Recent Development of Abstract Convex Geometries

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Combinatorics
Geometry
Computation

Biased introduction to (abstract) convex geometries

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$\boxed{\text{Setup}}$

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$\mathcal{L} \subseteq 2^E$ a family of subsets of E.

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$\mathcal{L} \subseteq 2^E$ a family of subsets of E.

Def.

$\mathcal{L}$ is called a **convex geometry** on E

if $\mathcal{L}$ satisfies the following conditions.

(1) $\emptyset \in \mathcal{L}$, $\mathbf{E} \in \mathcal{L}$.

(2) $\mathbf{X}, \mathbf{Y} \in \mathcal{L} \Rightarrow \mathbf{X} \cap \mathbf{Y} \in \mathcal{L}$.

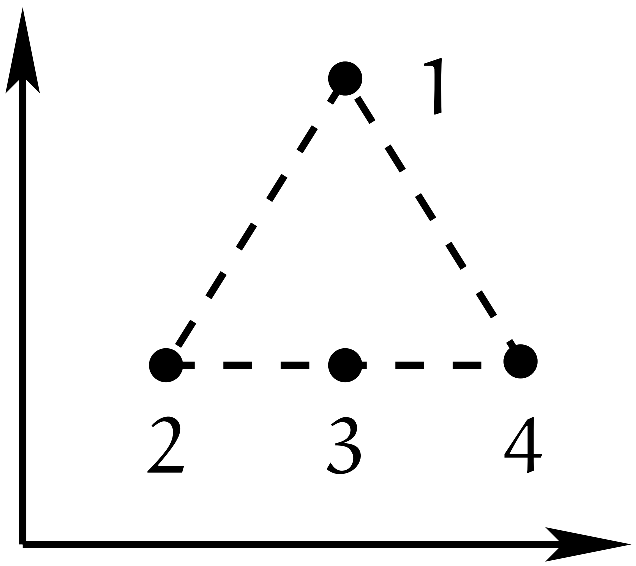
(3) $\mathbf{X} \in \mathcal{L} \setminus \{\mathbf{E}\} \Rightarrow \exists\, \mathbf{e} \in \mathbf{E} \setminus \mathbf{X}\colon \mathbf{X} \cup \{\mathbf{e}\} \in \mathcal{L}$.

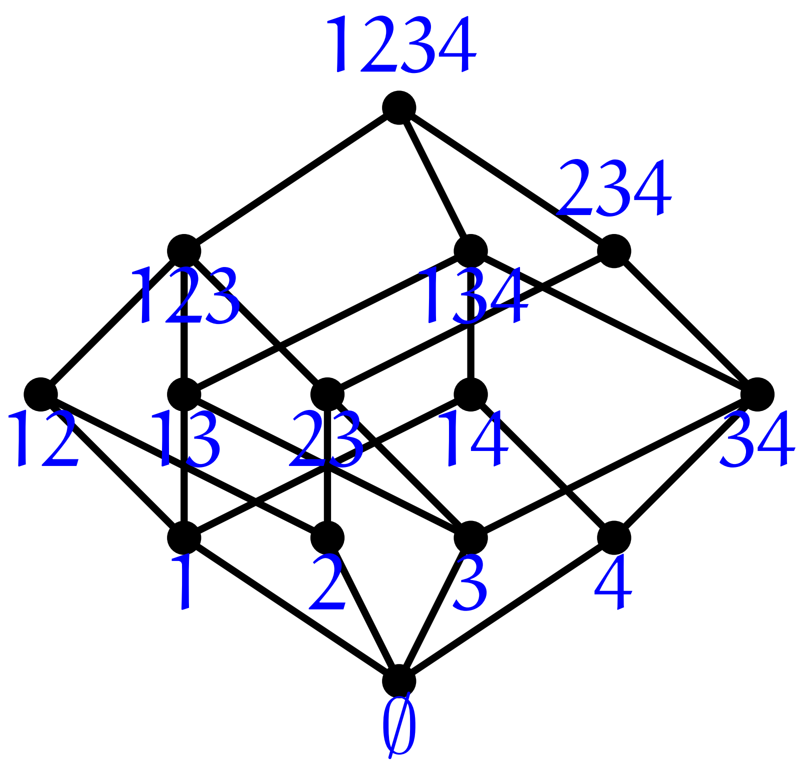
$X \subseteq E$ is called **convex** if $X \in \mathcal{L}$.

Given $\mathcal{P}$ a finite point set in $\mathbb{R}^d$

Def. $\mathcal{L}$ the **convex shelling** on $\mathcal{P}$:

$$\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$$





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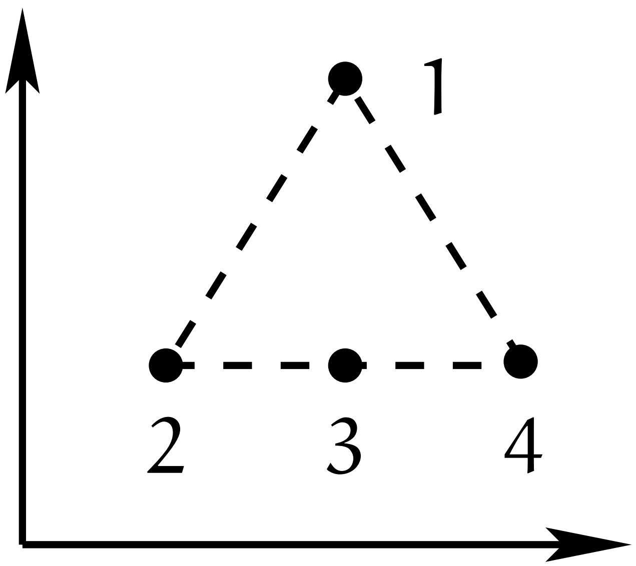

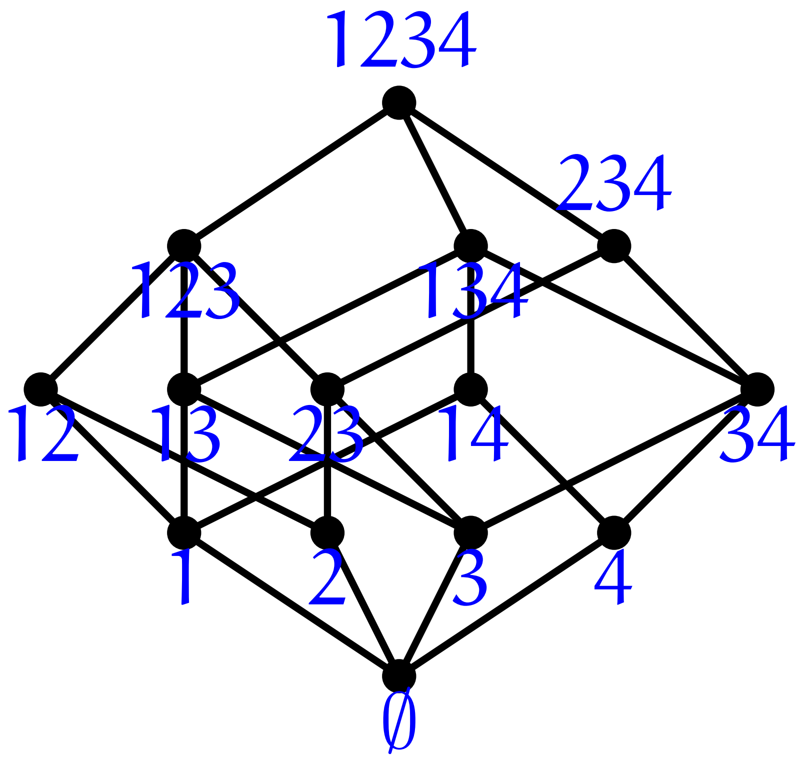

(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$

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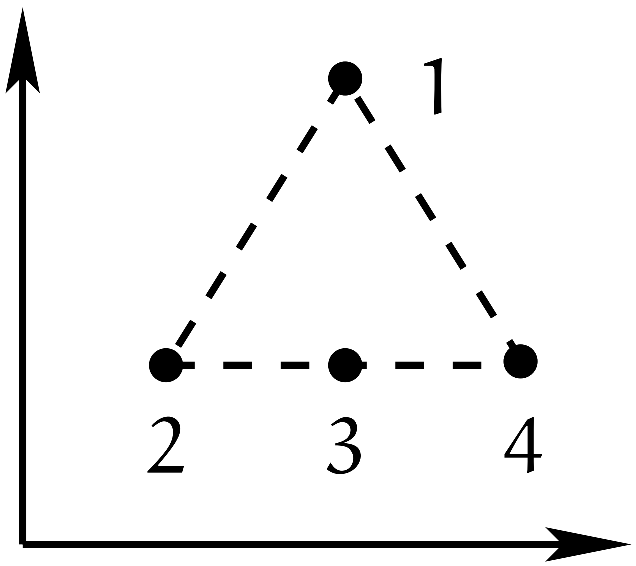

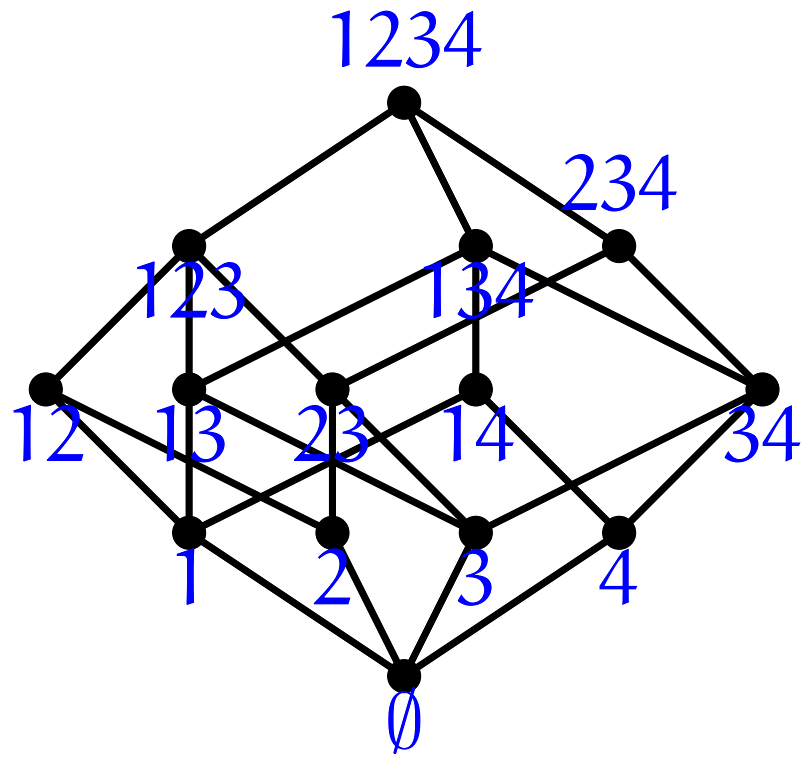

(2) $X, Y \in \mathcal{L} \Rightarrow X \cap Y \in \mathcal{L}$.

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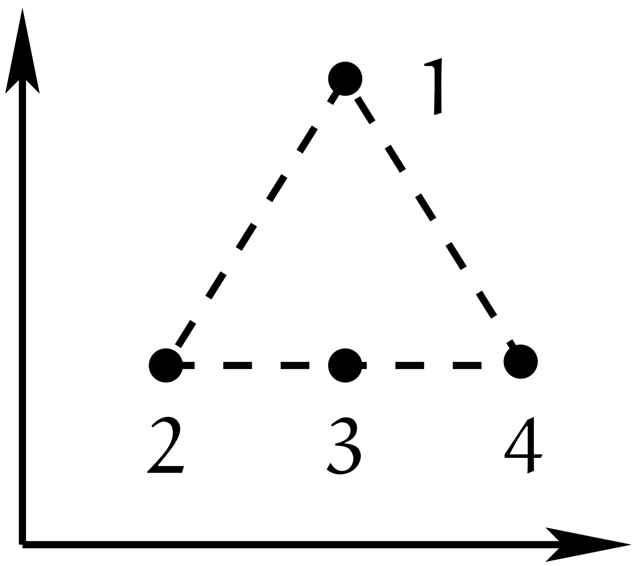

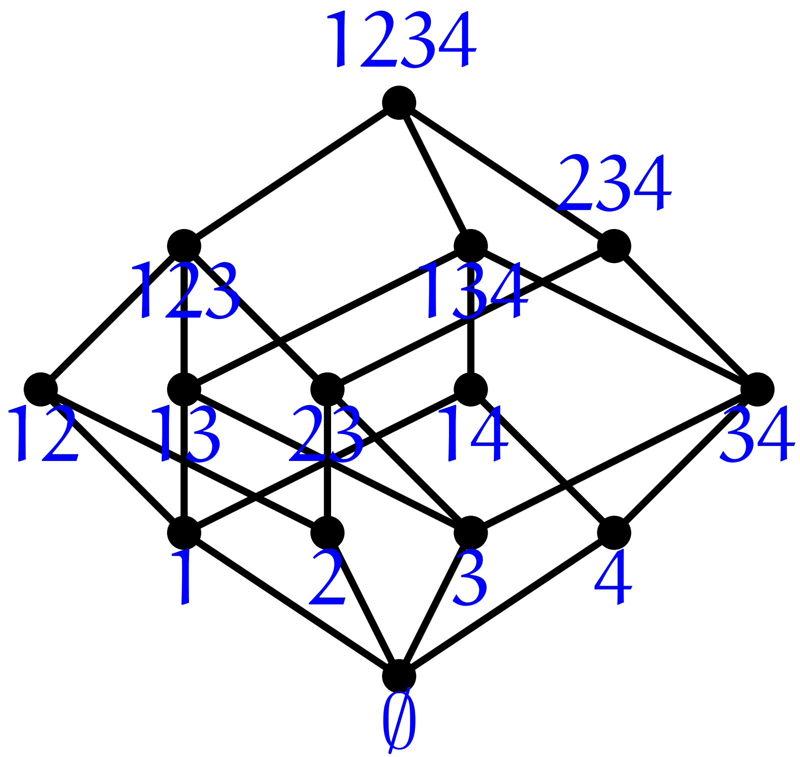

(3) $X \in \mathcal{L} \setminus \{E\} \Rightarrow \exists\, e \in E \setminus X$: $X \cup \{e\} \in \mathcal{L}$.

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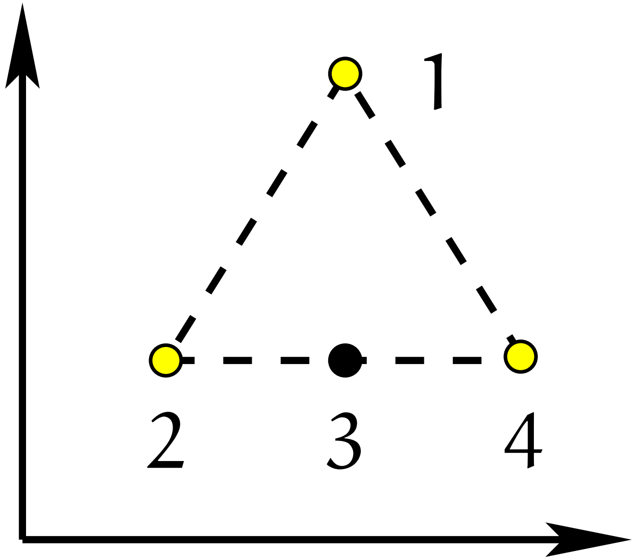

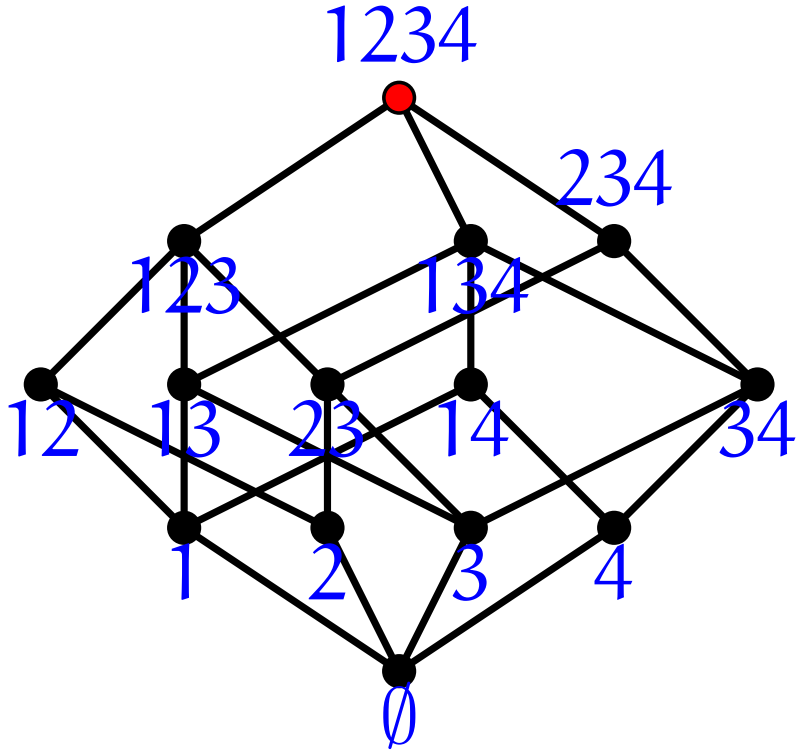

The extreme points of $\mathrm{conv}(\{1,2,3,4\}) = \{1,2,4\}$

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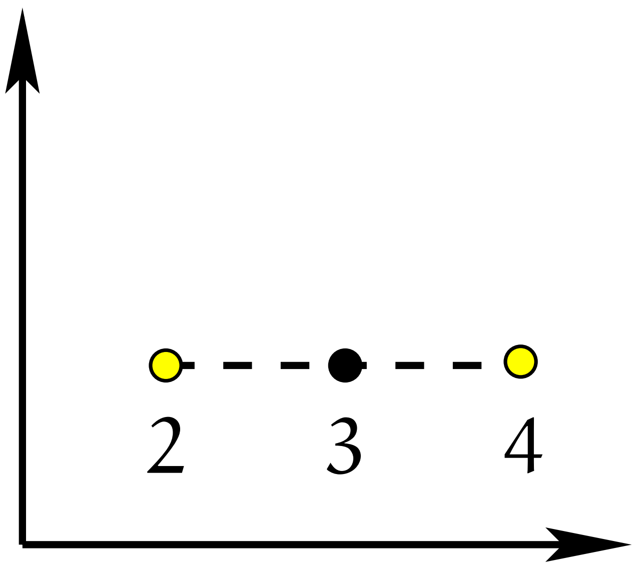

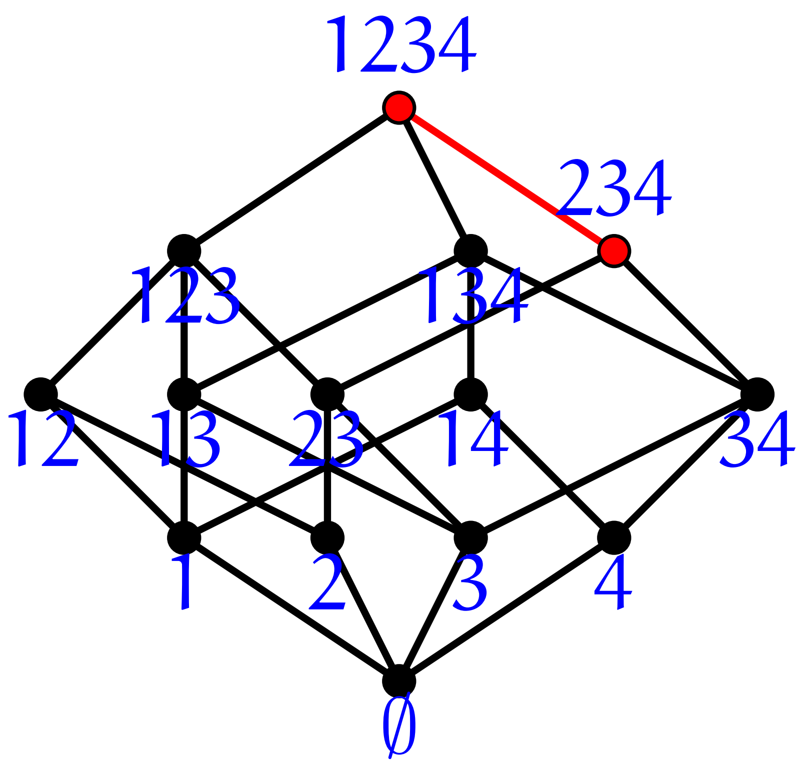

The extreme points of conv$(\{2,3,4\}) = \{2,4\}$

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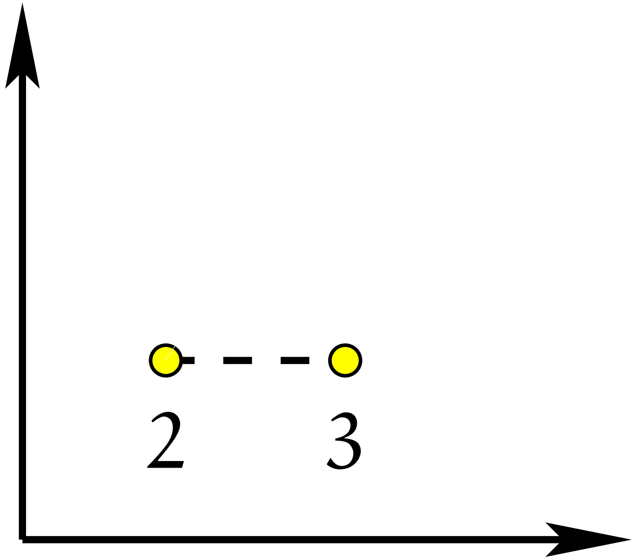

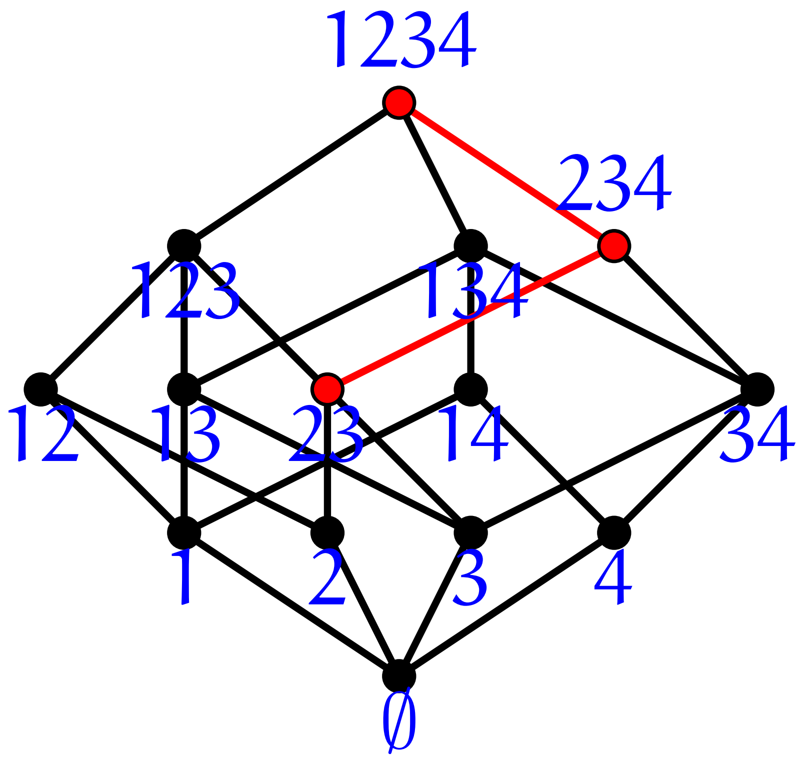

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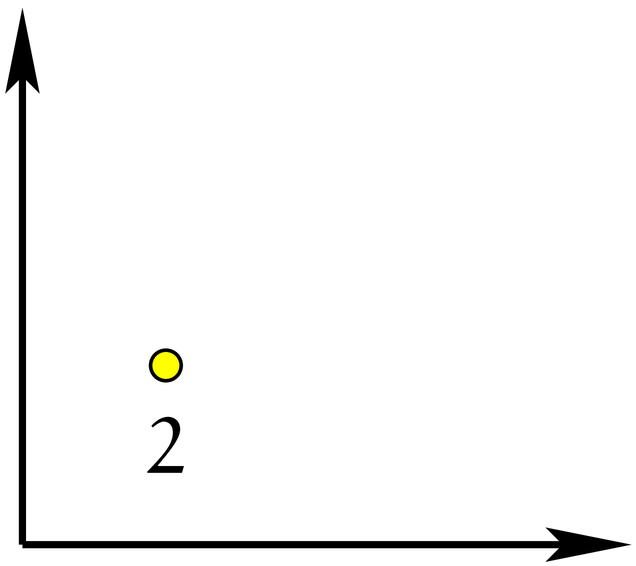

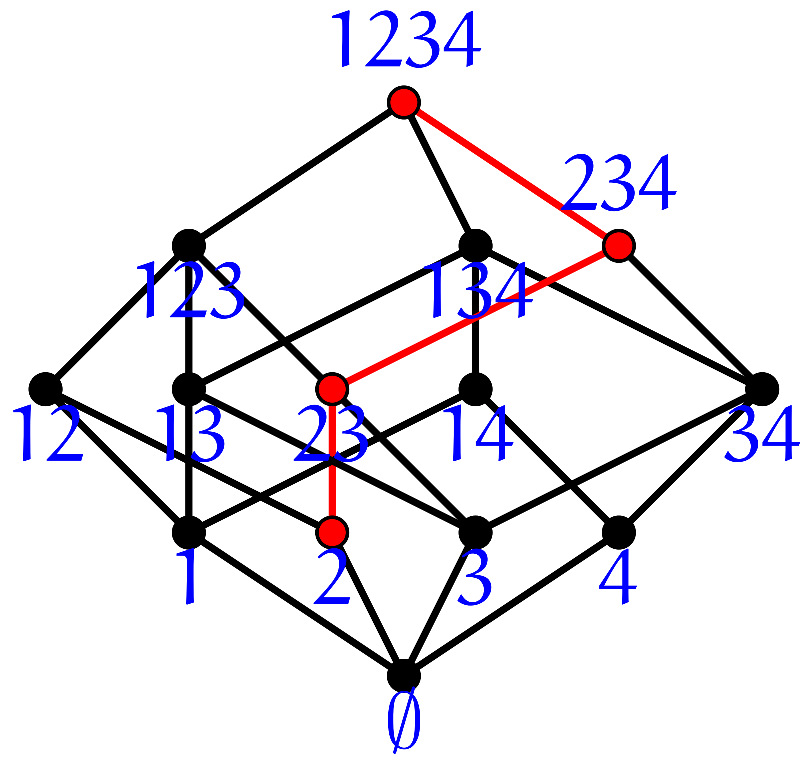

The extreme points of conv$(\{2\}) = \{2\}$

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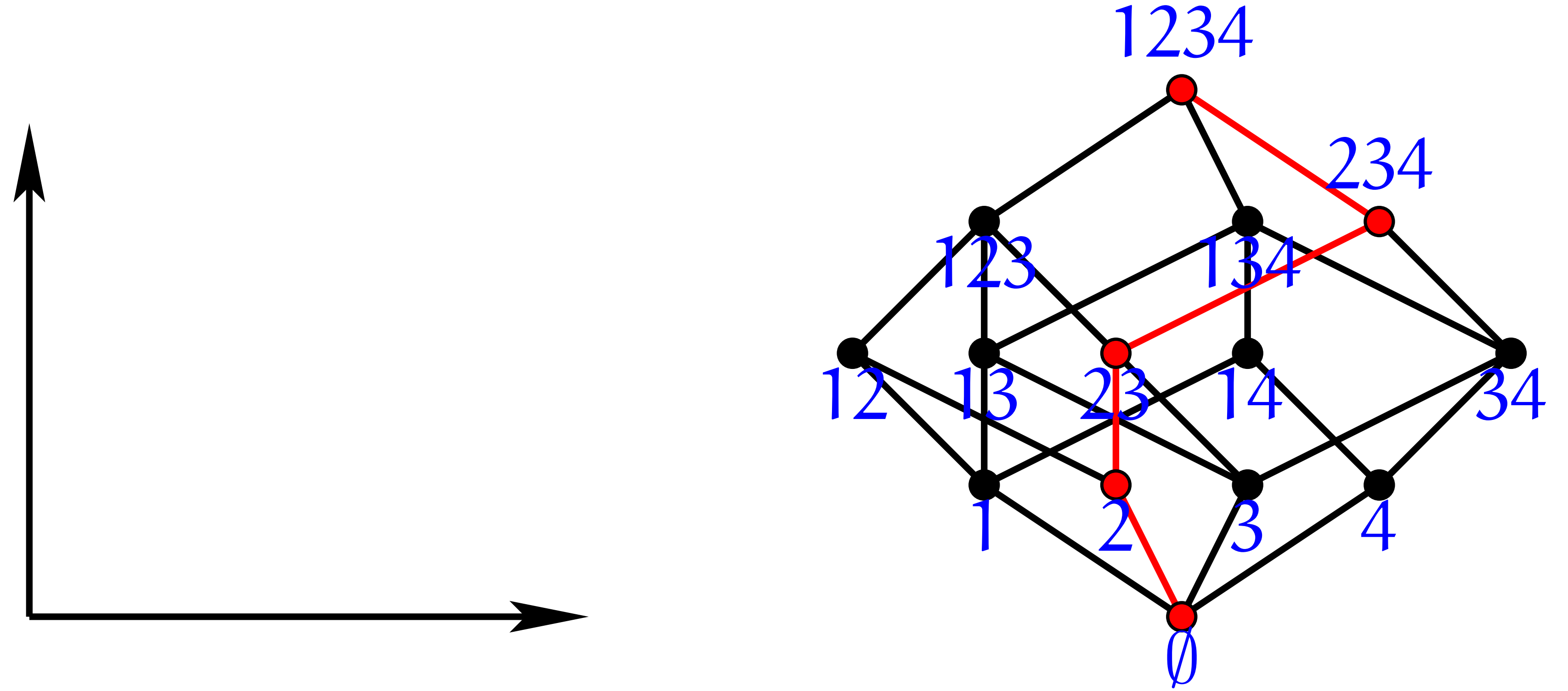

The extreme points of $\mathrm{conv}(\emptyset) = \emptyset$

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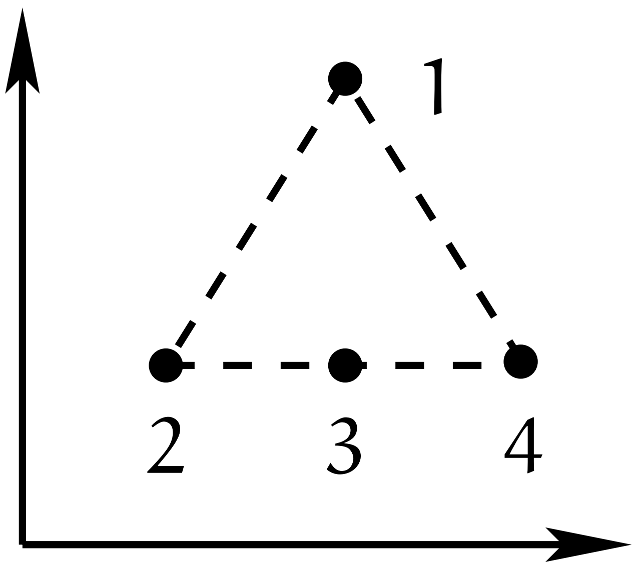

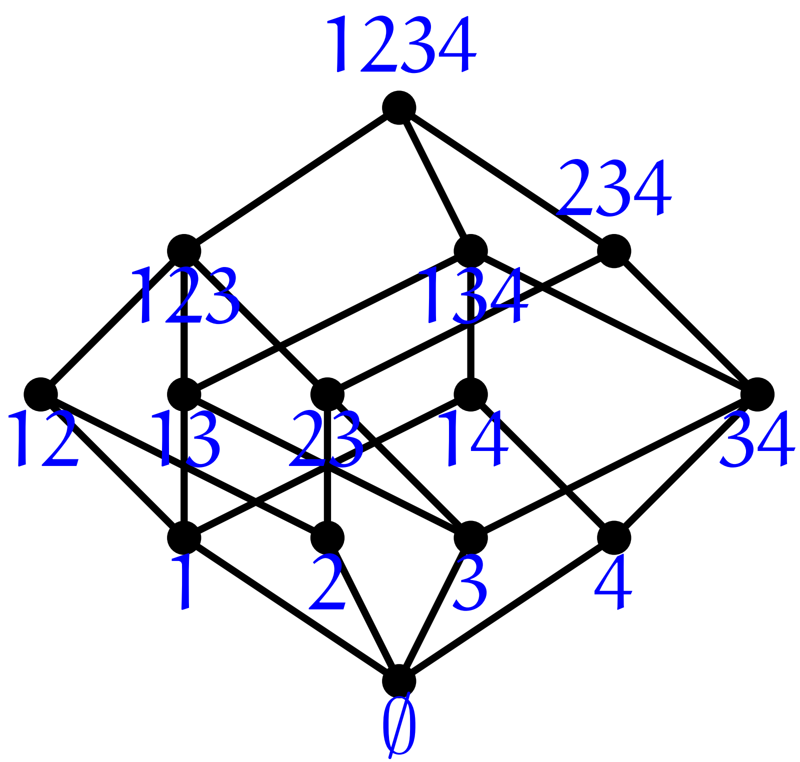

Given $P = (E, \leq)$ a partially ordered set

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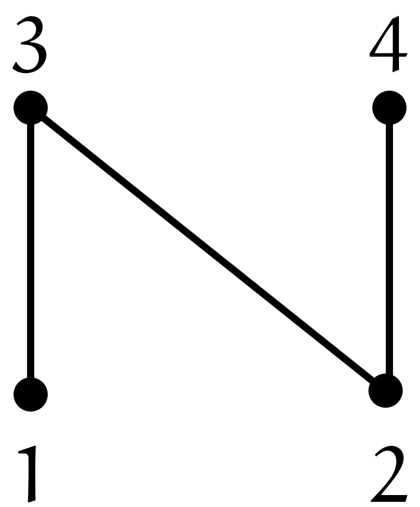

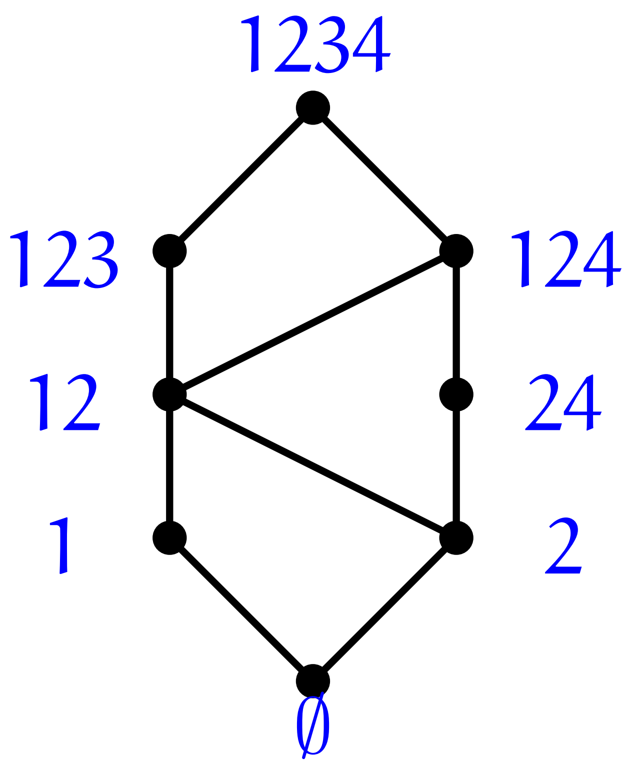

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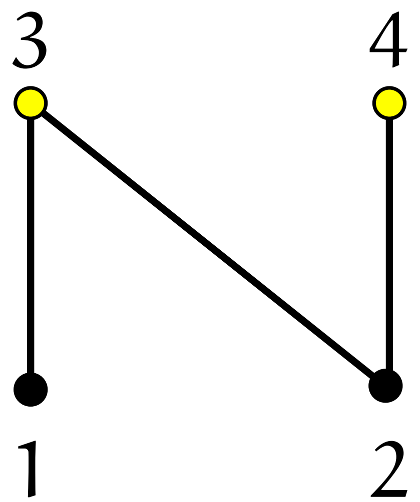

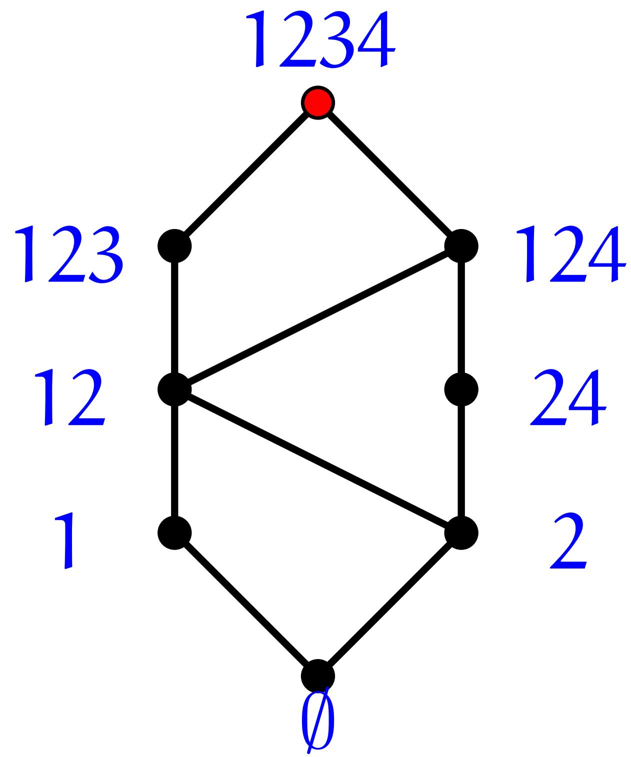

The maximal elements of $\{1, 2, 3, 4\} = \{3, 4\}$

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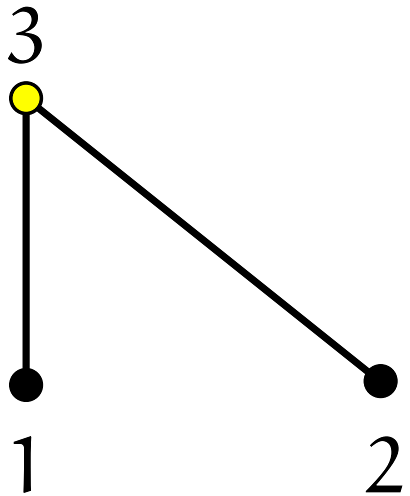

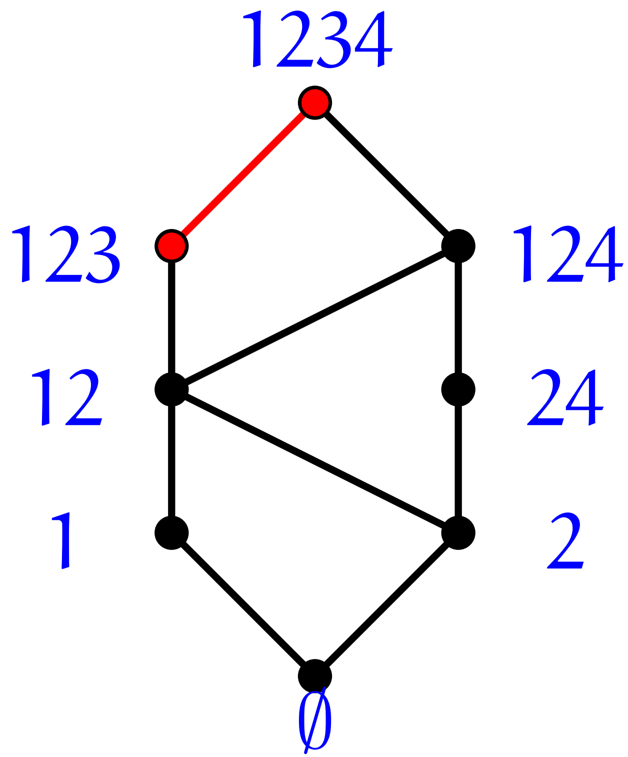

The maximal elements of $\{1, 2, 4\} = \{4\}$

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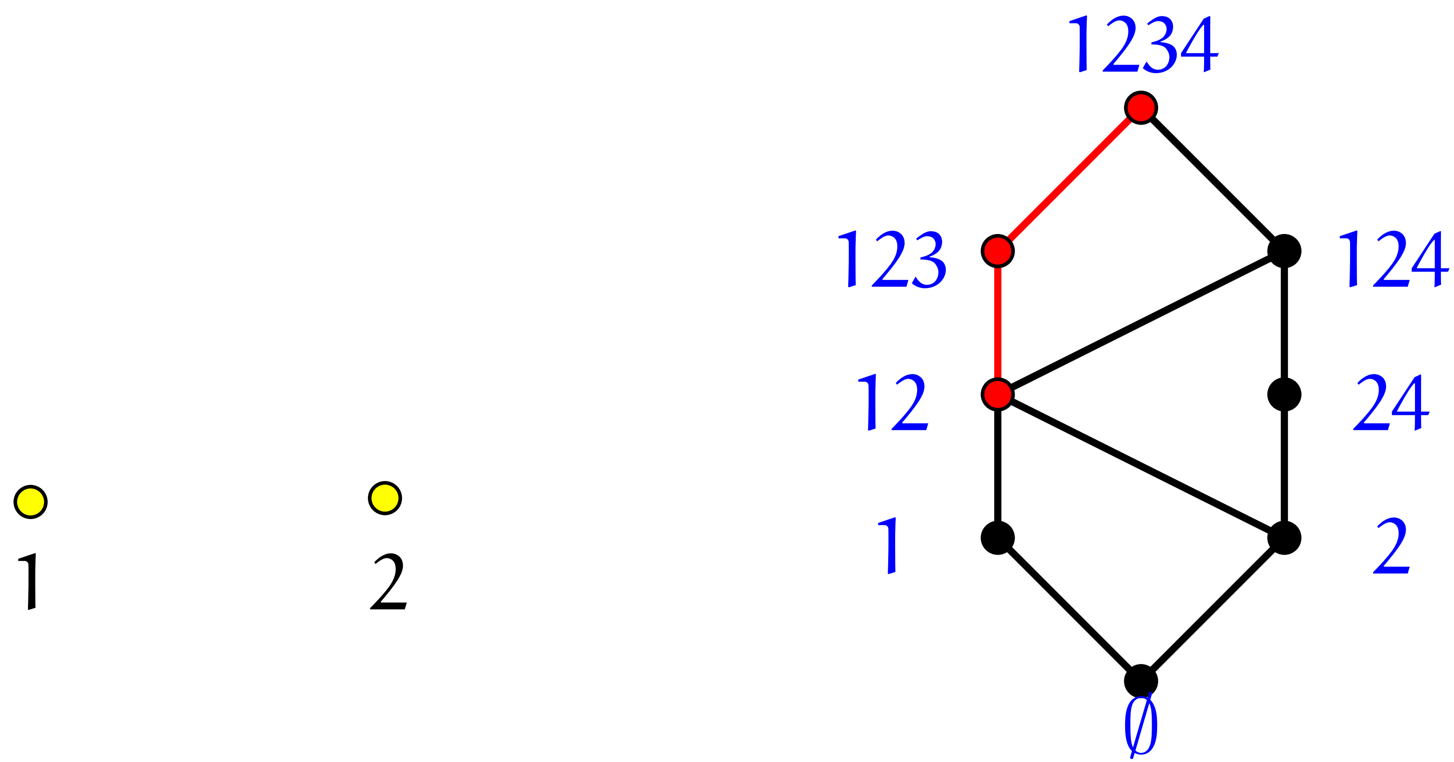

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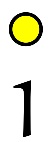

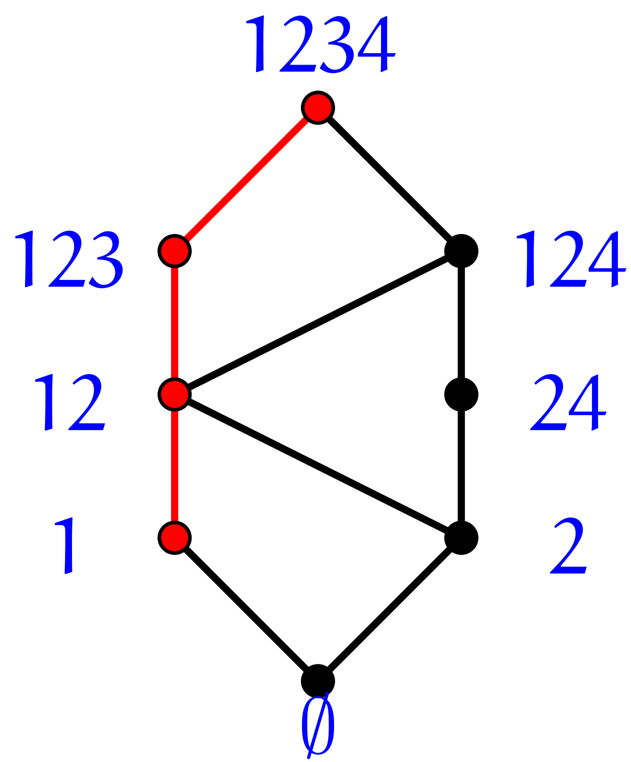

The maximal elements of $\{1\} = \{1\}$

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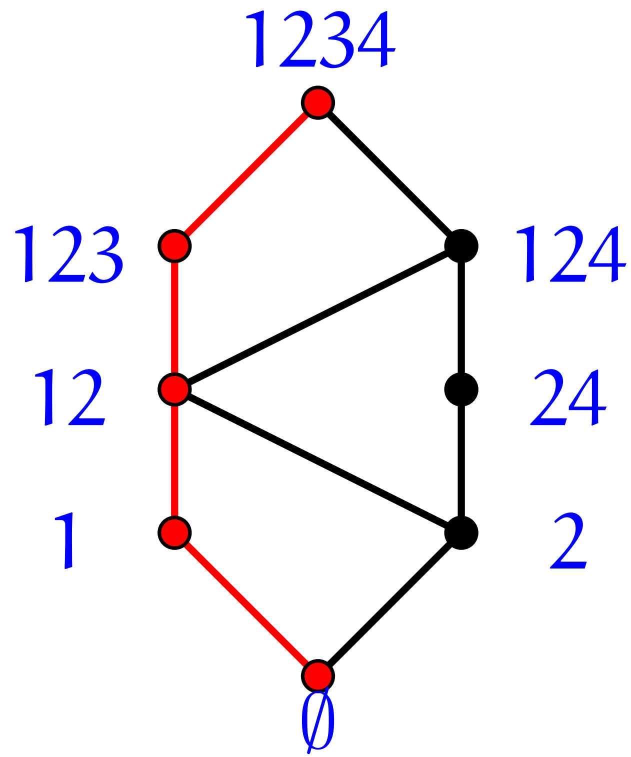

The maximal elements of $\emptyset = \emptyset$

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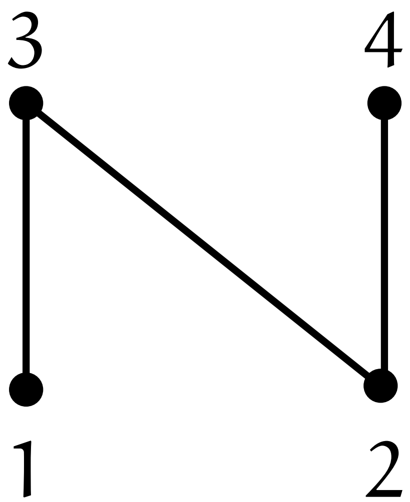
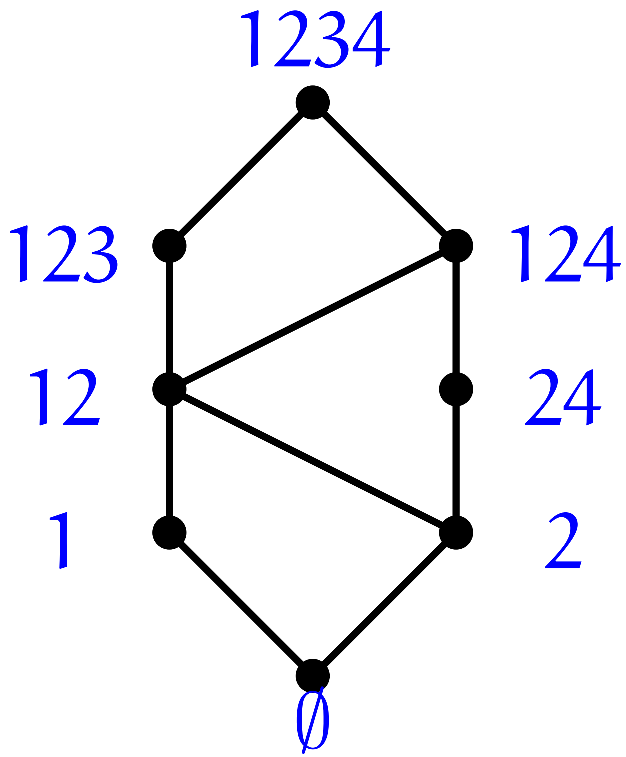
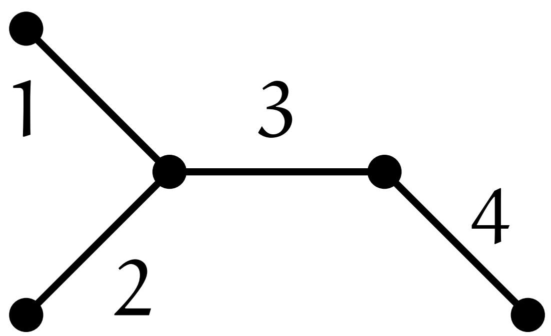


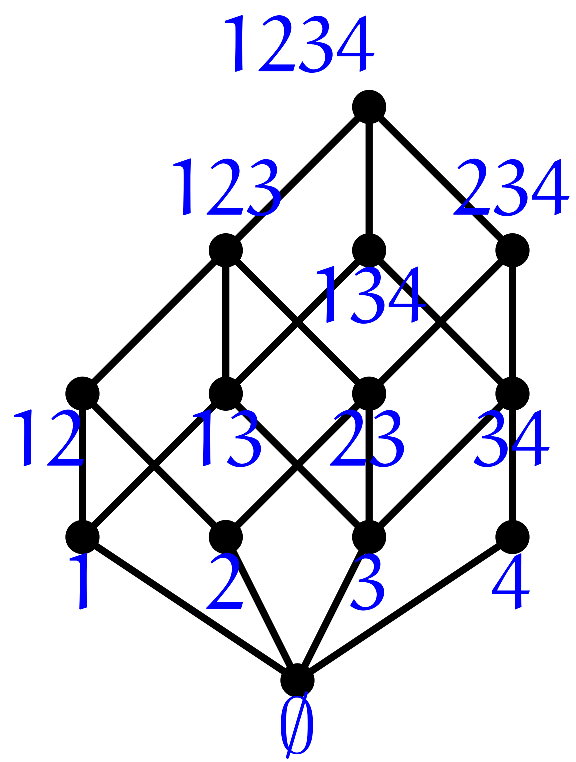


Given $T = (V, E)$ a tree

Def. $\mathcal{L}$ the **tree shelling** on T:

$\mathcal{L} = \{X \subseteq E : X$ **forms a subtree of** $T\}$





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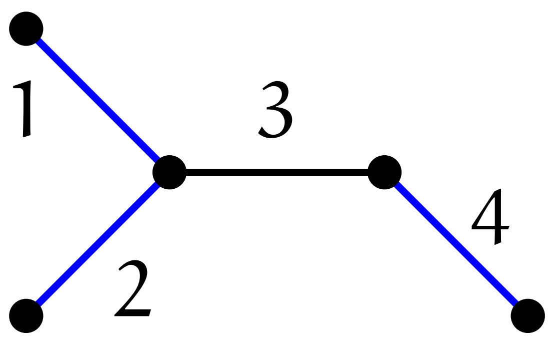

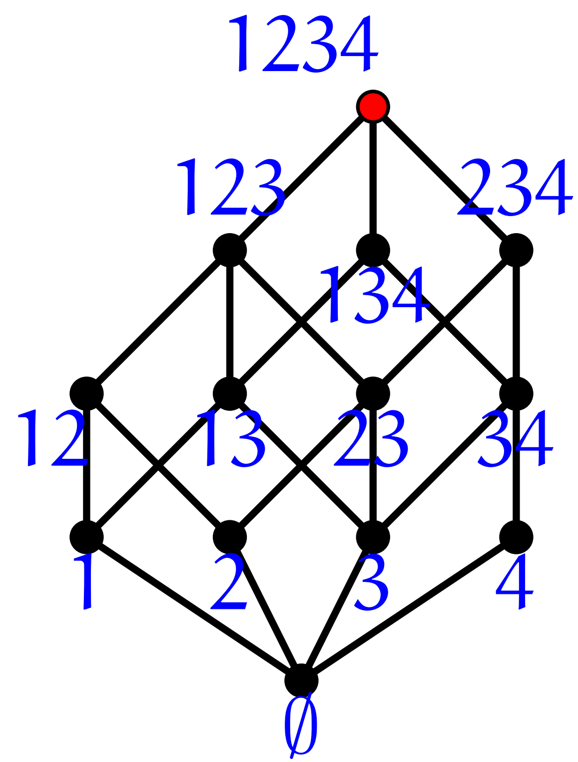

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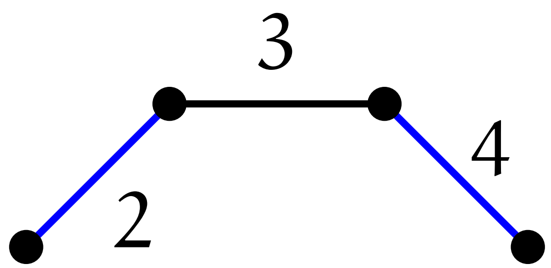

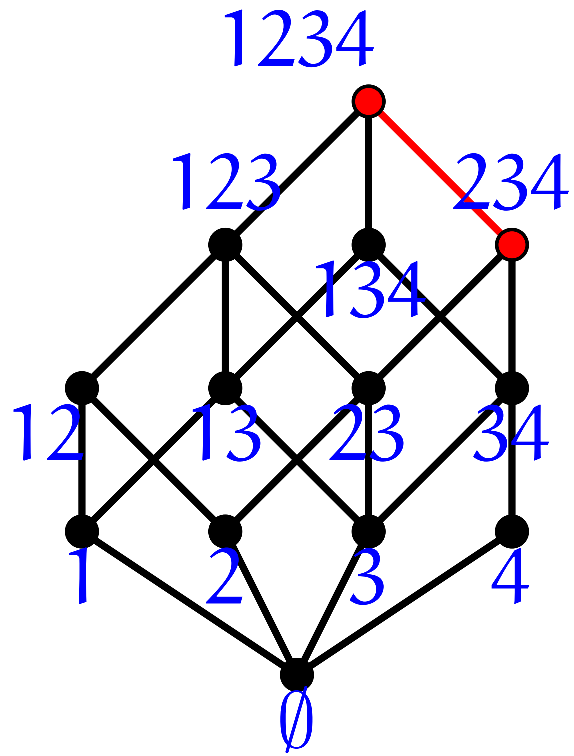

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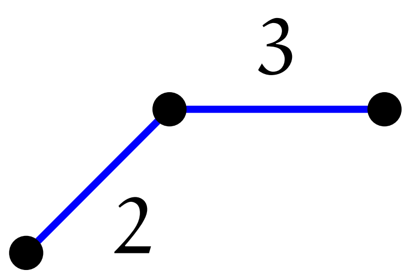

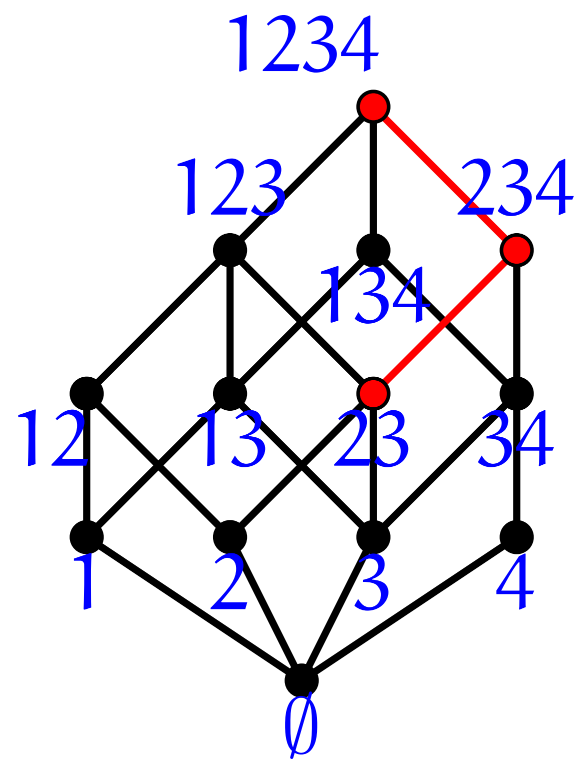

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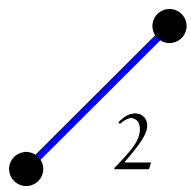

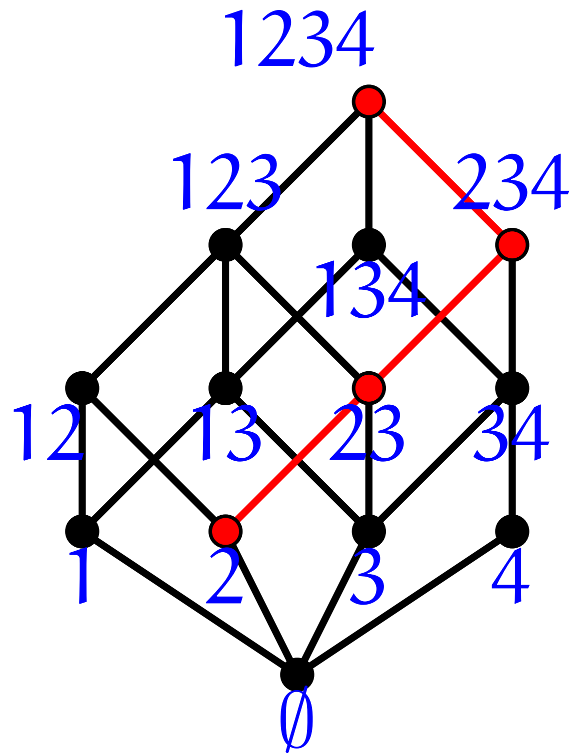

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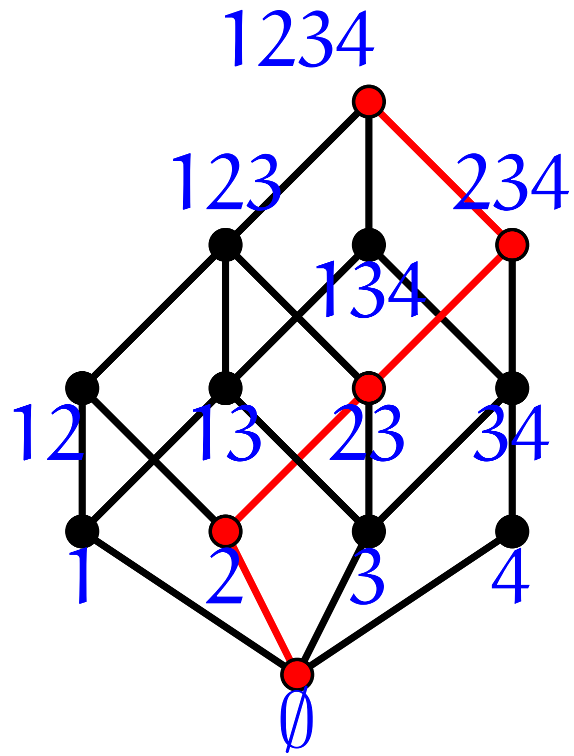

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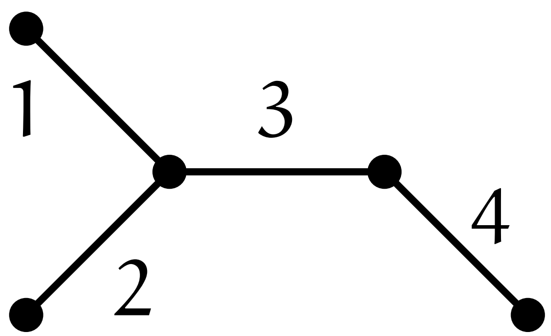
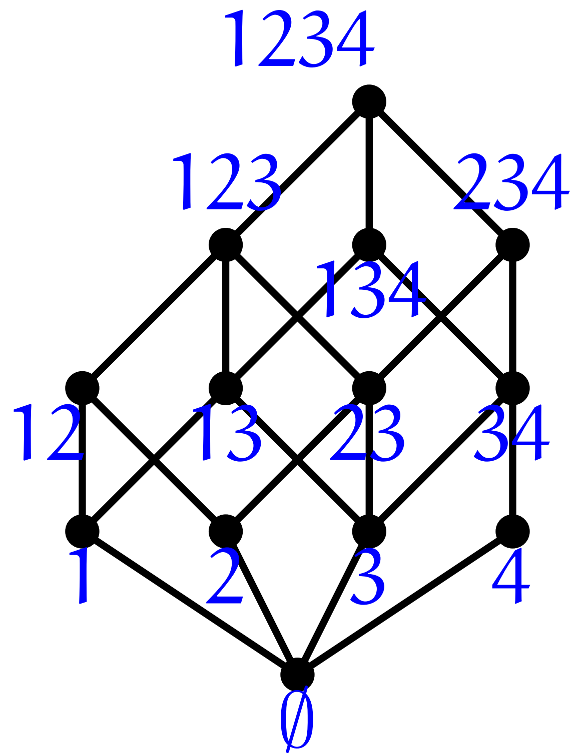


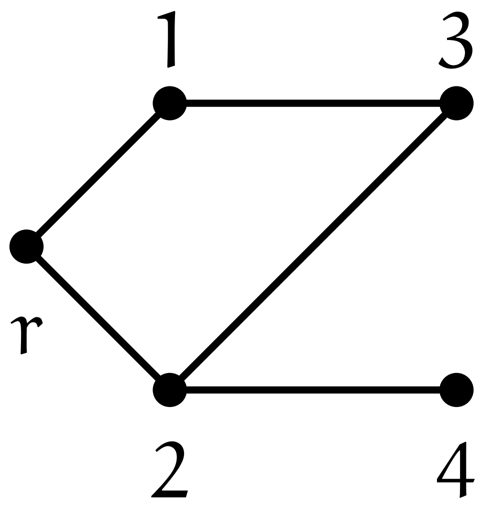


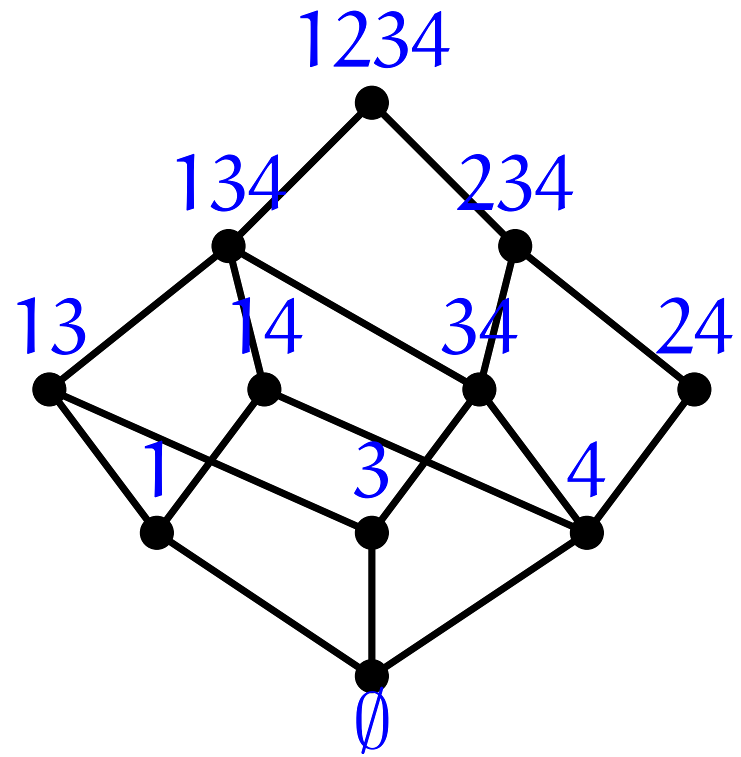
(Remove an edge incident to a leaf one by one)

Given $G = (V \cup \{r\}, E)$ a graph, r a root

Def. $\mathcal{L}$ the **graph search** on G:

$$\boldsymbol{\mathcal{L} = \{V \setminus X : \forall v \in X \ \exists \text{ an } (r,v)\text{-path in } G[X \cup \{r\}]\}}$$





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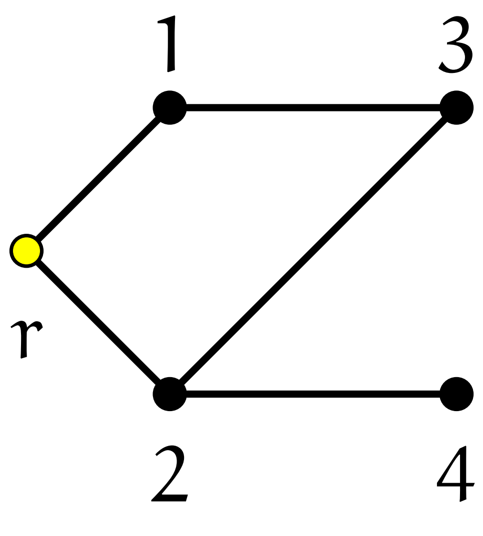

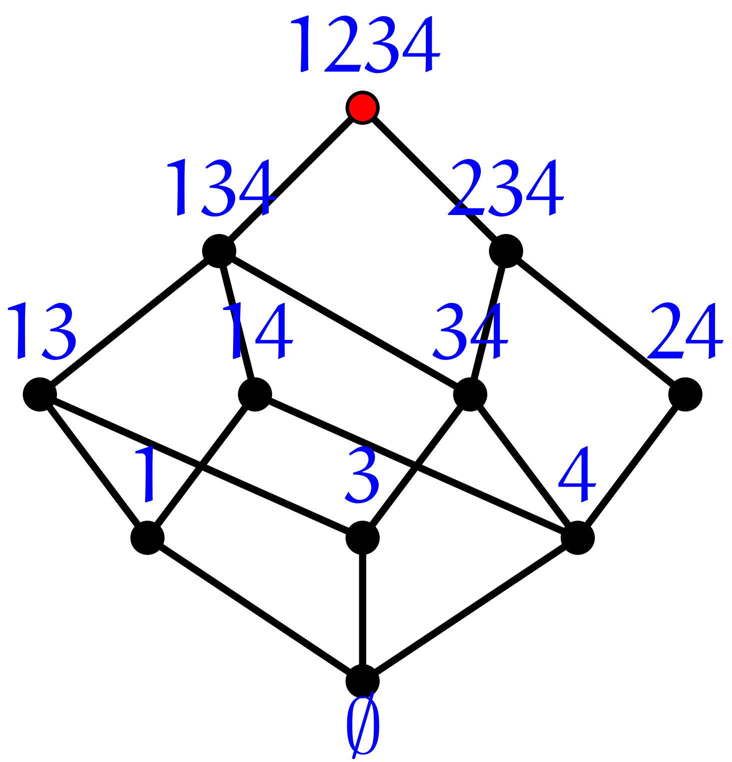

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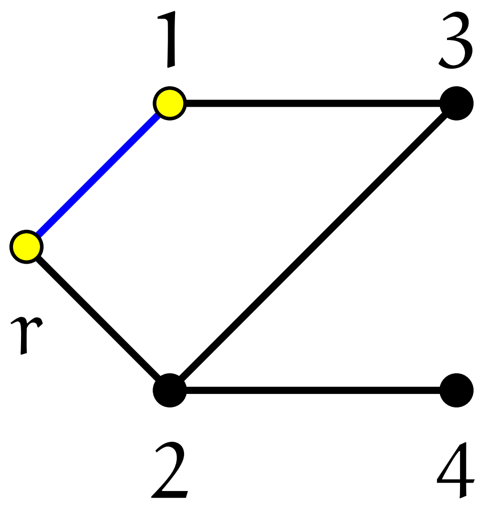

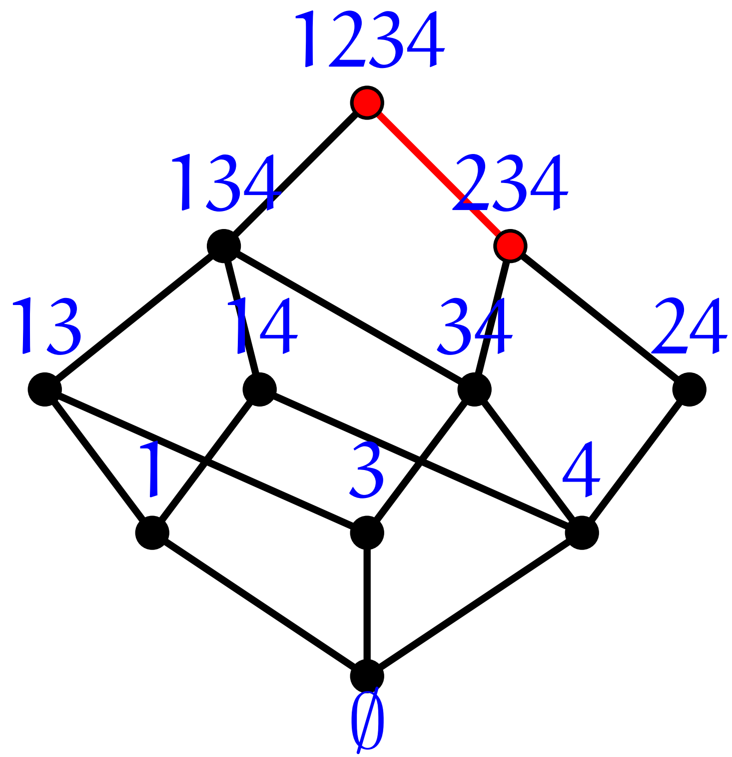

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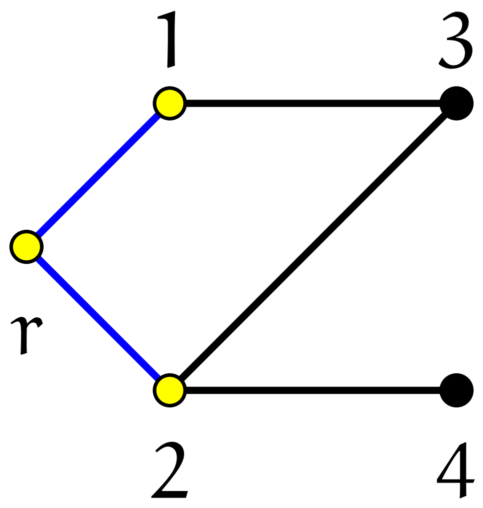

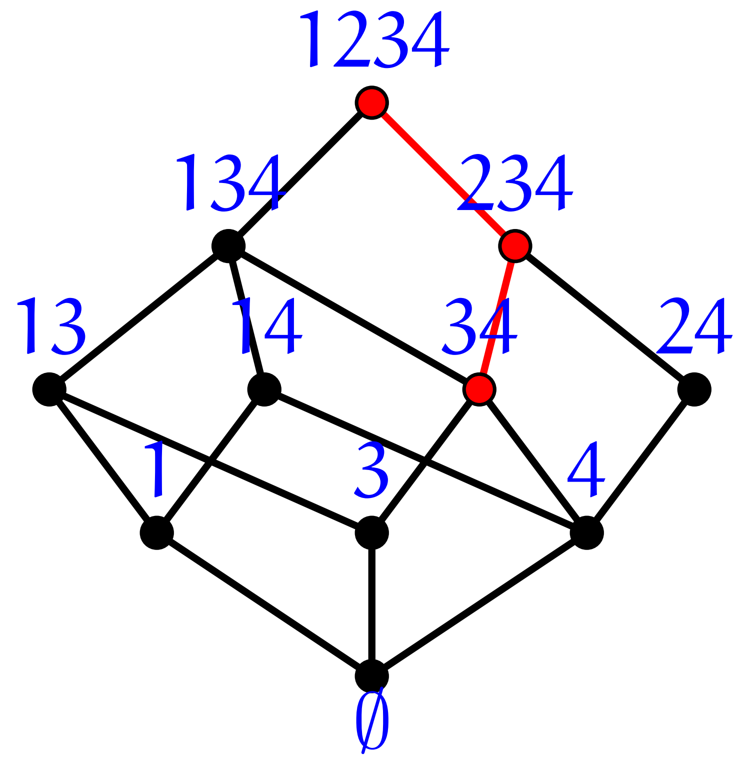

(Consider the search on a graph)

Given $\quad G = (V \cup \{r\}, E)$ a graph, r a root

Def. $\quad \mathcal{L}$ the **graph search** on G:

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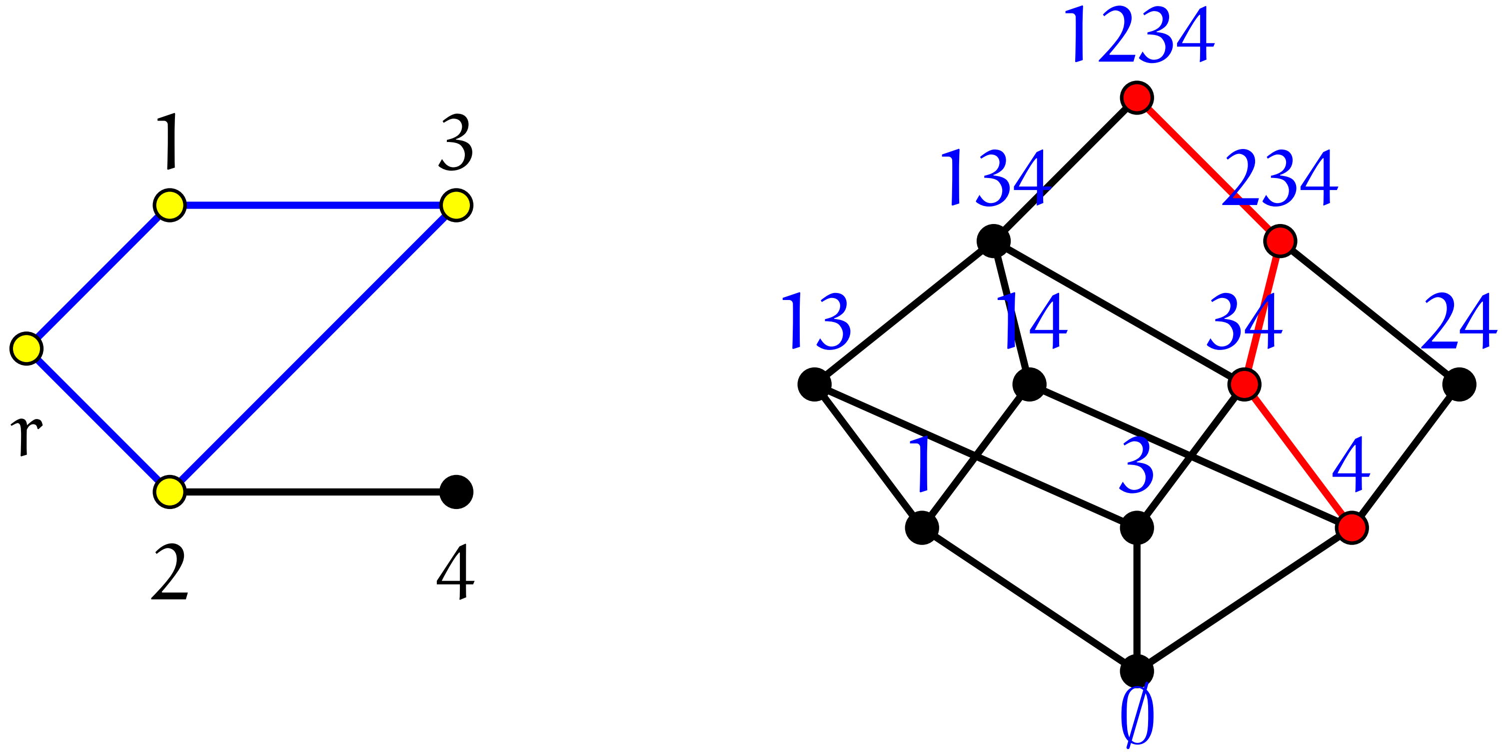

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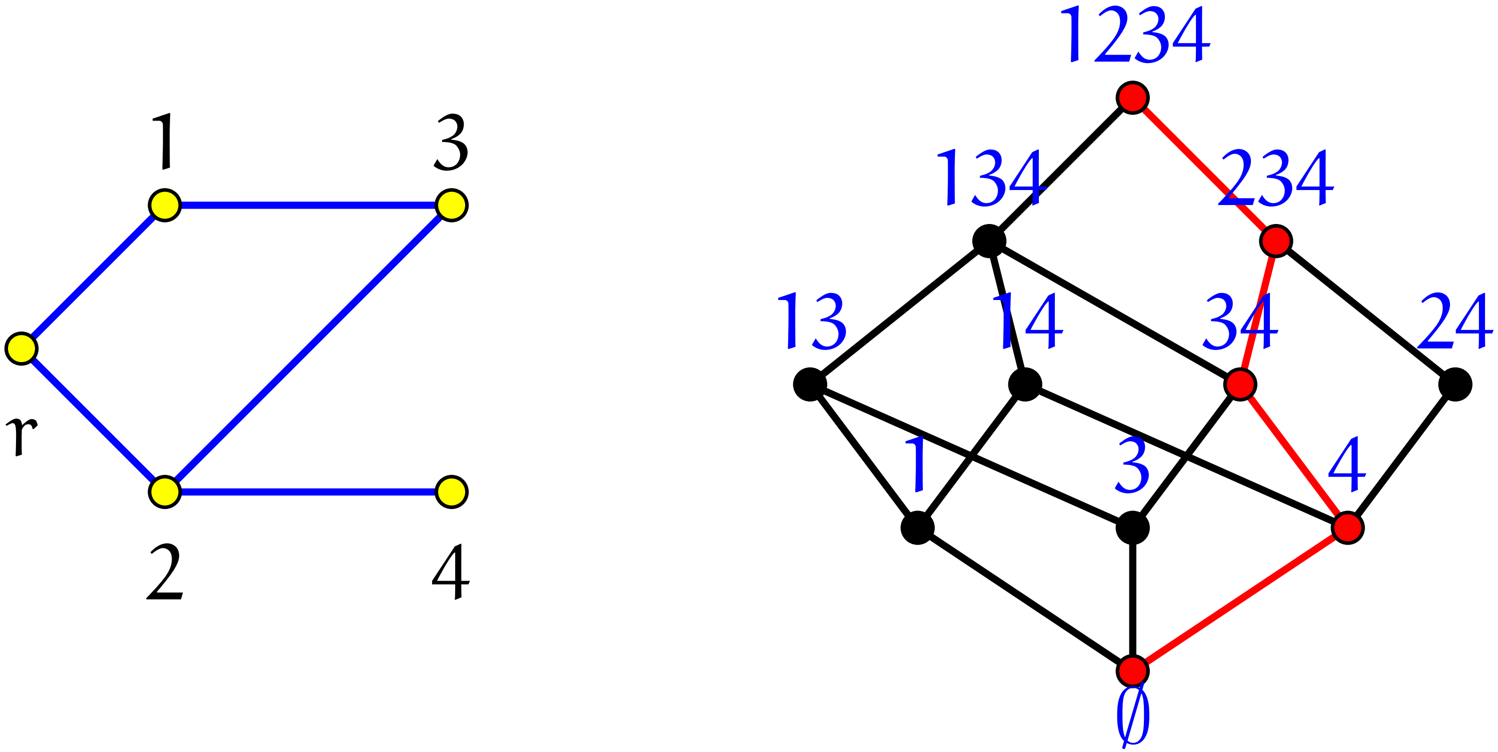

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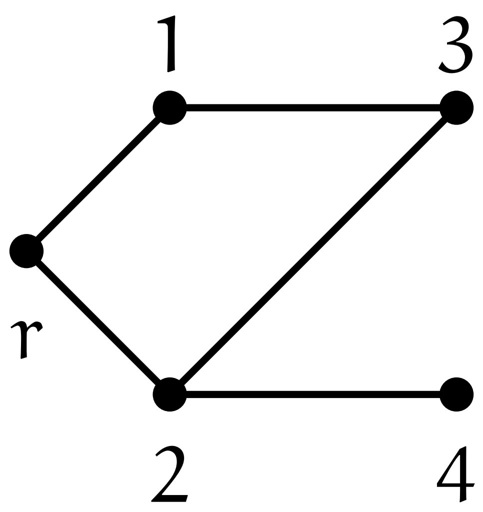

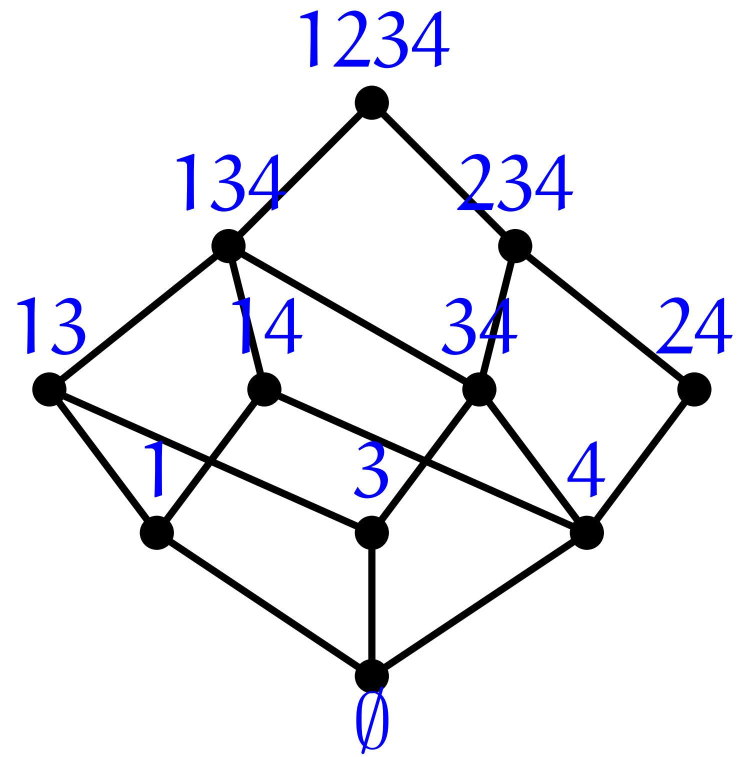

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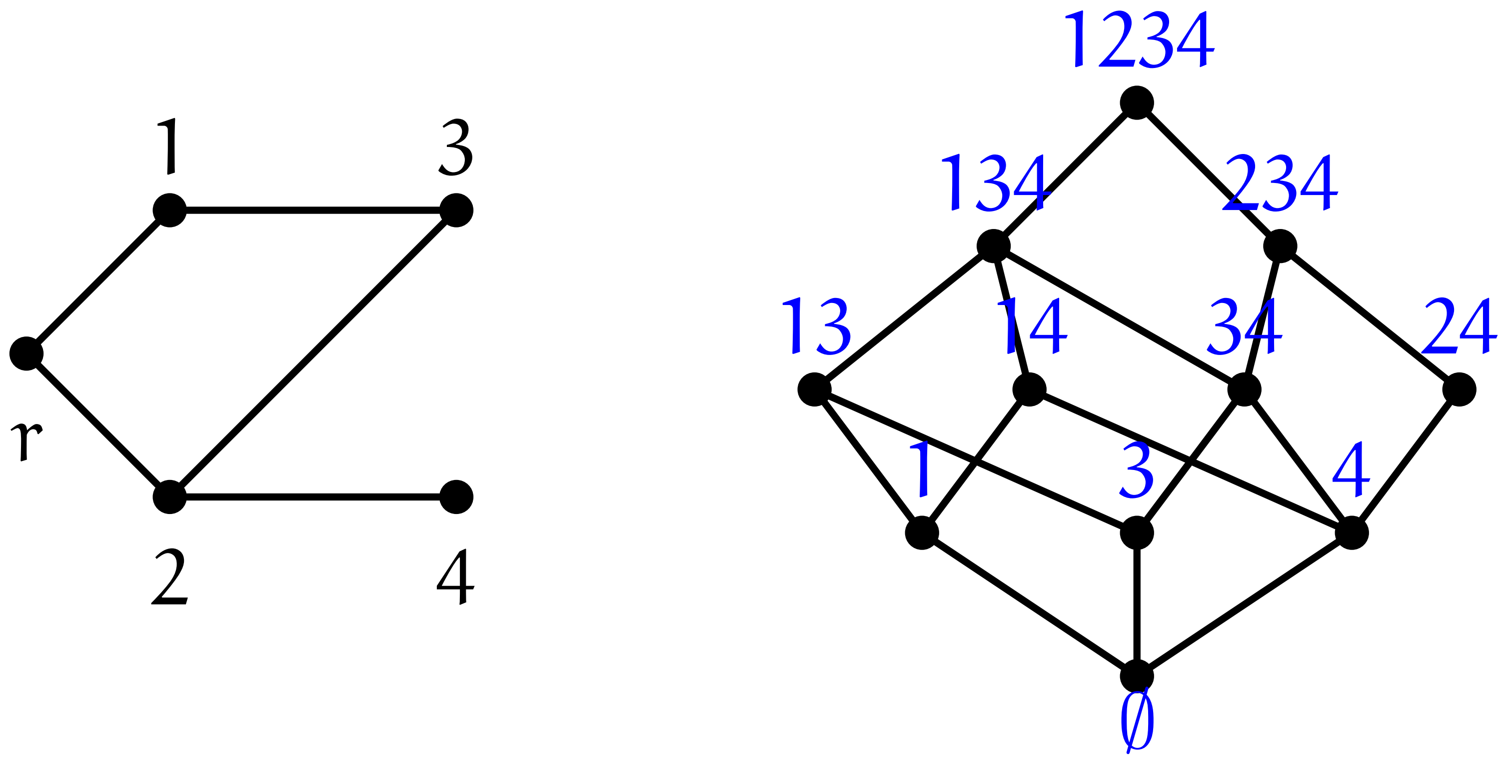

Rem. undirected/directed graph, point/line-search

There are many other examples...

- From graphs
  - The family of connected subgraphs in a block graph (Jamison-Waldner '81)
  - The family of monophonically convex sets in a chordal graph (Farber & Jamison '86)
  - The family of geodecically convex sets in a Ptolemaic graph (Farber & Jamison '86)
  - The family of m^3-convex sets in an HDDA-free graph (Dragan, Nicolai & Branstädt '99)
- From partially ordered sets
  - The family of order convex sets in a poset
  - The family of k-antichains in a poset (Greene & Kleitman '76)
  - The family of subsemilattices in a semilattice (Jamison-Waldner '78)
- From geometry
  - Lower convex shelling on a finite point set
  - Convex shelling on an acyclic oriented matroid (Edelman '82)
- From matroids
  - Line-search in a matroid (Goecke, Korte & Lovász '89)

and more!

- Understanding “Convexity”
  in an axiomatized combinatorial setting
- Counterpart of matroids
- Equivalent to antimatroids (and others)
- Mathematical social science, mathematical psychology
- AND/OR networks, scheduling, project planning
- Directed hypergraphs
- Horn CNF formulas
- Lattice theory...

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Biased introduction to (abstract) convex geometries

Setup $\mathcal{L}$ a convex geometry on E

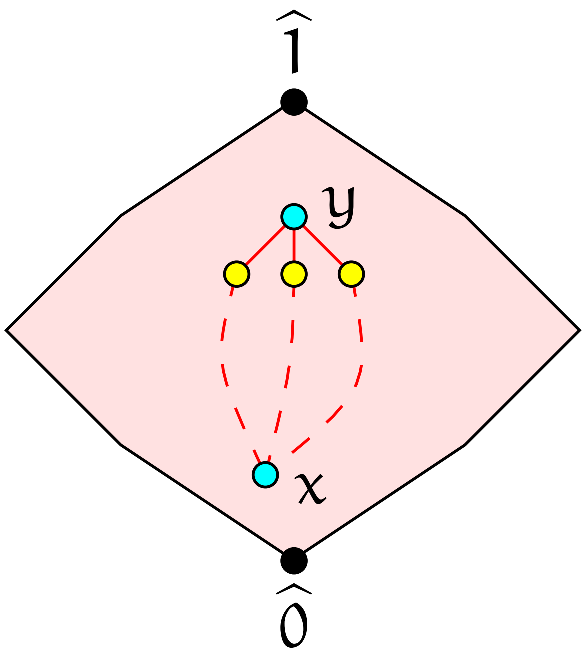
Obs. **$\mathcal{L}$ forms a lattice with $\subseteq$.**

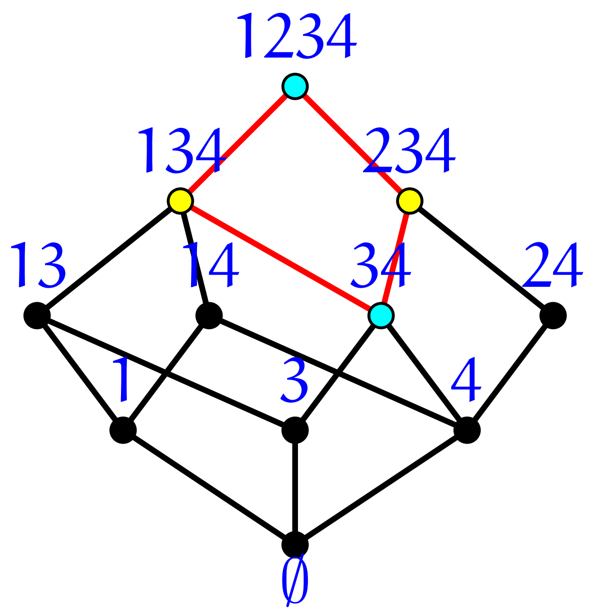
The maximum element $= E$.
The minimum element $= \emptyset$.
The meet of $X, Y \in \mathcal{L}$ $= X \cap Y \in \mathcal{L}$ (by **(2)**).

Setup $\mathcal{L}$ a convex geometry on E

Obs. **$\mathcal{L}$ forms a lattice with $\subseteq$.**

Def. A lattice L is **meet-distributive** $\Leftrightarrow$ for every $x, y \in L$ such that x is the meet of elements y covers, $[x, y]$ is Boolean





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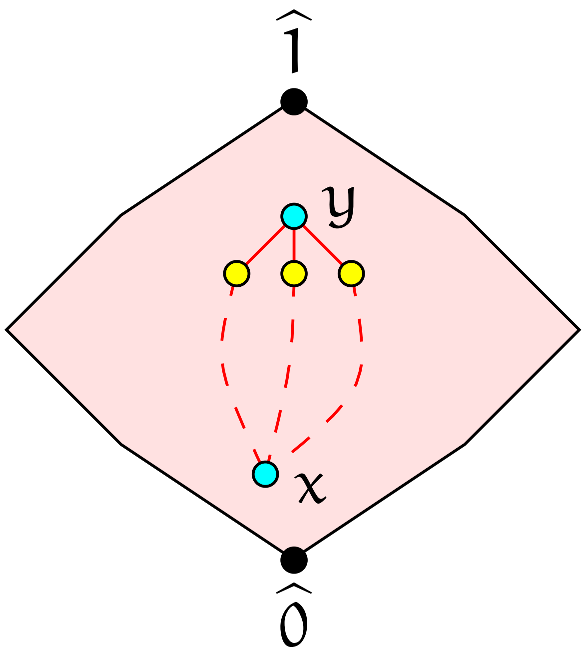

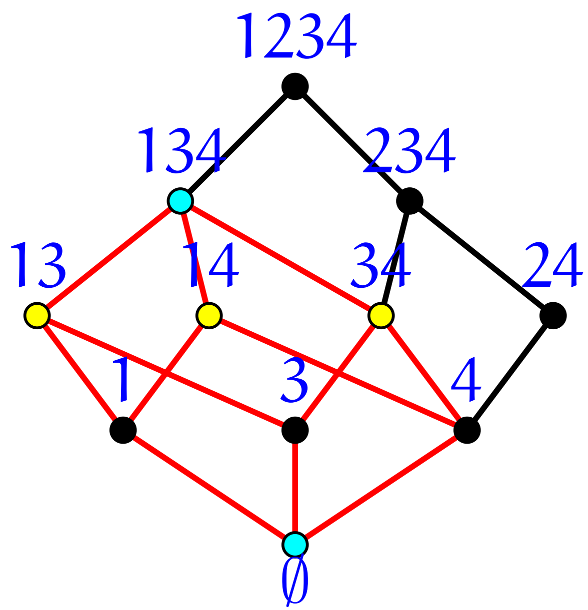

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Obs. $\mathcal{L}$ is meet-distributive.

Thm. (Edelman '80)
$\forall$ finite meet-distributive lattice L
$\exists$ a convex geometry $\mathcal{L}$ on E s.t. $L \cong \mathcal{L}$.

Convex geometries $\equiv$ Meet-distributive lattices

$\boxed{\text{Corresp.}}$ Set systems $\equiv$ Lattices

$\boxed{\text{Thm.}}$ (Edelman '80)

Convex geometries $\equiv$ Meet-distributive

Digression: similar relations in combinatorics

$\boxed{\text{Corresp.}}$	Set systems	$\equiv$	Lattices
$\boxed{\text{Thm.}}$	(Edelman '80)		
	Convex geometries	$\equiv$	Meet-distributive
$\boxed{\text{Thm.}}$	(Birkhoff '33)		
	Poset shellings	$\equiv$	Distributive

Digression: similar relations in combinatorics

Corresp.	Set systems	$\equiv$	Lattices
Thm.	(Edelman '80)		
	Convex geometries	$\equiv$	Meet-distributive
Thm.	(Birkhoff '33)		
	Poset shellings	$\equiv$	Distributive
Thm.	(Birkhoff '35; Whitney '35)		
	Matroids	$\equiv$	Geometric

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Thm.	(Campbell '43; Birkhoff & Frink '48)		
	Closure spaces	$\equiv$	General

Corresp.	Set systems	$\equiv$	Lattices	
Thm.	(Edelman '80)			
	Convex geometries	$\equiv$	Meet-distributive	
Thm.	(Birkhoff '33) [characterization of poset shellings]			
	Poset shellings	$\equiv$	Distributive	
Thm.	(Birkhoff '35; Whitney '35)			
	Matroids	$\equiv$	Geometric	
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Biased introduction to (abstract) convex geometries

Aim $\mathcal{L}$ a convex geometry

To find: $\mathcal{L}$ belongs to a certain class

$\Updownarrow$

$\mathcal{L}$ satisfies a certain property

Aim $\mathcal{L}$ a convex geometry

To find: $\mathcal{L}$ belongs to a certain class

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Certain classes??: in this talk

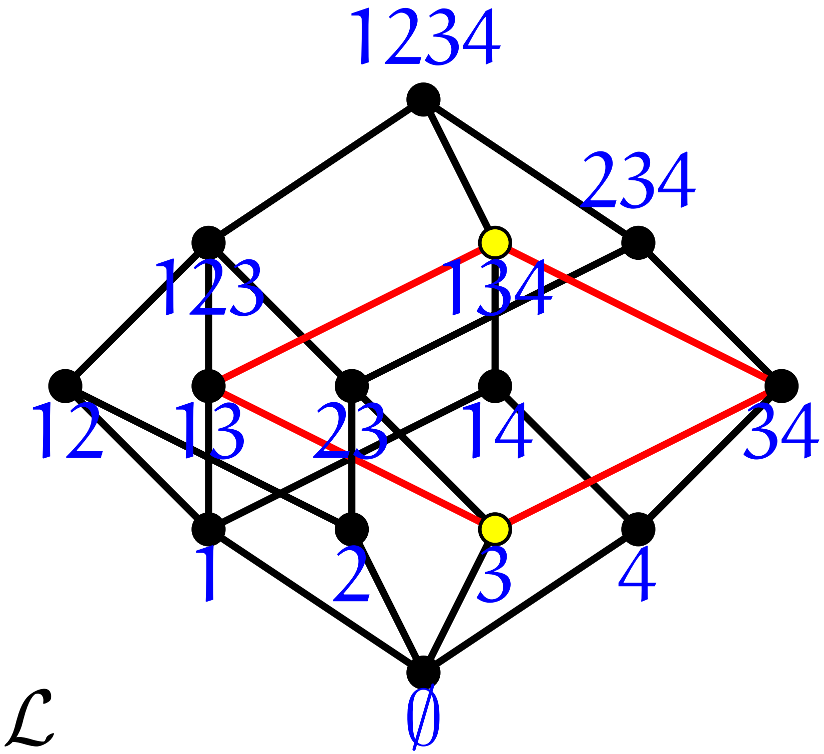
- convex shellings of finite point sets
- poset shellings
- tree shellings
- graph searches

Certain properties??: in this talk

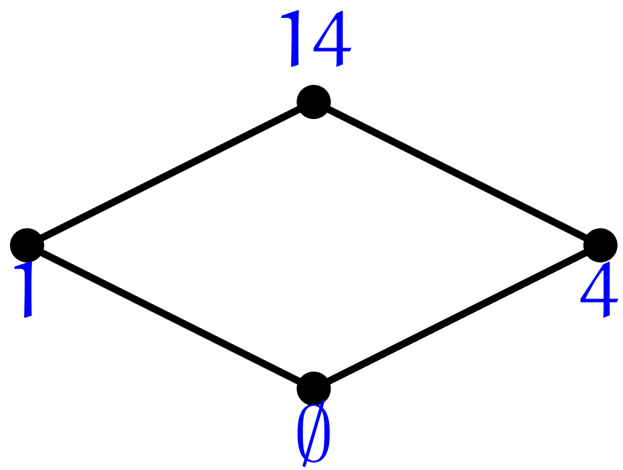
- forbidden minors

Setup $\quad \mathcal{L}$ a convex geometry on E, $A, B \in \mathcal{L}$, $A \subseteq B$

Def. $\quad$ The **minor** of $\mathcal{L}$ w.r.t. (A, B) is defined by
$\mathcal{L}[A, B] = \{X \setminus A : X \in \mathcal{L}, A \subseteq X \subseteq B\}$.



$\mathcal{L}$



$\mathcal{L}[3, 134]$

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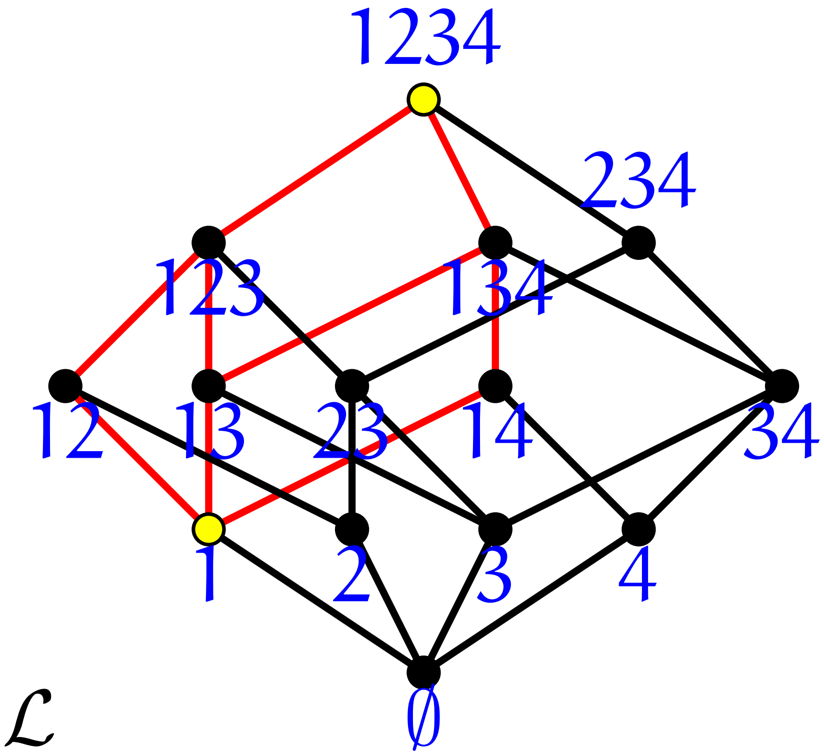

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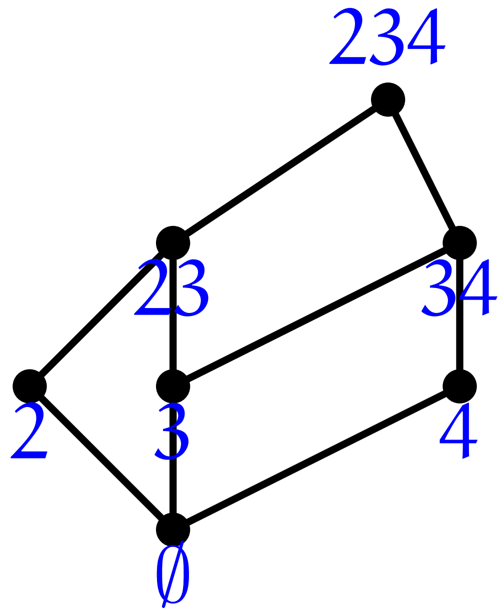

$\mathcal{L}[1, 1234]$

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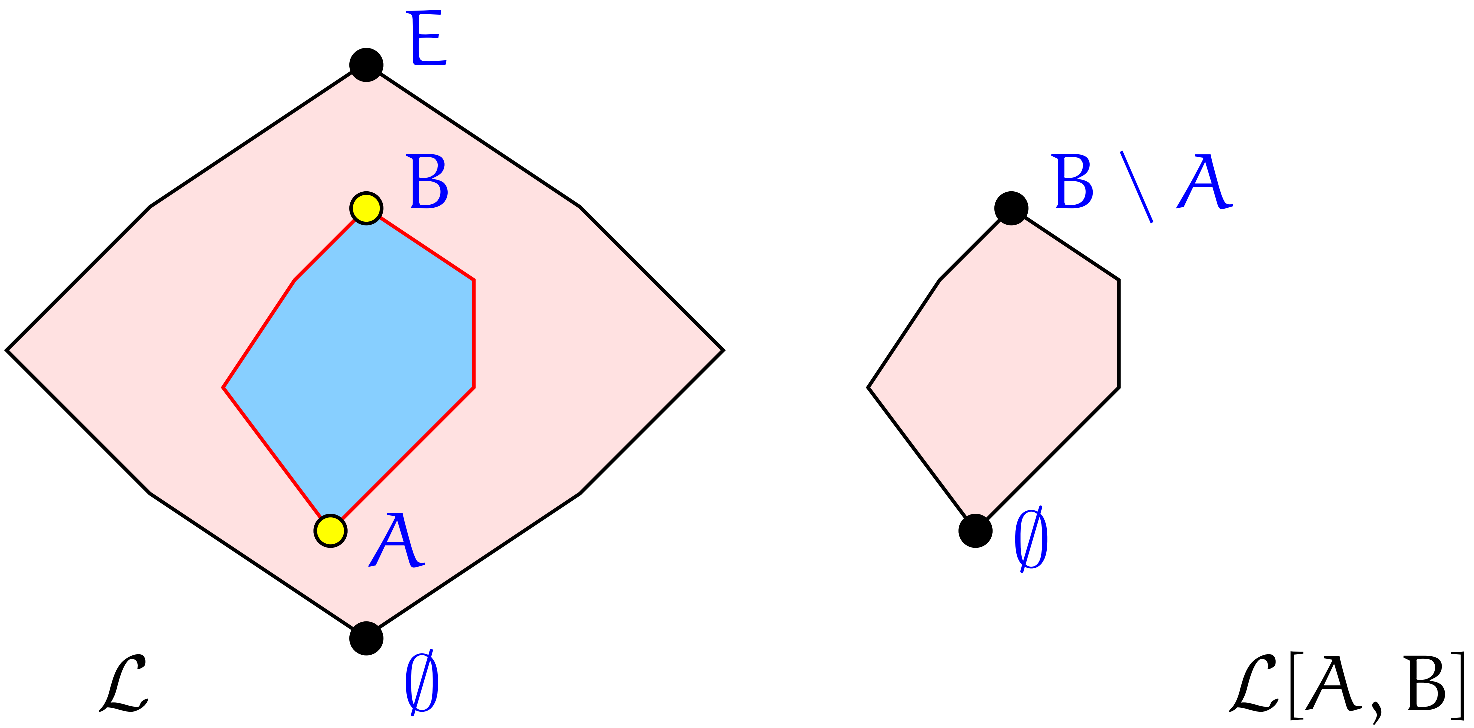


Rem. A minor is also a convex geometry.

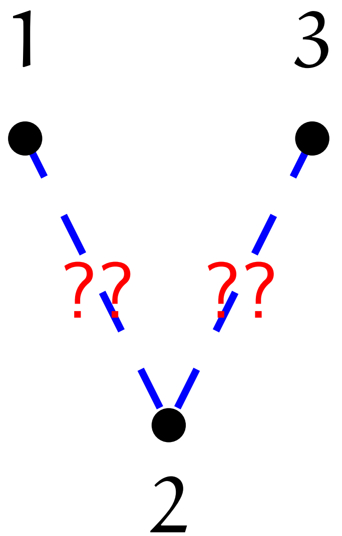
Classes closed under taking minors

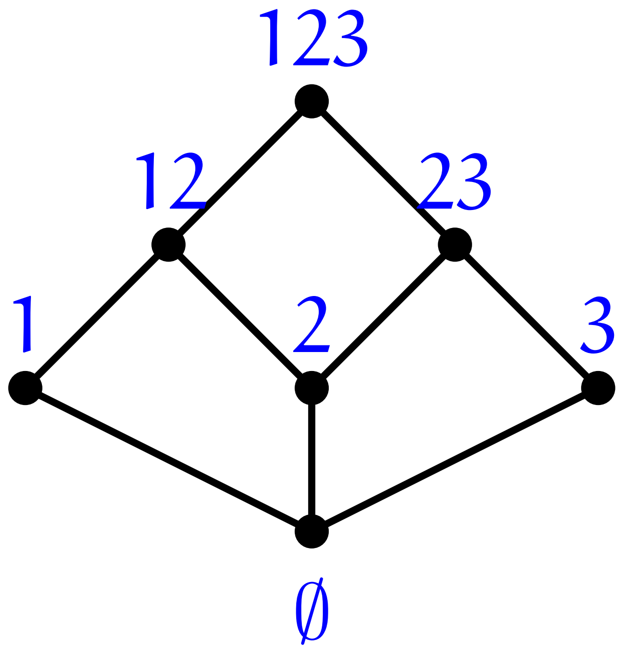
- Poset shellings
- Graph searches
  - directed/point
  - undirected/point
  - directed/line
  - undirected/line

Classes not closed under taking minors

- Convex shellings of finite point sets
- Tree shellings

Obs. This is not a poset shelling.





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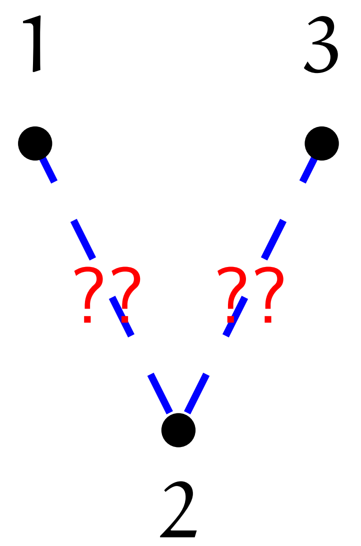
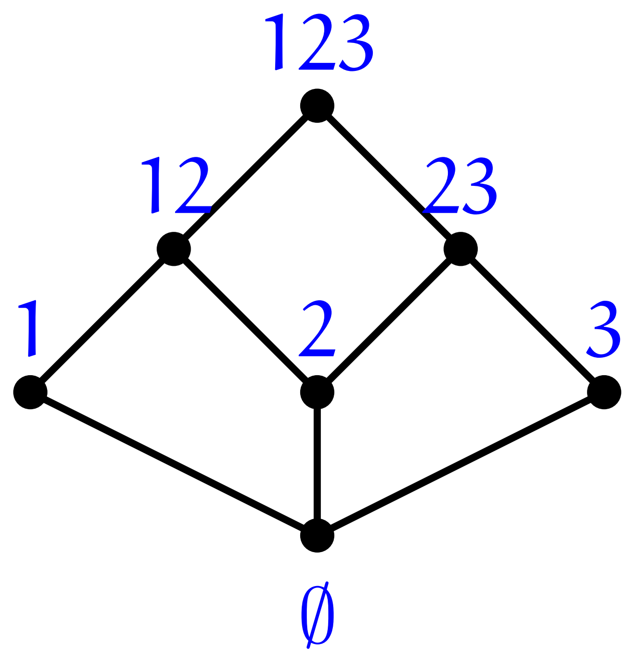
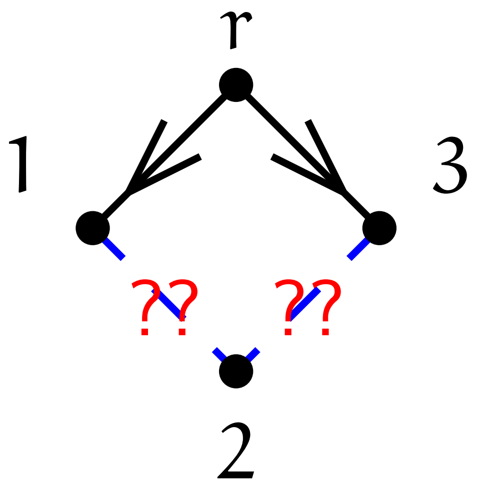


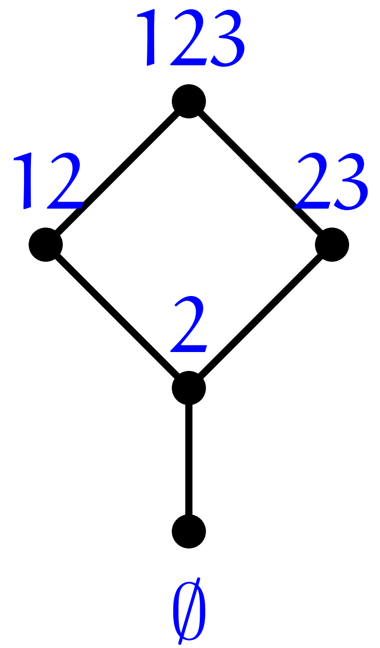


Thm. (Nakamura '03)

$$\mathcal{L} \text{ is a poset shelling}$$
$$\Updownarrow$$
$$\mathcal{L} \text{ contains no minor isomorphic to the one above.}$$

$\boxed{\text{Obs.}}$ This is not a digraph point-search.





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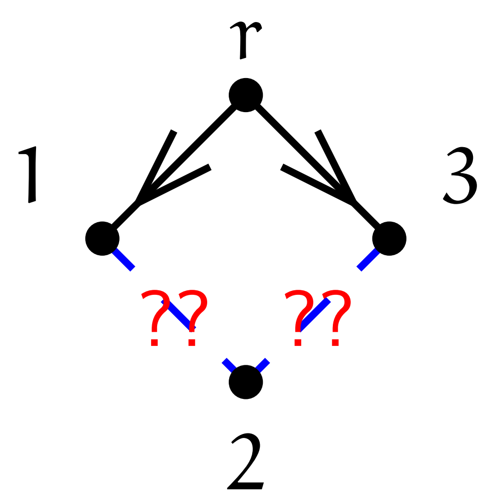

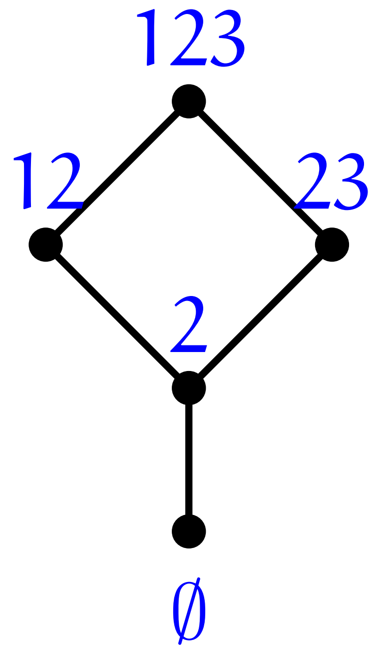

Thm. (Nakamura '03)

$\mathcal{L}$ is a digraph point-search

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$\mathcal{L}$ contains no minor isomorphic to the one above.

Forbidden-minor characterizations: List

- Poset shellings (Nakamura '03)
- Graph searches
  - directed/point (Nakamura '03)
  - undirected/point (Nakamura '03)
  - directed/line (Okamoto & Nakamura '03)
  - undirected/line (OPEN)

Open problem

Forbidden-minor characterization of undirected graph line-searches

Classes closed under taking minors

- Poset shellings
- Graph searches
  - directed/point
  - undirected/point
  - directed/line
  - undirected/line

Classes not closed under taking minors

- Convex shellings of finite point sets
- Tree shellings

Classes closed under taking minors

- Poset shellings
- Graph searches
  - directed/point
  - undirected/point
  - directed/line
  - undirected/line
- Minors of convex shellings of finite point sets
- Minors of tree shellings

Classes not closed under taking minors

- Convex shellings of finite point sets
- Tree shellings

$\boxed{\text{Thm}}$ (Kashiwabara, Nakamura & Okamoto '03)

$$\mathcal{L} \text{ is a minor of a convex shelling}$$
$$\Updownarrow$$
$$\mathcal{L} \text{ is a convex geometry.}$$

$\boxed{\text{Thm}}$ (Kashiwabara, Nakamura & Okamoto '03)

$$\mathcal{L} \text{ is a minor of a convex shelling}$$
$$\Updownarrow$$
$$\mathcal{L} \text{ is a convex geometry.}$$

In other words,

Every convex geometry is
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More precisely speaking, ...

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For any convex geometry $\mathcal{L}$,
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such that $\mathcal{L} \cong \mathcal{L}'[\mathrm{A}, \mathrm{B}]$,
where $\mathcal{L}'$ is the convex shelling on $\mathcal{P}$ and
$\mathrm{A}, \mathrm{B} \in \mathcal{L}'$, $\mathrm{A} \subseteq \mathrm{B}$.

Thm (Kashiwabara, Nakamura & Okamoto '03)

$\forall \mathcal{L}\ \exists \mathcal{P} = \mathcal{P}(\mathcal{L}) \subseteq \mathbb{R}^d$, for some $d = d(\mathcal{L})$,
such that $\mathcal{L} \cong \mathcal{L}'[A, B]$,
where $\mathcal{L}'$ is the conv shelling on $\mathcal{P}$
$A, B \in \mathcal{L}'$, $A \subseteq B$.

Open problem What is $\min d(\mathcal{L})$?

What's known: let $\mathcal{L}$ be on an n-element set

- $\forall \mathcal{L}$: $\min d(\mathcal{L}) \leq n - 1$
- $\exists \mathcal{L}$: $\min d(\mathcal{L}) \geq n - 2$

Thm (Kashiwabara, Nakamura & Okamoto '03)

$\forall \mathcal{L}\ \exists \mathcal{P} = \mathcal{P}(\mathcal{L}) \subseteq \mathbb{R}^{d}$, for some $d = d(\mathcal{L})$,
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Open problem What is $\min |\mathcal{P}(\mathcal{L})|$?

What's known: let $\mathcal{L}$ be on an n-element set

- $\forall \mathcal{L}$: $n \leq \min |\mathcal{P}(\mathcal{L})| \leq n + |\mathcal{C}(\mathcal{L})|$

  ($\mathcal{C}(\mathcal{L})$ the circuits of $\mathcal{L}$, ... defined later)

- Poset shellings (Nakamura '03)
- Graph searches
  - directed/point (Nakamura '03)
  - undirected/point (Nakamura '03)
  - directed/line (Okamoto & Nakamura '03)
  - undirected/line (OPEN)
- Minors of convex shellings (Kashiwabara, Nakamura & Okamoto '03)
- Minors of tree shellings (OPEN)

Open problem Forbidden-minor characterization of minors of tree shellings

$\boxed{\text{Setup}}$ Given a convex geometry $\mathcal{L}$ as the oracle

$$X \longrightarrow \boxed{\text{Oracle}} \longrightarrow \begin{cases} \text{“Yes”} & \text{if } X \in \mathcal{L} \\ \text{“No”} & \text{if } X \notin \mathcal{L} \end{cases}$$

Measure: # of oracle calls

$\boxed{\text{Thm.}}$ (Enright '01)

(1) There is a poly-time algorithm to recognize a poset shelling.

(2) There is no poly-time algorithm to recognize a graph search (directed/point).

(3) There is no poly-time algorithm to recognize a graph search (undirected/point).

Biased introduction to (abstract) convex geometries

Framework

Convex geometries	Matroids
Convex sets	Flats
Closure	Closure
Extreme points	isthmus
Independent sets	Independent sets
Circuits	Circuits
$\vdots$	$\vdots$

Setup $\mathcal{L} \subseteq 2^E$ a convex geometry on E

Def. The **closure operator** of $\mathcal{L}$ is

a mapping $\tau_{\mathcal{L}} : 2^E \to 2^E$ defined as

$$\boldsymbol{\tau_{\mathcal{L}}(A) = \bigcap \{X \in \mathcal{L} : A \subseteq X\}}$$

$$= \textbf{smallest convex set containing } \boldsymbol{A}.$$

Analogy $X \subseteq \mathbb{R}^d$

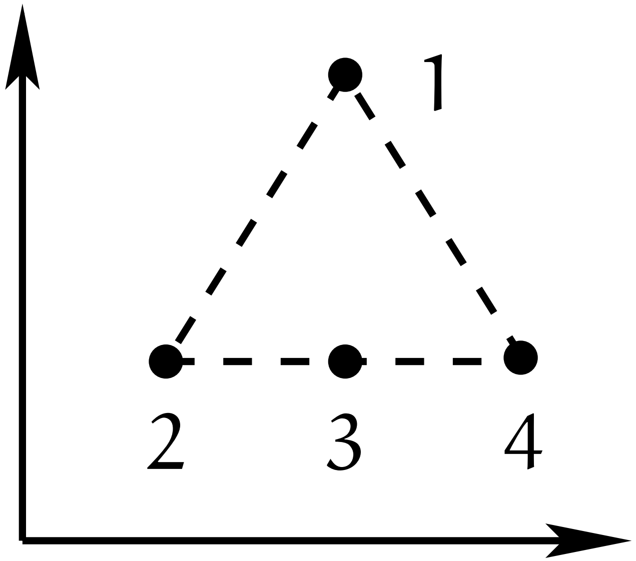
$\mathrm{conv}(X) =$ smallest convex set containing X.

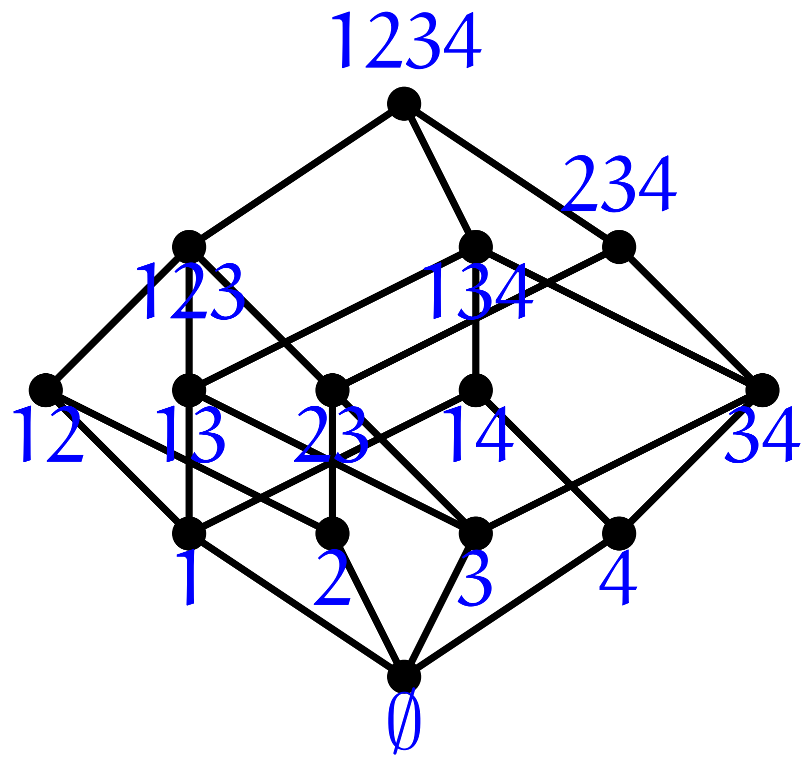
Obs. $A \subseteq \tau_{\mathcal{L}}(A)$ for all $A \subseteq E$

$\boxed{\text{Def.}}$ $\boldsymbol{\tau_{\mathcal{L}}(A)}$ **= smallest convex set containing** $\boldsymbol{A}$**.**

$\boxed{\text{Def.}}$ $\tau_{\mathcal{L}}(A)$ = **smallest convex set containing** A.

$\boxed{\text{E.g.}}$ $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$

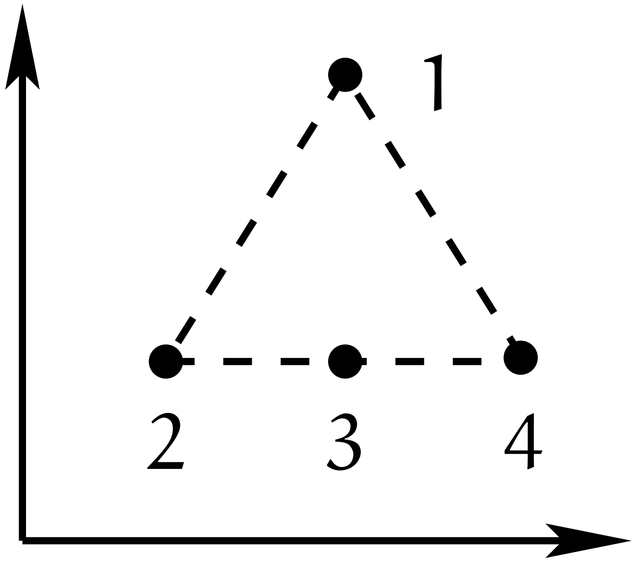


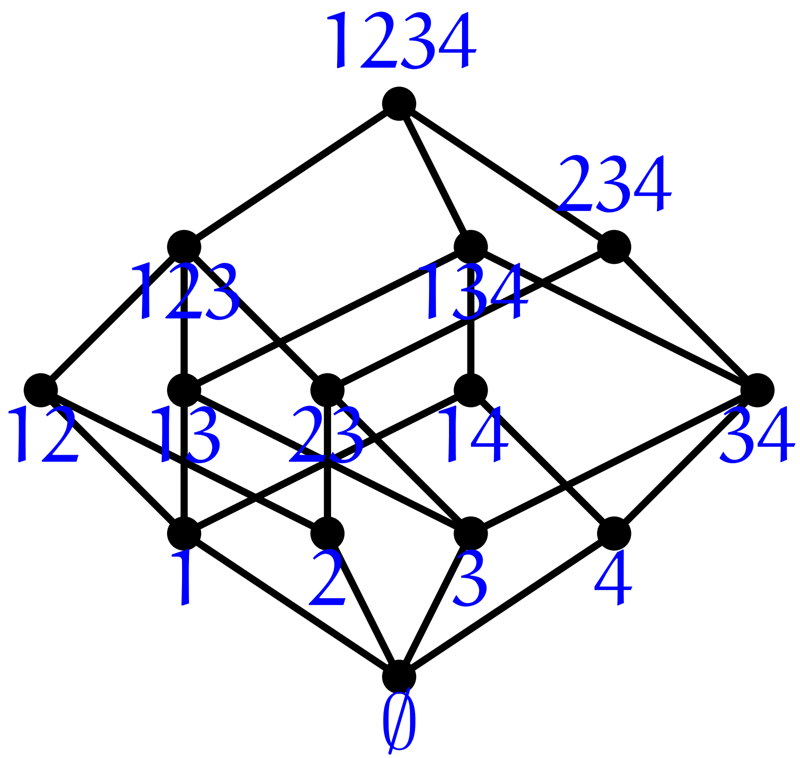


$\tau_{\mathcal{L}}(\{2,4\}) = ??$

Def. $\tau_{\mathcal{L}}(A)$ = **smallest convex set containing** A.

E.g. $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$

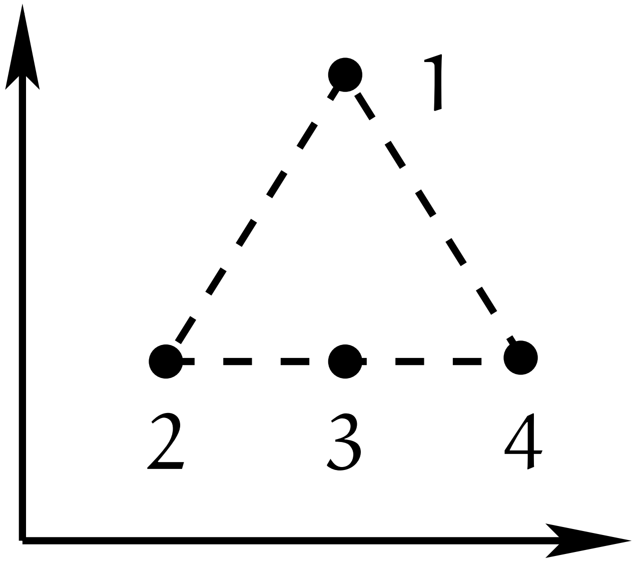


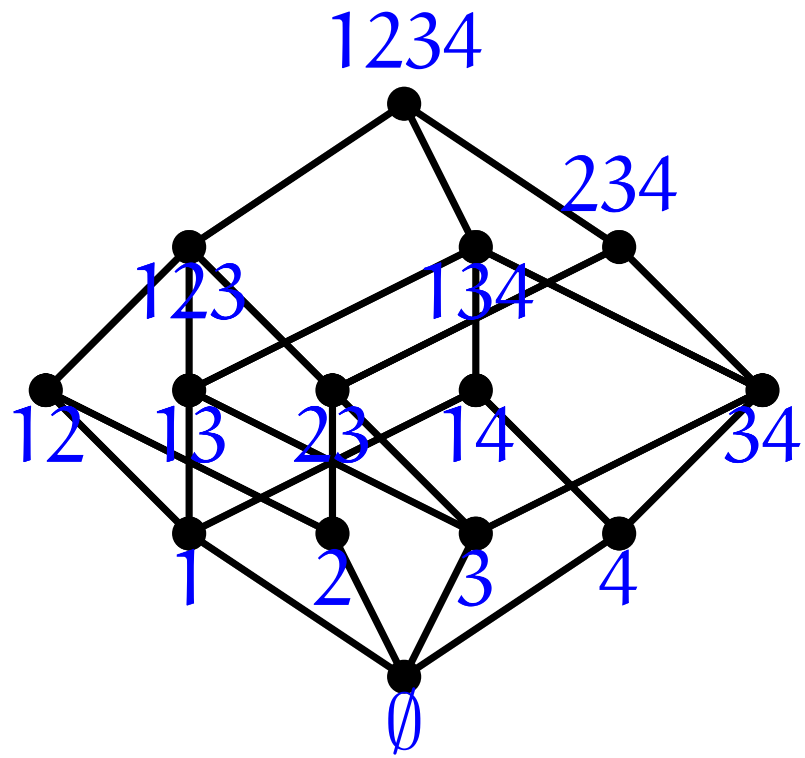


$\tau_{\mathcal{L}}(\{2,4\}) = \{2,3,4\}$.

Def. $\tau_{\mathcal{L}}(A)$ = **smallest convex set containing** A.

E.g. $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$





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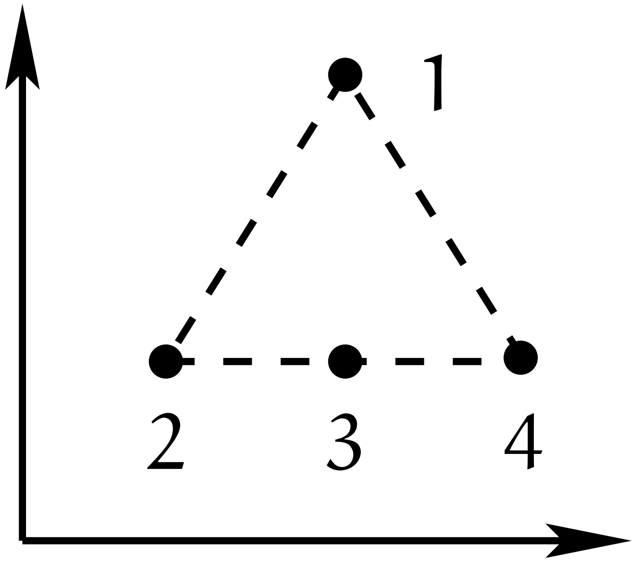

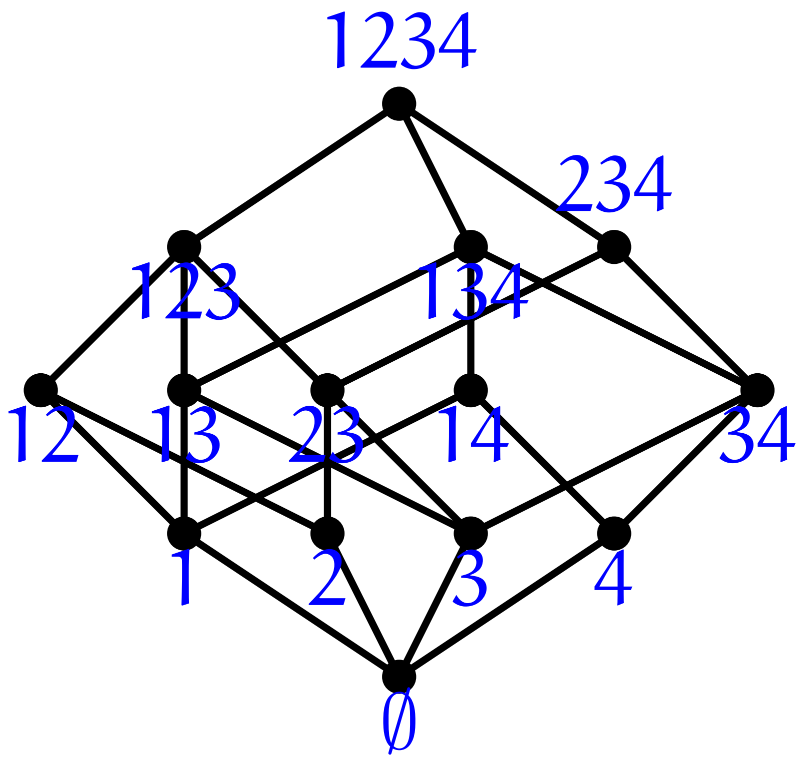

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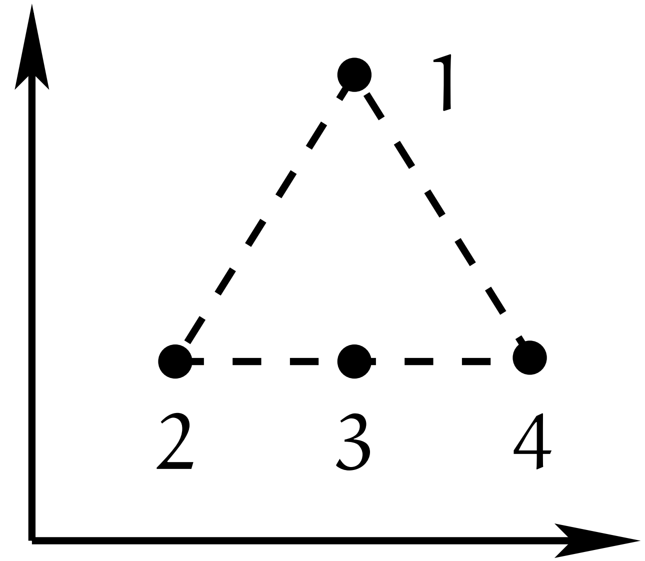

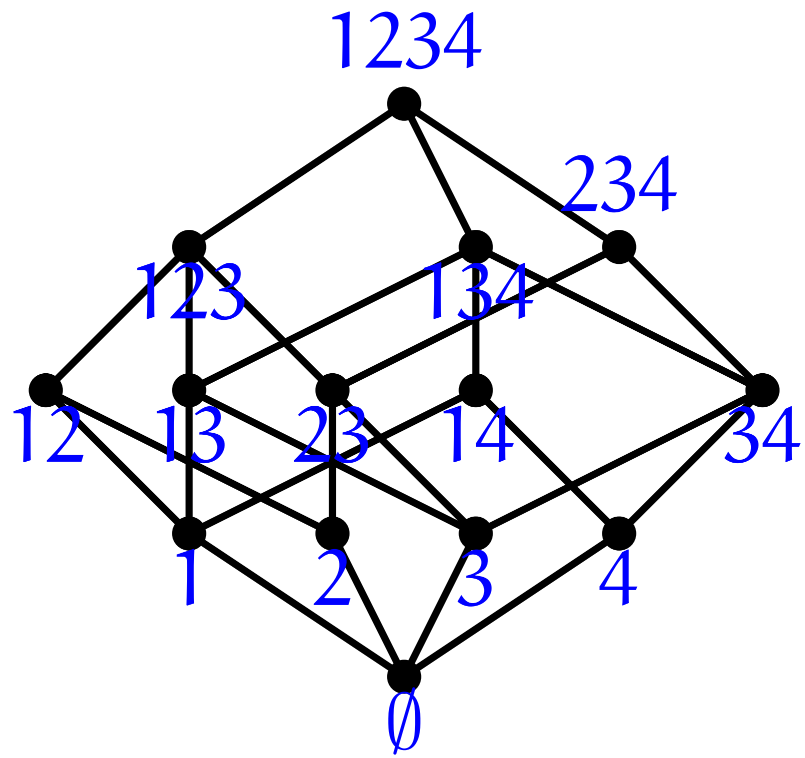

Rem $\mathcal{L} = \{X \subseteq E : X = \tau_{\mathcal{L}}(X)\}$ (in general).

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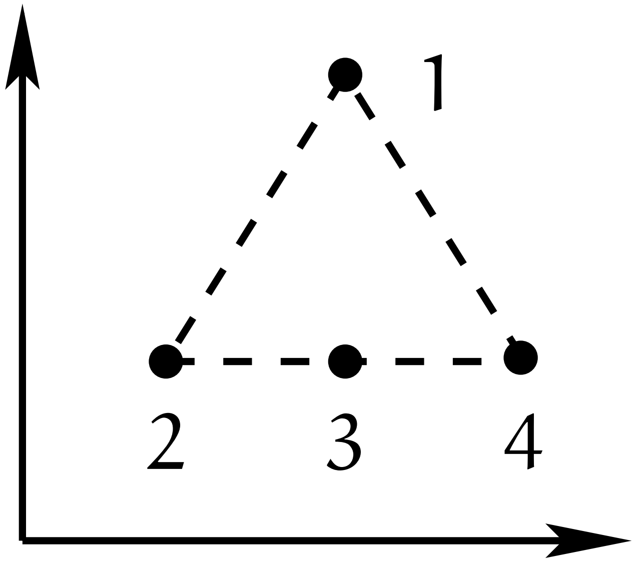

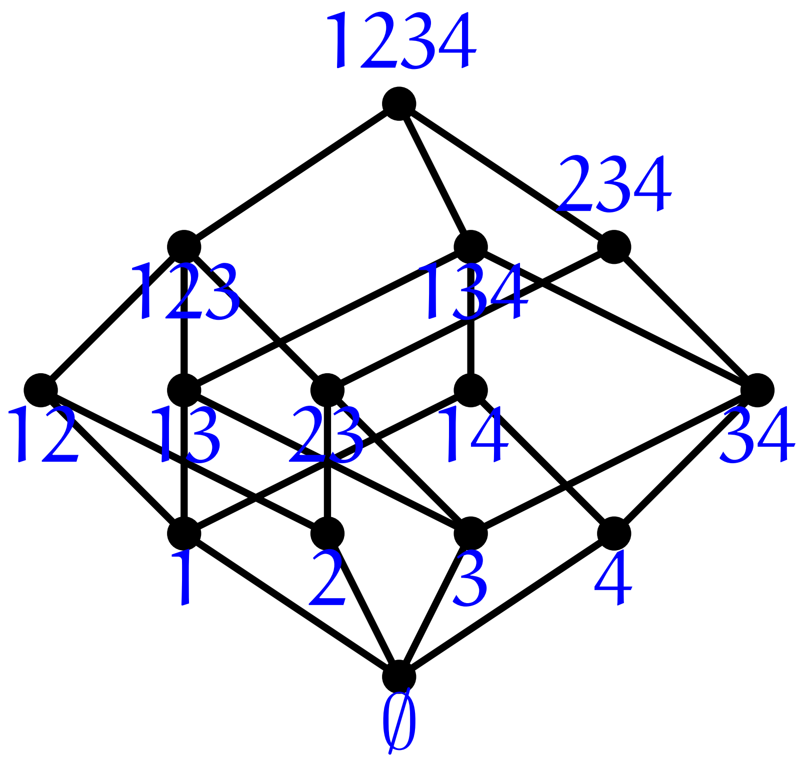

Rem Characterizations of convex geometries by closure operators (Edelman & Jamison '85)

Setup $\tau : 2^E \to 2^E$ a map

Thm (Edelman & Jamison '85)

τ is the closure operator of some convex geometry

$\Updownarrow$

(1) $\tau(\emptyset) = \emptyset$.

(2) $A \subseteq \tau(A)$ **for all** $A \subseteq E$

(3) $A \subseteq B \subseteq E \Rightarrow \tau(A) \subseteq \tau(B)$.

(4) $\tau(\tau(A)) = \tau(A)$ **for all** $A \subseteq E$.

(5) $A \subseteq E, e, f \notin \tau(A), e \neq f,$
$e \in \tau(A \cup \{f\}) \Rightarrow f \notin \tau(A \cup \{e\})$.

Setup $\tau : 2^E \to 2^E$ a map

Cf.

τ is the closure operator of some matroid

$\Updownarrow$

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Setup $\mathcal{L} \subseteq 2^{\mathrm{E}}$ a convex geometry on E

Def. The **extreme point operator** of $\mathcal{L}$ is
a mapping $\mathrm{ex}_{\mathcal{L}} : 2^{\mathrm{E}} \to 2^{\mathrm{E}}$ defined as

$$\mathbf{ex}_{\mathcal{L}}(\boldsymbol{A}) = \{\boldsymbol{e} \in \boldsymbol{A} : \boldsymbol{e} \notin \boldsymbol{\tau}_{\mathcal{L}}(\boldsymbol{A} \setminus \{\boldsymbol{e}\})\}.$$

Analogy $\mathrm{X} \subseteq \mathbb{R}^{\mathrm{d}}$ a convex polyhedron (pointed)
$\mathrm{vert}(\mathrm{X})$ = the set of vertices of X.

Obs. $\mathrm{ex}_{\mathcal{L}}(A) \subseteq A$ for all $A \subseteq \mathrm{E}$

$\boxed{\text{Def.}}$ $$\mathbf{ex}_{\mathcal{L}}(\boldsymbol{A}) = \{\boldsymbol{e} \in \boldsymbol{A} : \boldsymbol{e} \notin \boldsymbol{\tau}_{\mathcal{L}}(\boldsymbol{A} \setminus \{\boldsymbol{e}\})\}.$$

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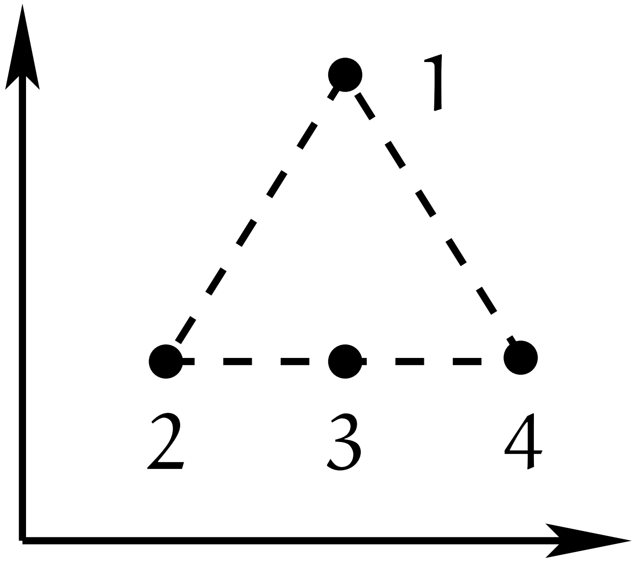

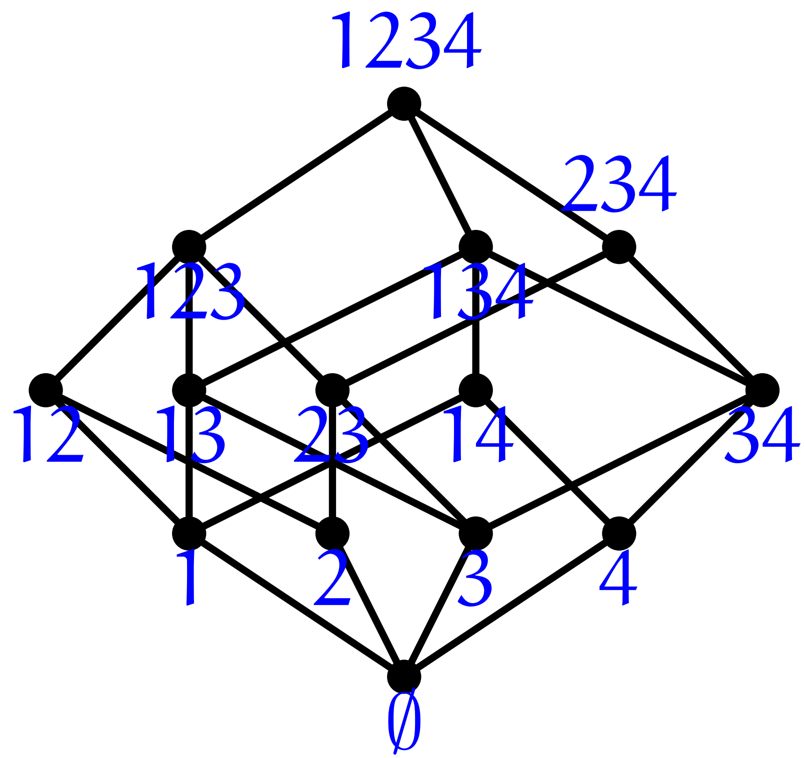

$\mathrm{ex}_{\mathcal{L}}(\{1,2,3,4\}) = ??$

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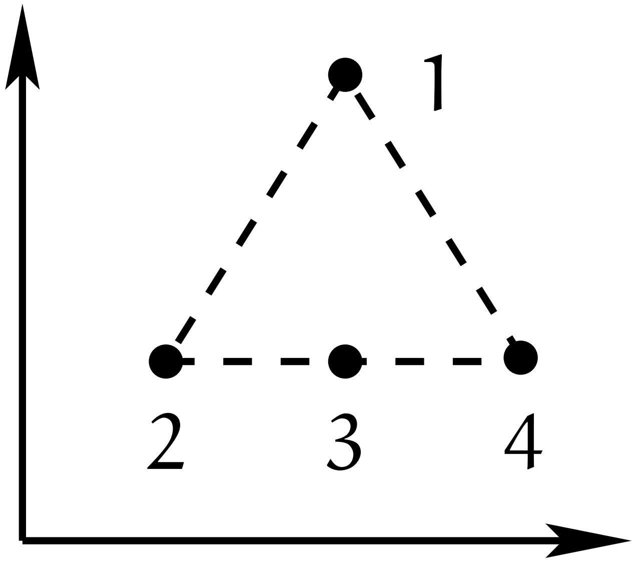

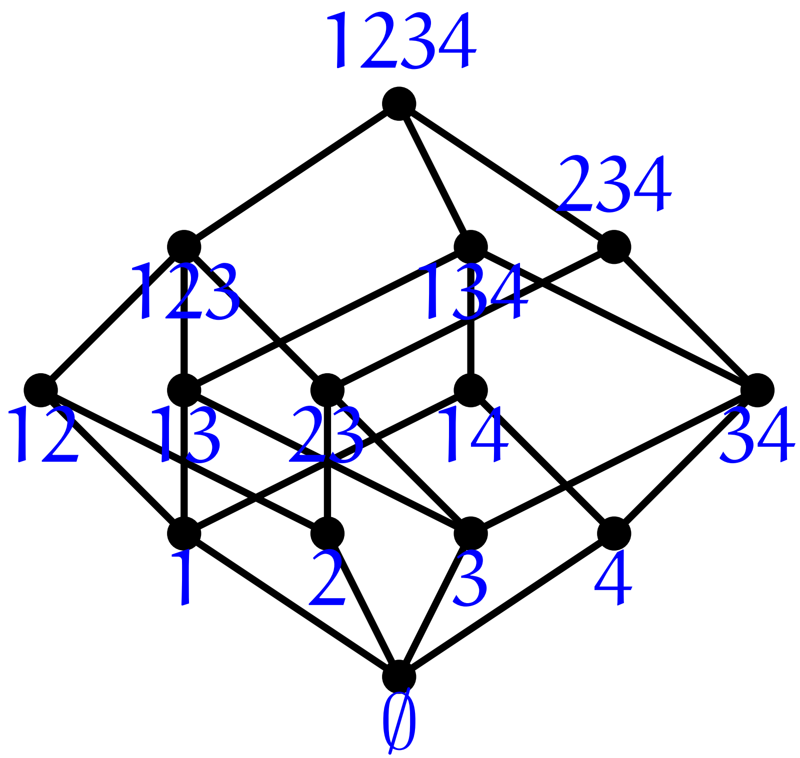

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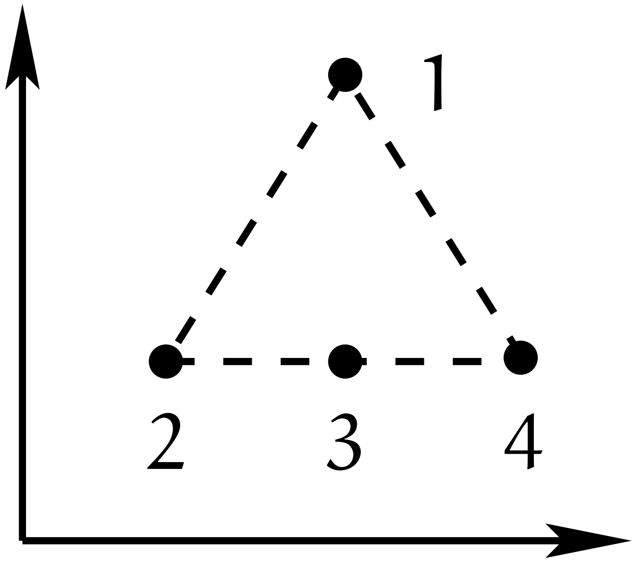

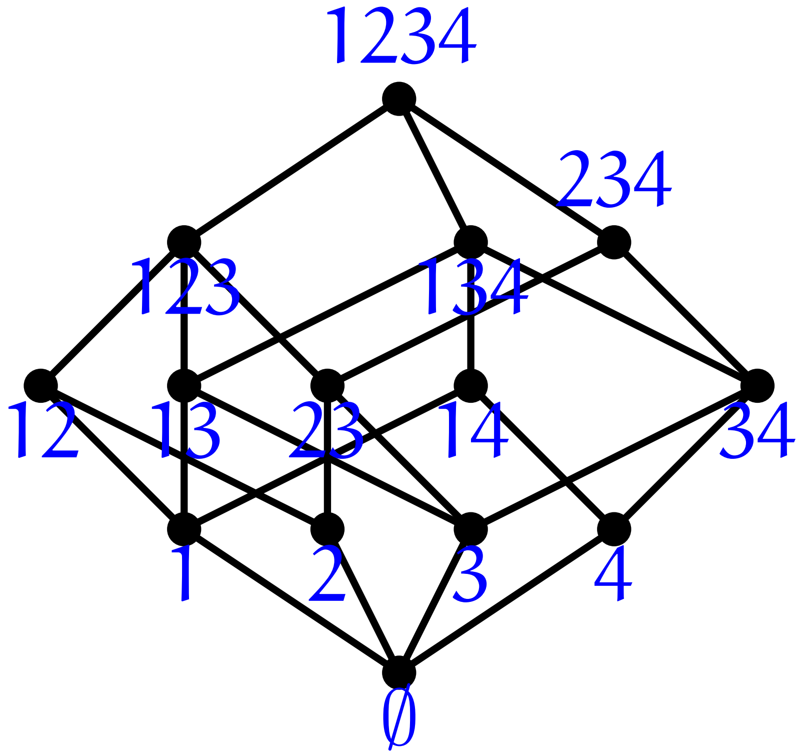

$\mathrm{ex}_{\mathcal{L}}(\{2,4\}) = \{2,4\}$.

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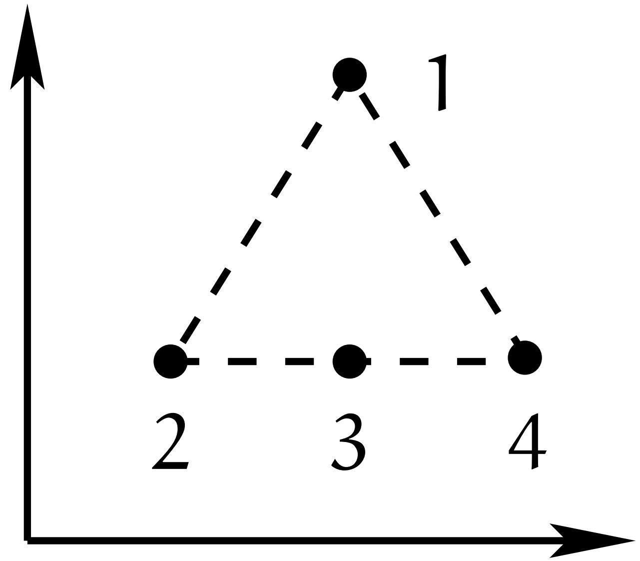

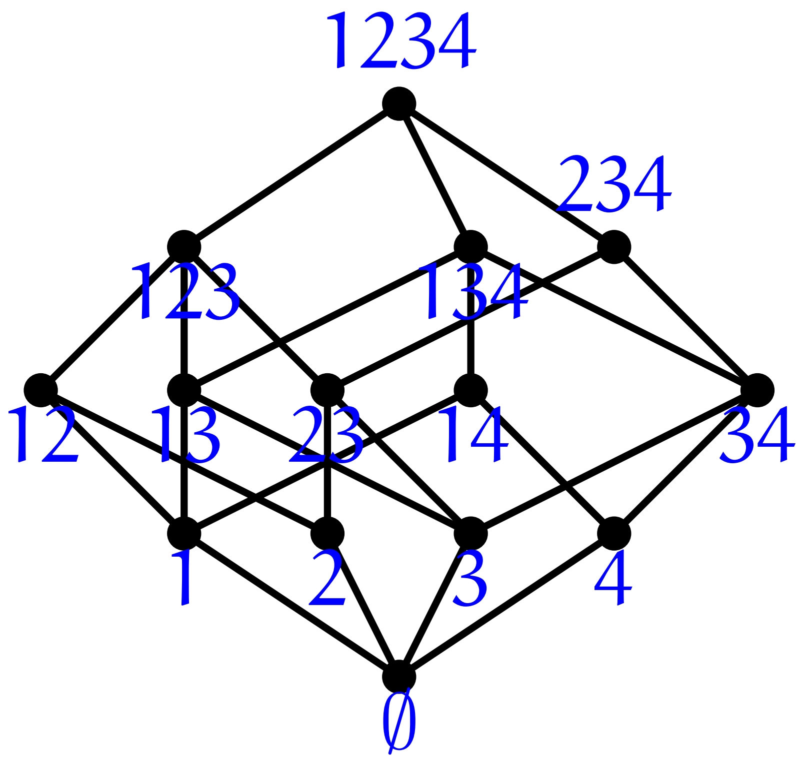

$\mathrm{ex}_{\mathcal{L}}(\{1\}) = \{1\}$

Def. $\mathbf{ex}_{\mathcal{L}}(\boldsymbol{A}) = \{\mathbf{e} \in \boldsymbol{A} : \mathbf{e} \notin \boldsymbol{\tau}_{\mathcal{L}}(\boldsymbol{A} \setminus \{\mathbf{e}\})\}.$

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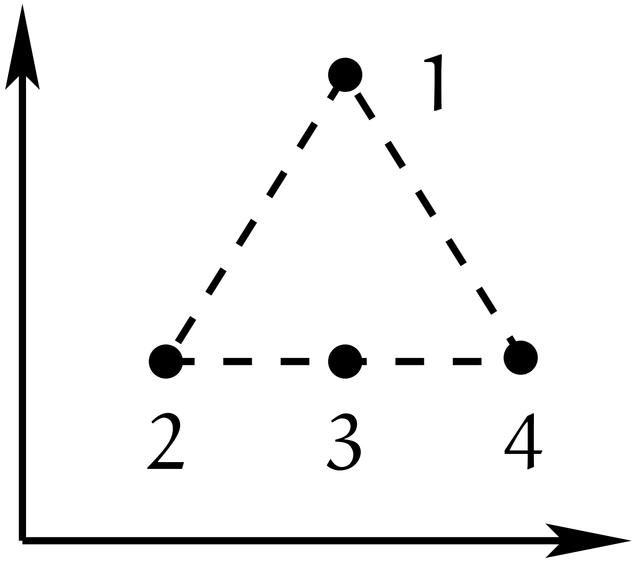

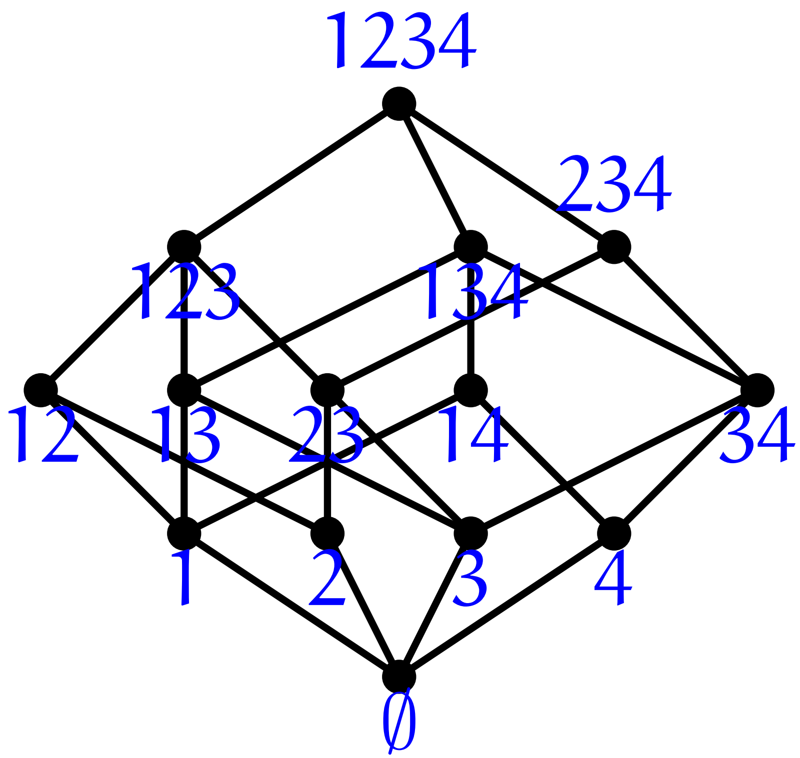

Rem $\mathcal{L} = \{\mathbf{X} \subseteq \mathbf{E} : \mathbf{e} \in \mathbf{ex}_{\mathcal{L}}(\mathbf{X} \setminus \{\mathbf{e}\})\ \forall \mathbf{e} \in \mathbf{E} \setminus \mathbf{X}\}.$

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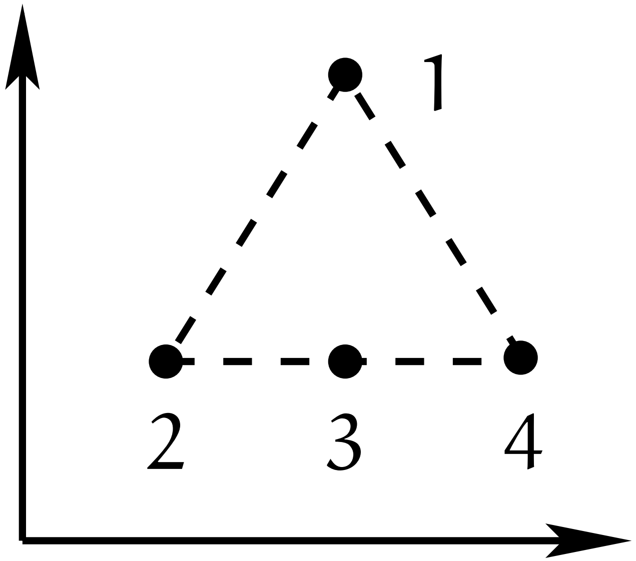

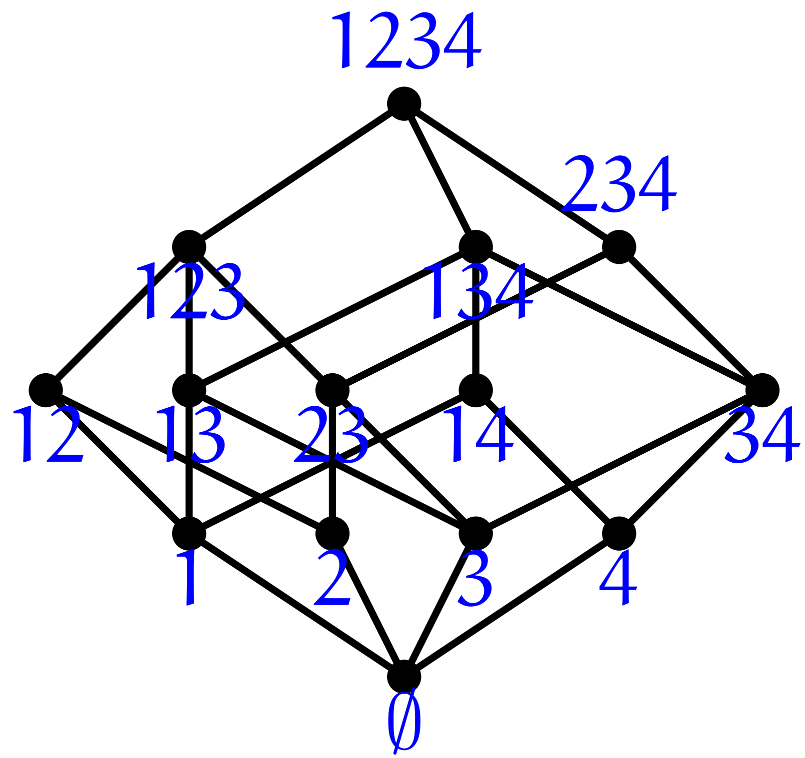

Rem Characterizations of convex geometries by extreme point operators (Koshevoy '99, Ando '02)

Def. $\mathbf{ex}_{\mathcal{L}}(\boldsymbol{A}) = \{\mathbf{e} \in \boldsymbol{A} : \mathbf{e} \notin \boldsymbol{\tau}_{\mathcal{L}}(\boldsymbol{A} \setminus \{\mathbf{e}\})\}$.

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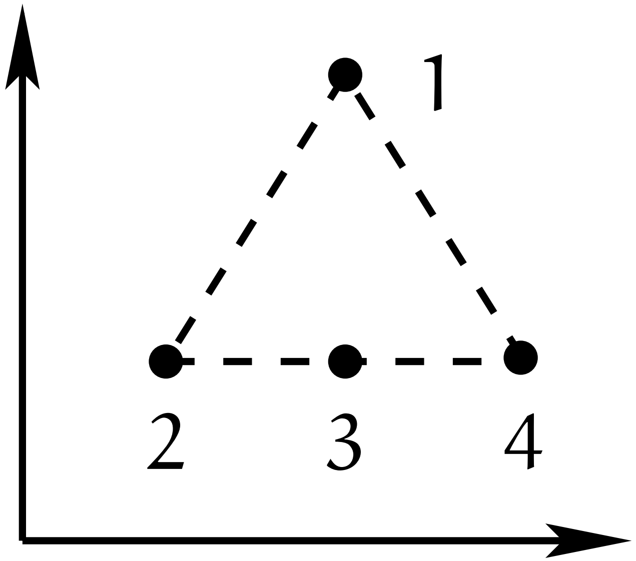
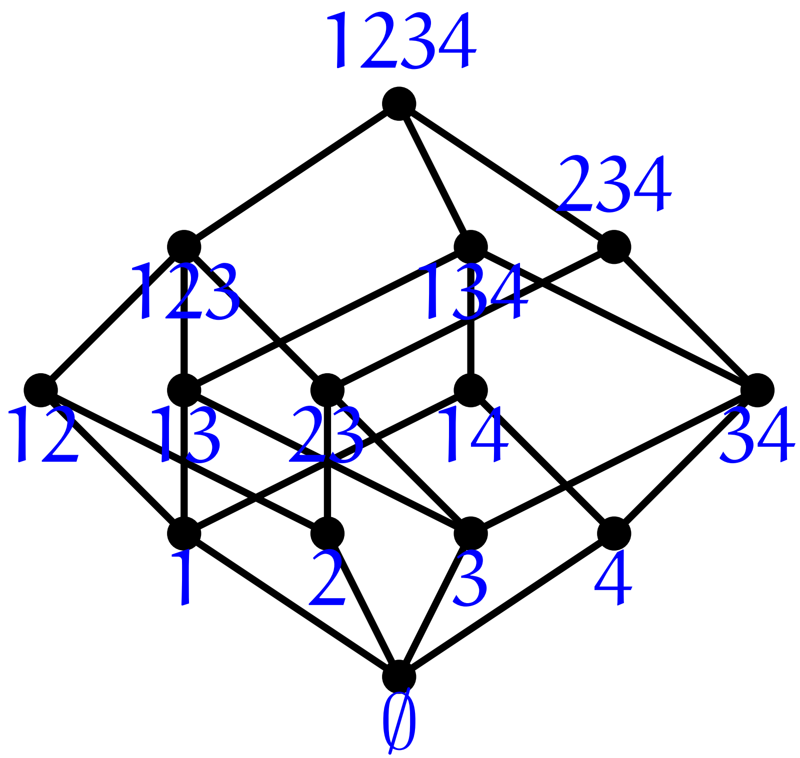




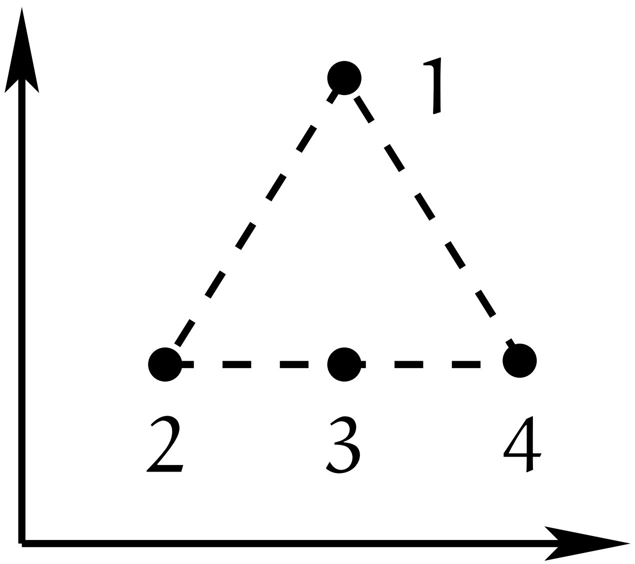
Rem. Characterizations of matroids by extreme point operators (Ando '02)

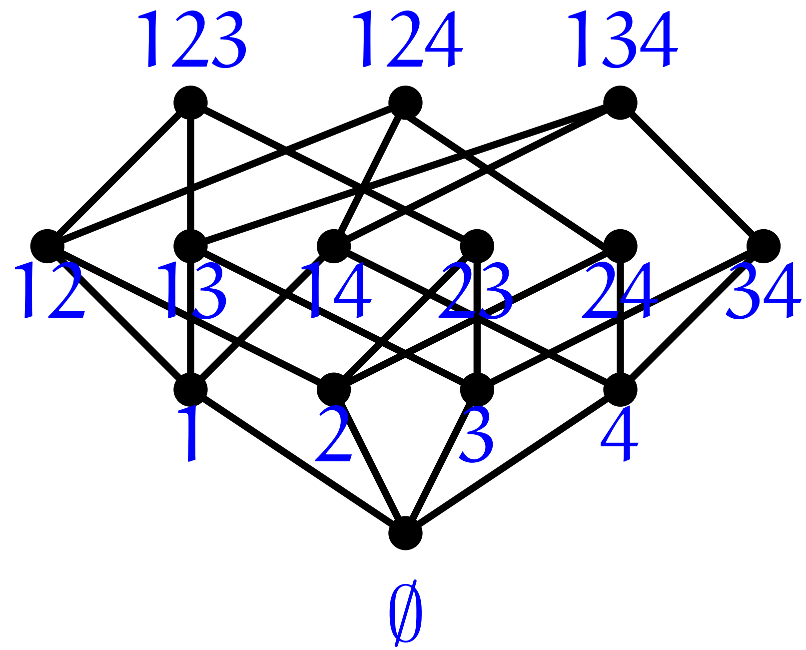
Setup $\mathcal{L} \subseteq 2^E$ a convex geometry on E

Def. A set $I \subseteq E$ is **independent** in $\mathcal{L}$ if $\mathbf{ex}_{\mathcal{L}}(\mathbf{I}) = \mathbf{I}$.

E.g. $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$

$\mathrm{ex}_{\mathcal{L}}(A)$ = the set of extreme points of $\mathrm{conv}(A)$





$\boxed{\text{Def.}}$ $\mathsf{Ind}(\mathcal{L})$ = the family of independent sets in $\mathcal{L}$

$\boxed{\text{Lem.}}$ $\mathrm{I} \subseteq \mathrm{J},\ \mathrm{J} \in \mathsf{Ind}(\mathcal{L}) \Rightarrow \mathrm{I} \in \mathsf{Ind}(\mathcal{L})$

Def. $\mathsf{Ind}(\mathcal{L})$ = the family of independent sets in $\mathcal{L}$

Lem. $\mathrm{I} \subseteq \mathrm{J},\ \mathrm{J} \in \mathsf{Ind}(\mathcal{L}) \Rightarrow \mathrm{I} \in \mathsf{Ind}(\mathcal{L})$

Open problem characterization of the family of independent sets in a convex geometry

Def. $\mathrm{Ind}(\mathcal{L})$ = the family of independent sets in $\mathcal{L}$

Lem. $\mathrm{I} \subseteq \mathrm{J},\ \mathrm{J} \in \mathrm{Ind}(\mathcal{L}) \Rightarrow \mathrm{I} \in \mathrm{Ind}(\mathcal{L})$

Open problem characterization of the family of independent sets in a convex geometry

Cf. $\mathcal{M}$ a matroid
$\mathrm{Ind}(\mathcal{M})$ is the family of independent sets in $\mathcal{M}$

$$\Updownarrow$$

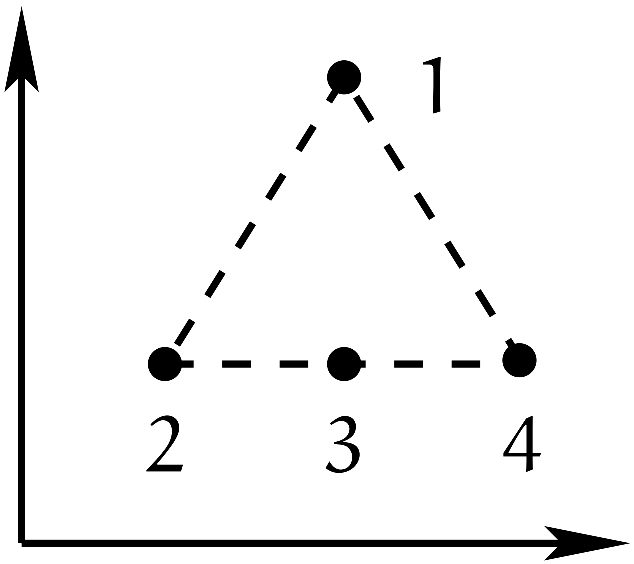
(1) $\mathbf{I} \subseteq \mathbf{J},\ \mathbf{J} \in \mathbf{Ind}(\mathcal{M}) \Rightarrow \mathbf{I} \in \mathbf{Ind}(\mathcal{M})$.

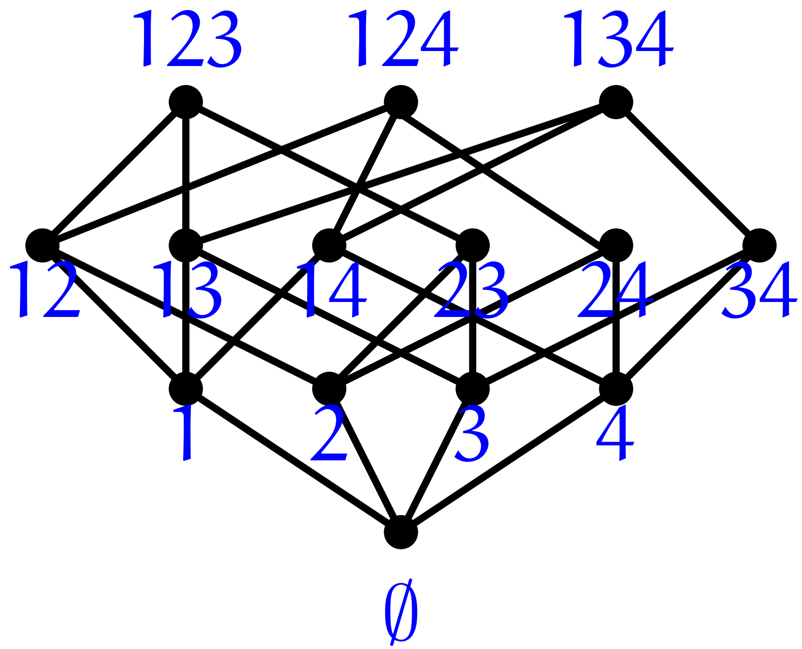
(2) $\mathbf{I}_1, \mathbf{I}_2 \in \mathbf{Ind}(\mathcal{M}),\ |\mathbf{I}_1| > |\mathbf{I}_2|$
$\Rightarrow \exists\ e \in \mathbf{I}_1 \setminus \mathbf{I}_2$: $\mathbf{I}_2 \cup \{e\} \in \mathbf{Ind}(\mathcal{M})$.

Setup $\mathcal{L} \subseteq 2^E$ a convex geometry on E

Def. A **circuit** of $\mathcal{L}$ is a minimal dependent set.

E.g. $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X) \cap \mathcal{P} = X\}$





$\mathcal{C}(\mathcal{L}) = \{234\}$.

Def. A **circuit** of $\mathcal{L}$ is a minimal dependent set.

$\mathcal{C}(\mathcal{L})$ = the family of circuits of $\mathcal{L}$

Rem Characterization of convex geometries
by the family of circuits (Dietrich '87)

Biased introduction to (abstract) convex geometries

Optimization on convex geometries

- Goecke '86
  - Linear optimization on convex geometries is NP-hard.
- Boyd & Faigle '90
  - Greedy algorithm for bottleneck optimization (Generalization of Lawler '73)
  - Algorithmic characterization of convex geometries
- Nakamura '00, Kempner & Levit '03
  - Further study on algor char of convex geometries
- Kashiwabara & Okamoto '03
  - Submodular-type optimization for convex geometries
  - Introduction of "c-submodular functions" (Based on Krüger '00, Ando '02)
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Cooperative games

Every set of players can form a coalition

Cooperative games on convex geometries (Bilbao)

A member in a convex geometry can only be a coalition (based on Faigle & Kern '92)

- Power indices (Edelman '97, Bilbao, Jiménez & Lopéz '98)
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- Cores (Bilbao, Lebrón & Jiménez '99, Okamoto '02)
- Selectopes (Bilbao, Jiménez, Lebrón & Peters '00)

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Enumeration from the viewpoint of convex geometries

- Background
  - Gray code for the linear extensions of a poset
    $\Rightarrow$ an efficient enumeration!!
  - Linear extensions of a poset
    $=$ Removing sequences in a poset shelling
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Biased introduction to (abstract) convex geometries

Biased introduction to abstract convex geometries

Summary

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Biased introduction to abstract convex geometries

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Summary

Theory of convex geometries

- appears in many guises (by different names).
- is activated by some recent results.
- leaves many open questions.
- sheds light to matroid theory as well.
- does not have enough people working on.
- is looking forward to your contributions.

Thank you very much.

Slides will be obtained from

http://www.inf.ethz.ch/personal/okamotoy/

or by email to

okamotoy@inf.ethz.ch