we will make some remarks.

6.1 Non-necessity of the Monge property for total balancedness

We have proved that a traveling salesman game is totally balanced if the distance matrix is a Monge matrix. However, the Monge property is not necessary for the total balancedness of traveling salesman games. For example, let $N_0 = \{0, 1, 2, 3\}$ and

$$D = \begin{bmatrix} 0 & 3 & 1 & 3 \\ 3 & 0 & 3 & 6 \\ 1 & 3 & 0 & 3 \\ 3 & 6 & 3 & 0 \end{bmatrix}.$$

Here, we have $c_D(\{1\}) = c_D(\{3\}) = 6$, $c_D(\{2\}) = 2$, $c_D(\{1,2\}) = c_D(\{2,3\}) = 7$, $c_D(\{1,3\}) = c_D(\{1,2,3\}) = 12$. Therefore, we can see that this game is totally balanced. However, D is not a permuted Monge matrix.

6.2 Non-sufficiency of the Monge property for submodularity

We have an example which says that the Monge property does not imply the submodularity of the traveling salesman game, as follows: $N_0 = \{0, 1, 2, 3, 4\}$ and

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 2 & 1 & 0 \end{bmatrix}.$$

Choose $S = \{1, 2, 4\}$ and $T = \{1, 3, 4\}$. Then we have $c_D(S) = c_D(T) = c_D(S \cap T) = 8$ and $c_D(S \cup T) = 9$. Therefore, $c_D(S) + c_D(T) < c_D(S \cup T) + c_D(S \cap T)$, but it can be easily verified that D has the Monge property.

6.3 Other polynomially solvable special cases

Propositions 2 and 6 say that there exists a shortest tour which is pyramidal for the TSP with the Monge property or the Kalmanson property. In addition, there are other classes of distance matrices which derive a similar result. We can find some of them in [1, 4, 14, 25, 52]. So, we might ask if these other classes also yield the associated traveling salesman games with non-empty cores. For a symmetric TSP, we have symmetric Demidenko matrices and van der Veen matrices as polynomially solvable cases, for example. A matrix D is a *symmetric Demidenko matrix* if D is symmetric and satisfies for any $i < j < j+1 < l$

$$d[i,j] + d[j+1,l] \leq d[i,j+1] + d[j,l]. \tag{6}$$

Similarly, a symmetric matrix D is a *van der Veen matrix* if for any $i < j < j+1 < l$

$$d[i,j] + d[j+1,l] \leq d[i,l] + d[j,j+1]. \tag{7}$$

Consider a symmetric TSP on the cities N_0 with a symmetric distance matrix D. If D is a symmetric Demidenko matrix, then there exists a shortest tour which is pyramidal [11]. Also, if D is a van der Veen matrix, then there exists a shortest tour which is pyramidal [68]. Here we remark that there exists an instance of traveling salesman games with a symmetric Demidenko matrix or a van der Veen matrix such that the core is empty. Here is such an instance: $N_0 = \{0, 1, 2, 3\}$ and

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & 3 & 3 & 0 \end{bmatrix}.$$

We can see that D is symmetric and satisfies the conditions (6) and (7). So D is a symmetric Demidenko matrix and also a van der Veen matrix. We now show that D implies core-emptiness. Suppose that the core is non-empty, that is, we have a vector $x \in \mathbb{R}^3$ such that $x(\{1,2,3\}) = c_D(\{1,2,3\})$ and $x(S) \leq c_D(S)$ for all $S \subseteq \{1,2,3\}$. We can observe that $c_D(\{1\}) = 2$, $c_D(\{2,3\}) = 5$ and $c_D(\{1,2,3\}) = 8$. Therefore, $x(\{1,2,3\}) = x(\{1\}) + x(\{2,3\}) \leq c_D(\{1\}) + c_D(\{2,3\}) = 2+5 < 8 = c_D(\{1,2,3\})$, which contradicts $x(\{1,2,3\}) = c_D(\{1,2,3\})$. Hence, the core is empty.

6.4 Relationship with the prize-collecting traveling salesman problem

As a consequence of Theorem 8, we can see that the Kalmanson property forms a polynomially solvable case of the prize-collecting traveling salesman problem. Let N_0 be the cities and D be an $N_0 \times N_0$ distance matrix. In addition, we have a non-negative vector $p \in \mathbb{R}^N$ which represents a reward or a prize associated with each city. Let (N, c_D) be a traveling salesman game. A *prize-collecting traveling salesman problem* is the problem to find a subtour starting from 0 which maximizes the sum of the prizes on the visited cities minus the total length of the subtour; more formally, to find $\max\{p(S) - c_D(S) : S \subseteq N\}$, where $p(S) = \sum_{i \in S} p[i]$. Define a function $g : 2^N \to \mathbb{R}$ as $g(S) = c_D(S) - p(S)$ for every $S \in 2^N$. Then the problem is equivalent to $-\min\{g(S) : S \subseteq N\}$. If D is a permuted Kalmanson matrix, then c_D is submodular (Theorem 8), therefore g is also submodular. So a minimizer of g can be obtained by an algorithm for the submodular function minimization problem. See [30, 33, 34, 60] for algorithms to solve the submodular function minimization problem in strongly polynomial time. Algorithms for the submodular function minimization problem are surveyed by Fleischer [22], and Fujishige [24].

6.5 Open problems

Here, we will state some open problems related to the work in this paper.

6.5.1 The asymmetric Monge property

In this paper, we have proved that a traveling salesman game with the symmetric Monge property has a non-empty core. So a natural question is about the asymmetric case. We did not know that a traveling salesman game with the asymmetric Monge property has a non-empty core or not.

6.5.2 Characterizations of totally balanced and submodular traveling salesman games

Another open problem is to characterize totally balanced traveling salesman games or submodular traveling salesman games in terms of distance matrices. Possibly, the decision problems like "is a given traveling salesman game totally balanced?" or "is a given traveling salesman game submodular?" are intractable, which implies that the good characterizations are beyond reach. For a Steiner tree game (which is also called a minimum cost spanning network game) introduced by Megiddo [49], Fang–Cai–Deng [20] proved that deciding the total balancedness of a given Steiner tree game is $\mathcal{NP}$-hard. This is the only known result on hardness of deciding the total balancedness of a class of combinatorial optimization games. Furthermore, as far as the author knows, there is no hardness result on deciding the submodularity (or the supermodularity) of a class of combinatorial optimization games. Note that the supermodularity of assignment games is characterized by Solymosi–Raghavan [63], and the submodularity of minimum coloring games and minimum vertex cover games is characterized by Okamoto [53].

6.5.3 Testing membership in the core

Faigle–Fekete–Hochstättler–Kern [15] studied the complexity of testing membership in the core of minimum cost spanning tree games and showed that this problem is co$\mathcal{NP}$-complete. Also, Fang–Cai–Deng [20] showed that testing membership in the core of Steiner tree games with non-empty cores is co$\mathcal{NP}$-hard. For us, how about testing membership in the core of traveling salesman games? Even in the general case we do not know the complexity. To be precise, we will state what is the problem exactly.

Problem. TESTING MEMBERSHIP IN THE CORE OF TRAVELING SALESMAN GAMES.

Instance. Cities N_0, an $N_0 \times N_0$ distance matrix D and a vector $x \in \mathbb{R}^N$ satisfying $x(N) = c_D(N)$.

Question. Does x belong to the core of the traveling salesman game (N, c_D)?

We leave the complexity issue of this problem as an open problem.

6.5.4 Core non-emptiness for the metric case

As Theorem 1, we proved that deciding the core non-emptiness of a given traveling salesman game is $\mathcal{NP}$-hard. However, the reduction in the proof was not adapted to the metric case (in which the distance matrix satisfies the triangle inequality) or the 2-dimensional Euclidean case. It is very plausible that the metric case and the 2-dimensional Euclidean case are also $\mathcal{NP}$-hard. So, we will describe that as a conjecture.

Conjecture 10. *The problem of testing the core non-emptiness of a given metric traveling salesman game is $\mathcal{NP}$-hard. This is $\mathcal{NP}$-hard even for the 2-dimensional Euclidean case.*

6.5.5 Naturally balanced and naturally totally balanced graphs

Herer [31], Herer–Penn [32] and Granot–Granot–Zhu [27] studied the underlying graph structure which always yields a submodular traveling salesman game. This kind of graphs are called naturally submodular. So, it is interesting to investigate "naturally balanced" graphs or "naturally totally balanced" graphs. To state a problem precisely, we will give the definitions. We are given a graph $G = (V, E)$ and a nonnegative weight function $f : E \to \mathbb{R}$ associated with each edge of the graph. Then, we construct a traveling salesman game (N, c_D) as follows. First we fix a vertex $v \in V$ as the home, then $N = V \setminus \{v\}$. Each entry $d[i, j]$ of the distance matrix D is determined as the length of a shortest path from i to j in G. The characteristic function c_D is defined as explained in Section 2 where the home is now v. A graph $G = (V, E)$ is called *naturally submodular* if for any $v \in V$ and any nonnegative weight function f the game (N, c_D) is submodular. Natural submodularity is characterized by Herer–Penn [32] for undirected graphs and by Granot-Granot-Zhu [27] for directed graphs and bidirected graphs. Analogously we may define a *naturally balanced* graph as a graph which yields a traveling salesman game with a non-empty core for any choice of the home $v \in V$ and any nonnegative function f. A *naturally totally balanced* graph can be defined similarly. Now, our open problem is to characterize the naturally balanced graphs and the naturally totally balanced graphs. Note that a similar investigation was provided for delivery games by Granot–Hamers–Tijs [28].

Acknowledgements

The author thanks Yoshiaki Oda for introducing the Monge property to him, and Masahiro Hachimori, Kenji Kashiwabara, Yasuko Matsui, Masataka Nakamura, and three anonymous referees for their valuable comments. Also the author is grateful to Satoru Iwata for pointing out the relationship between the prize-collecting traveling salesman problem and the submodular function minimization problem, and to Tomomi Matsui for telling the proof of the $\mathcal{NP}$-completeness to decide a given bin packing game has a non-empty core [48].

References

[1] Md.F. Baki and S.N. Kabadi, Pyramidal traveling salesman problem, Comput. Oper. Res. 26 (1999) 353–369.

[2] J.M. Bilbao, Cooperative games on combinatorial structures (Kluwer Academic Publishers, Dordrecht, 2000).

[3] C.G. Bird, On cost allocation for a spanning tree: a game theoretic approach, Networks 6 (1976) 335–350.

[4] R.E. Burkard, V.G. Deĭneko, R. van Dal, J.A.A. van der Veen and G.J. Woeginger, Well-solvable cases of the travelling salesman problem: a survey, SIAM Rev. 40 (1998) 496–546.

[5] R.E. Burkard, B. Klinz and R. Rudolf, Perspectives of Monge properties in optimization, Discrete Appl. Math. 70 (1996) 95–161.

[6] P. Chardaire, On the core of facility location games, submitted.

[7] G. Christopher, M. Farach and M. Trick, The structure of circular decomposable metrics, in: Proc. 4th ESA, Lect. Notes Comput. Sci. 1136 (Springer-Verlag, New York, 1996) 406–418.

[8] I.J. Curiel, Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems, (Kluwer Academic Publishers, Dordrecht, 1997)

[9] V.G. Deĭneko and V.L. Filonenko, On the reconstruction of specially structured matrices, Aktualnyje Problemy EVM i programmirovanije, Dnepropetrovsk, DGU, 1979 (in Russian).

[10] V.G. Deĭneko, R. Rudolf and G.J. Woeginger, Sometimes travelling is easy: the master tour problem, SIAM J. Discrete Math. 11 (1998) 81–93.

[11] V.M. Demidenko, A special case of travelling salesman problems, Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Mat. Nauk 5 (1976) 28–32, (in Russian).

[12] X. Deng, T. Ibaraki and H. Nagamochi, Algorithmic aspects of the core of combinatorial optimization games, Math. Oper. Res. 24 (1999) 751–766.

[13] X. Deng and C.H. Papadimitriou, On the complexity of cooperative solution concepts, Math. Oper. Res. 19 (1994) 257–266.

[14] H. Enomoto, Y. Oda and K. Ota, Pyramidal tours with step-backs and the asymmetric traveling salesman problem, Discrete Appl. Math. 87 (1998) 57–65.

[15] U. Faigle, S.P. Fekete, W. Hochstättler and W. Kern, On the complexity of testing membership in the core of min-cost spanning tree games, Int. J. Game Theory 26 (1997) 361–366.

[16] U. Faigle, S.P. Fekete, W. Hochstättler and W. Kern, On approximately fair cost allocation in Euclidean TSP games, OR Spektrum 20 (1998) 29–37.

[17] U. Faigle and W. Kern, On some approximately balanced combinatorial cooperative games, ZOR 38 (1993) 141-152.

[18] U. Faigle and W. Kern, On the core of ordered submodular cost games, Math. Program. 87 (2000) 483–499.

[19] U. Faigle, W. Kern and J. Kuipers, On the computation of the nucleolus of a cooperative game, Int. J. Game Theory 30 (2001) 79–98.

[20] Q. Fang, M. Cai and X. Deng, Total balancedness condition for Steiner tree games, Discrete Appl. Math. 127 (2003) 555–563.

[21] P.C. Fishburn and H.O. Pollak, Fixed route cost allocation, Am. Math. Mon. 90 (1983) 366–378.

[22] L. Fleischer, Recent progress in submodular function minimization, OPTIMA 64 (2000) 1–11.

[23] S. Fujishige, Submodular Functions and Optimization, Ann. Discrete Math. 47 (North-Holland, Amsterdam, 1991).

[24] S. Fujishige, Submodular function minimization and related topics, Optim. Methods Softw. 18 (2003) 167–180.

[25] P.C. Gilmore, E.L. Lawler and D.B. Shmoys, Well-solved special cases, in: Lawler, E.L., Lenstra, J.K., Rinnooy Kan, A.H.G., Shmoys, D.B., eds., The Traveling Salesman Problem – A Guided Tour of Combinatorial Optimization (Wiley, Chichester, 1985) 87–143.

[26] M.X. Goemans and M. Skutella, Cooperative facility location games, J. Algorithms, to appear. An extended abstract version has appeared in: Proc. 11th SODA, 2000, 76–85.

[27] D. Granot, F. Granot and W.R. Zhu, Naturally submodular digraphs and forbidden digraph configurations, Discrete Appl. Math. 100 (2000) 67–84.

[28] D. Granot, H. Hamers and S. Tijs, On some balanced, totally balanced and submodular delivery games, Math. Program. 86 (1999) 355–366.

[29] D. Granot and G. Huberman, Minimum cost spanning tree games, Math. Program. 21 (1981) 1–18.

[30] M. Grötschel, L. Lovász and A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Algorithms Comb. 2 (Springer-Verlag, Berlin Heidelberg, 1988). The 2nd edition, 1993.

[31] Y.T. Herer, Submodularity and the traveling salesman problem, Eur. J. Oper. Res. 114 (1999) 489–508.

[32] Y. Herer and M. Penn, Characterization of naturally submodular graphs: a polynomial solvable class of the TSP, Proc. Am. Math. Soc. 123 (1995) 613–619.
[33] S. Iwata, Fully combinatorial algorithm for submodular function minimization, J. Comb. Theory Ser. B 84 (2002) 203–212. Corrections are on his webpage (http://www.sr3.t.u-tokyo.ac.jp/~iwata/).
[34] S. Iwata, L. Fleischer and S. Fujishige, A combinatorial strongly polynomial-time algorithm for minimizing submodular functions, J. Assoc. Comput. Mach. 48 (2001) 761–777.
[35] S.N. Kabadi, Polynomially solvable cases of the traveling salesman problem, in: Gutin, G., Punnen, A.P., eds., The Traveling Salesman Problem and Its Variations (Kluwer Academic Press, Dordrecht, 2002) 489–584.
[36] S.N. Kabadi, New polynomially solvable classes and a new heuristic for the traveling salesman problem and its generalizations, Discrete Appl. Math. 119 (2002) 149–168.
[37] S.N. Kabadi and Md.F. Baki, Gilmore–Gomory type traveling salesman problems, Comput. Oper. Res. 26 (1999) 329–351.
[38] K. Kalmanson, Edgeconvex circuits and the traveling salesman problem, Can. J. Math. 27 (1975) 1000–1010.
[39] Y. Kannai, The core and balancedness, in: Aumann, R.J., Hart, S., eds., Handbook of Game Theory, Volume 1 (Elsevier, Amsterdam, 1992) 355–395.
[40] A. Kolen, Solving covering problems and the uncapacitated plant location algorithms, European Journal of Operational Research 12 (1983) 266–278.
[41] A. Kolen and A. Tamir, Covering problems, in: P.B. Mirchandani and R.L. Francis eds., Discrete Location Theory (John Wiley and Sons, New York, 1990) 263–304.
[42] B. Korte and J. Vygen, Combinatorial Optimization: Theory and Algorithms, Algorithms Comb. 21 (Springer-Verlag, Berlin Heidelberg, 2000). The 2nd edition, 2002.
[43] J.B. Kruskal, On the shortest spanning subtree of a graph and the travelling salesman problem, Proc. Am. Math. Soc. 7 (1956) 48–50.
[44] H.W. Kuhn, The Hungarian method for the assignment problem, Nav. Res. Logist. Q. 2 (1955) 83–97.
[45] J. Kuipers, A note on the 5-person traveling salesman game, ZOR 38 (1993) 131–139.
[46] J. Kuipers, A polynomial time algorithm for computing the nucleolus of convex games, Report M 96-12, Maastricht University, 1996.
[47] M. Maschler, B. Peleg and L. Shapley, The kernel and bargaining set of convex games, Int. J. Game Theory 2 (1972) 73–93.
[48] T. Matsui, Personal communication.
[49] N. Megiddo, Cost allocation for Steiner trees, Networks 8 (1978) 1–6.
[50] K. Murota, Discrete Convex Analysis, (SIAM, Philadelphia, 2003).
[51] J. Nešetřil, E. Milkov and H. Nešetřilov, Otakar Borůvka on minimum spanning tree problem: translation of both the 1926 papers, comments, history, Discrete Math. 233 (2001) 3–36.
[52] Y. Oda, An asymmetric analogue of van der Veen conditions and the traveling salesman problem, Discrete Appl. Math. 109 (2001) 279–292.
[53] Y. Okamoto, Submodularity of some classes of the combinatorial optimization games, Math. Methods Oper. Res. 58 (2003) 131–139.
[54] C.H. Papadimitriou, The Euclidean travelling salesman problem is NP-complete, Theor. Comput. Sci. 4 (1977) 237–244.
[55] J.K. Park, A special case of the n-vertex traveling salesman problem that can be solved in $O(n)$ time, Inf. Process. Lett. 40 (1991) 247–254.
[56] B. Peleg, Axiomatizations of the core, in: Aumann, R.J., Hart, S., eds., Handbook of Game Theory, Volume 1 (Elsevier, Amsterdam, 1992) 397–412.
[57] J.A.M. Potters, A class of traveling salesman games, Methods Oper. Res. 59 (1989) 263–276.

[58] J.A.M. Potters, I.J. Curiel and S.H. Tijs, Traveling salesman games, Math. Program. 53 (1992) 199–211.
[59] R.C. Prim, Shortest connection networks and some generalizations, Bell Systems Tech. J. 36 (1957) 1389–1401.
[60] A. Schrijver, A combinatorial algorithm minimizing submodular functions in strongly polynomial time, J. Comb. Theory, Ser. B 80 (2000) 346–355.
[61] L. Shapley, Cores of convex games, Int. J. Game Theory 1 (1971) 11–26. (Errata, in the same volume, 1972, 111–130.)
[62] L. Shapley and M. Shubik, The assignment game I: the core, Int. J. Game Theory 1 (1972) 111–130.
[63] T. Solymosi and T.E.S. Raghavan, Assignment games with stable core, Int. J. Game Theory 30 (2001) 177–185.
[64] F. Supnick, Extreme Hamiltonian lines, Ann. Math. 66 (1957) 179–201.
[65] A. Tamir, On the core of a traveling salesman cost allocation game, Oper. Res. Lett. 8 (1988) 31–34.
[66] A. Tamir, On the core of cost allocation games defined on location problems, Transp. Sci. 27 (1992) 81–86.
[67] S. Tijs, Bounds for the core and the τ-value, in: Moeschlin, O., Pallaschke, P., eds., Game Theory and Mathematical Economics (North Holland, Amsterdam, 1981) 123–132.
[68] J.A.A. van der Veen, A new class of pyramidally solvable symmetric traveling salesman problems, SIAM J. Discrete Math. 7 (1994) 585–592.