The Free Complex of a Two-Dimensional Generalized Convex Shelling

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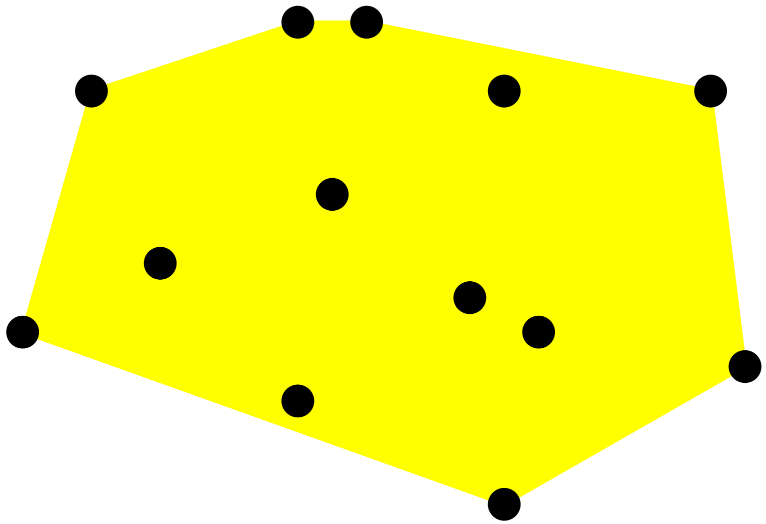
Combinatorics
Geometry
Computation

How many interior points are there in a finite point configuration $\mathcal{P}$?



An Euler-type formula:

$$\#\text{ of int. pts in } \mathcal{P} = (-1)^{d-1} \sum_{\text{free } A \subseteq \mathcal{P}} (-1)^{|A|} |A|.$$

$$\#\text{ of int. pts in } \mathcal{P} = (-1)^{\mathrm{d}-1} \sum_{\text{free } A \subseteq \mathcal{P}} (-1)^{|A|} |A|.$$

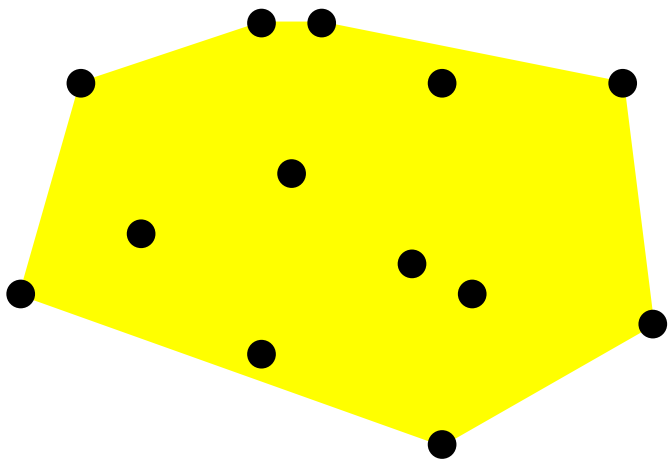
Proved by:

- Ahrens, Gordon & McMahon (DCG '99)
  for $\mathrm{d} = 2$, geometric proof
- Klain (Adv Math '99)
  for general d, using a valuation
- **Edelman & Reiner** (DCG '00)
  for general d, topological proof
  $\longrightarrow\longrightarrow\longrightarrow$ making use of **free complexes**

$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$.

Def. $X \subseteq \mathcal{P}$ is **free** if

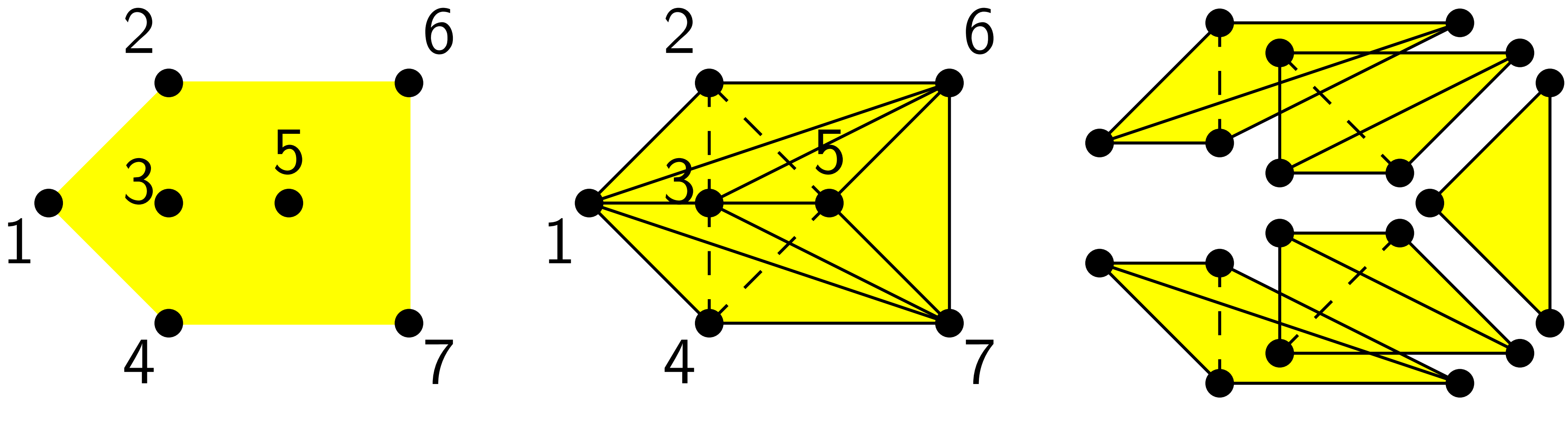
- $\mathrm{conv}(X) \cap \mathcal{P} = X$ (convexity)
- the extreme points of $\mathrm{conv}(X) = X$
  (the points of X lie in convex position)
  (independence).



Free complex of a point configuration

Def. The **free complex** of $\mathcal{P}$ is the family of all free sets in $\mathcal{P}$, denoted by $\mathrm{Free}(\mathcal{P})$.

Remark $\mathrm{Free}(\mathcal{P})$ is a simplicial complex.



$\mathcal{P}$ $\qquad$ $\mathrm{Free}(\mathcal{P})$ $\qquad$ the facets of $\mathrm{Free}(\mathcal{P})$

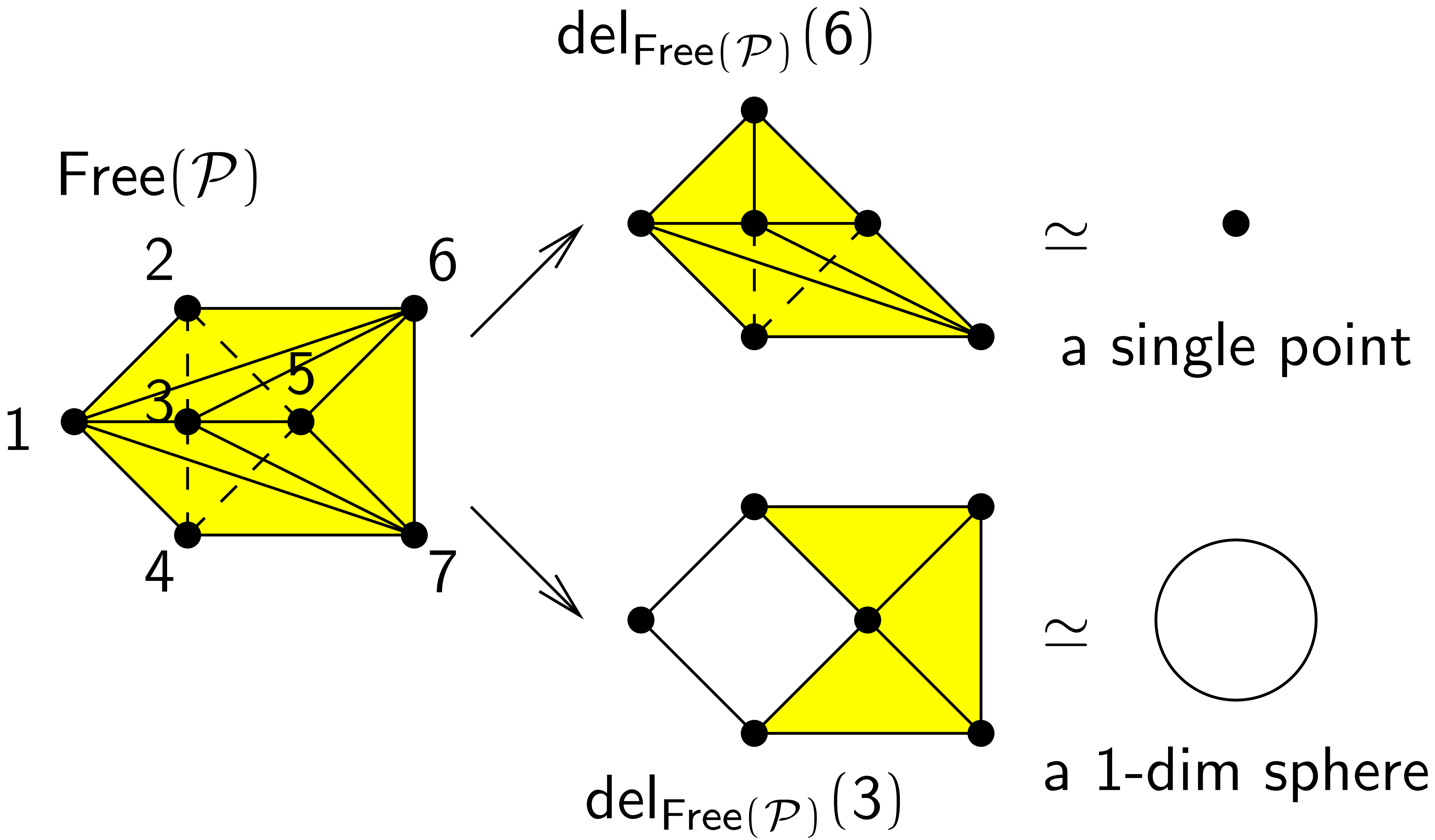
$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$

Consider the free complex $\mathsf{Free}(\mathcal{P})$ of $\mathcal{P}$.

In Edelman & Reiner's proof, it was a key that

- **$\mathrm{del}_{\mathsf{Free}(\mathcal{P})}(x)$ is contractible if $x \in \mathcal{P}$ lies on the bd of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathsf{Free}(\mathcal{P})}(x)) = 0$),
- **$\mathrm{del}_{\mathsf{Free}(\mathcal{P})}(x)$ has the integral homology of a $(d-1)$-dim sphere if $x \in \mathcal{P}$ lies in the interior of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathsf{Free}(\mathcal{P})}(x)) = (-1)^{d-1}$).

Proof by Edelman & Reiner: Example



Q.

How about a generalization to abstract convex geometries??

This work

- Study on their problems for a special case (2-dim generalized convex shellings).
- Result for this special case.

Talk plan

(1) (Abstract) convex geometries and Free complexes

(2) Questions by Edelman & Reiner for (abstract) convex geometries

(3) 2-dim generalized convex shellings

(4) Results

Convex geometries (Edelman & Jamison '85)

E a nonempty finite set,
$\mathcal{L} \subseteq 2^{\mathrm{E}}$ a family of subsets of E.

$\boxed{\text{Def.}}$ $\mathcal{L}$ is called a **convex geometry** on E
if $\mathcal{L}$ satisfies the following conditions.

(1) $\emptyset \in \mathcal{L}$, $\mathbf{E} \in \mathcal{L}$.

(2) $\mathbf{X}, \mathbf{Y} \in \mathcal{L} \Rightarrow \mathbf{X} \cap \mathbf{Y} \in \mathcal{L}$.

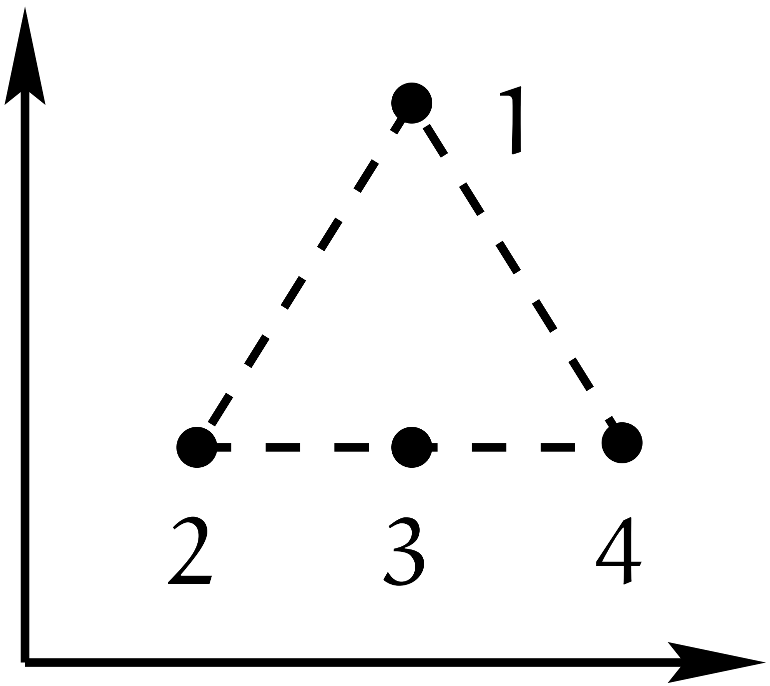
(3) $\mathbf{X} \in \mathcal{L} \setminus \{\mathbf{E}\} \Rightarrow \exists\, \mathbf{e} \in \mathbf{E} \setminus \mathbf{X}$: $\mathbf{X} \cup \{\mathbf{e}\} \in \mathcal{L}$.

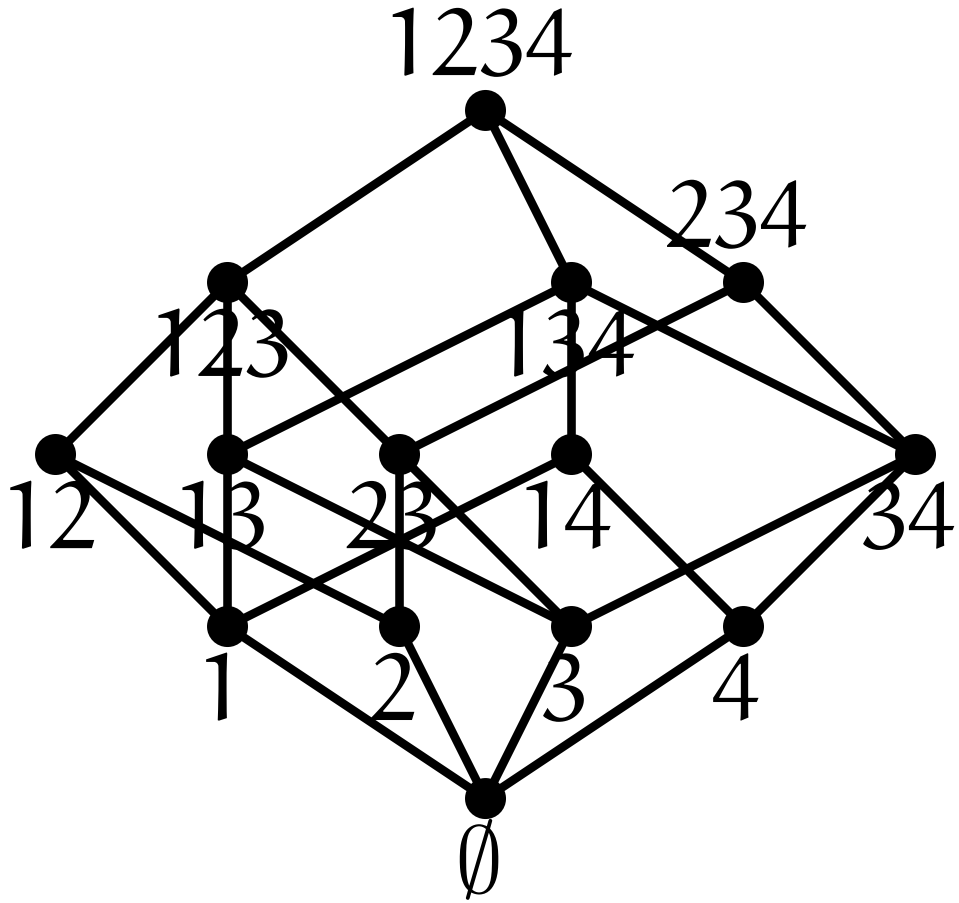
$X \subseteq E$ is called **convex** if $X \in \mathcal{L}$.

Example 1: convex shelling

$\mathcal{P}$ a finite point set in $\mathbb{R}^d$.

Define: $\mathcal{L} = \{X \subseteq \mathcal{P} : \mathbf{conv}(X) \cap \mathcal{P} = X\}$.

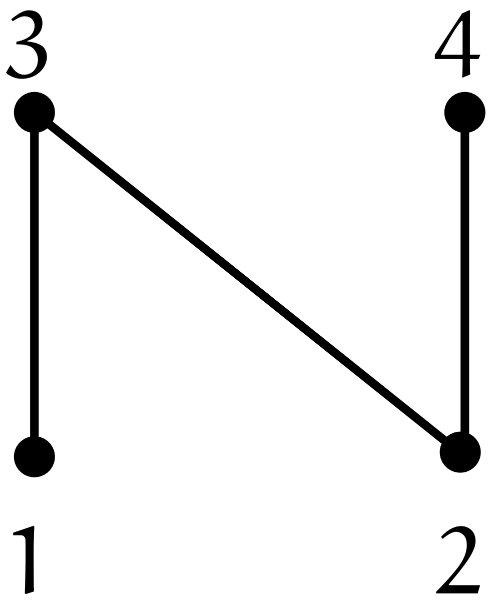


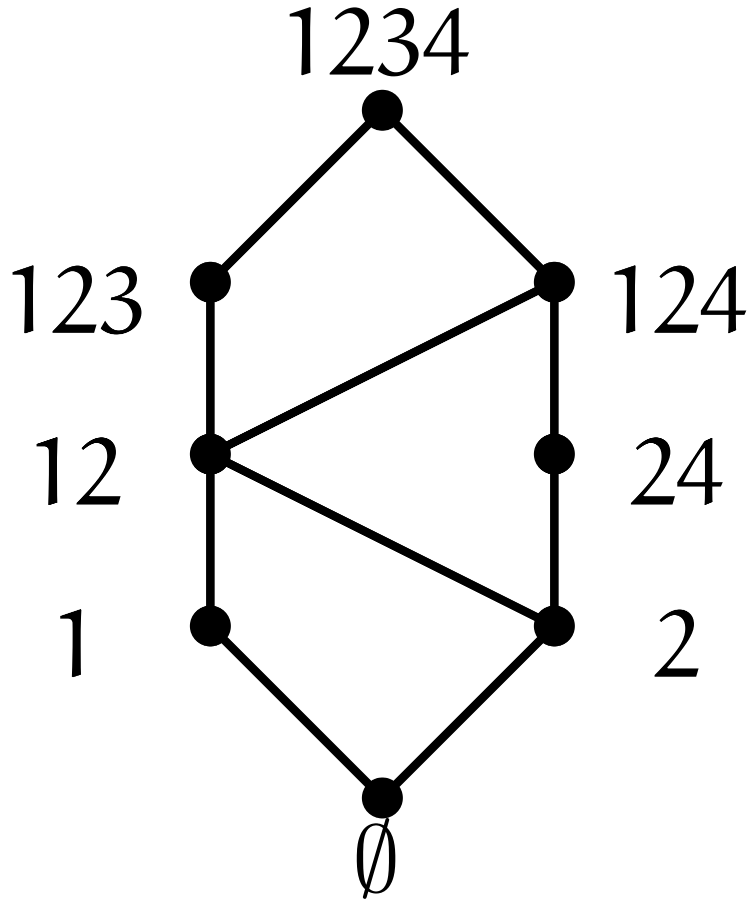


$\mathcal{L}$ is called the **convex shelling** on $\mathcal{P}$.

$P = (E, \leq)$ a partially ordered set.

Define: $\mathcal{L} = \{X \subseteq E : e \in X, f \leq e \Rightarrow f \in X\}$.





$\mathcal{L}$ is called the **poset shelling** of P.

Convex geometries arise from various objects.

- From graphs
  - Tree shellings
  - Graph searches
  - Simplicial elimination of chordal graphs
- From partially ordered sets
  - Poset double shellings
  - k-chains
- From finite point sets in $\mathbb{R}^d$
  - Lower convex shellings
- From oriented matroids
  - Convex shellings of acyclic OMs
- ...

Free sets in a convex geometry

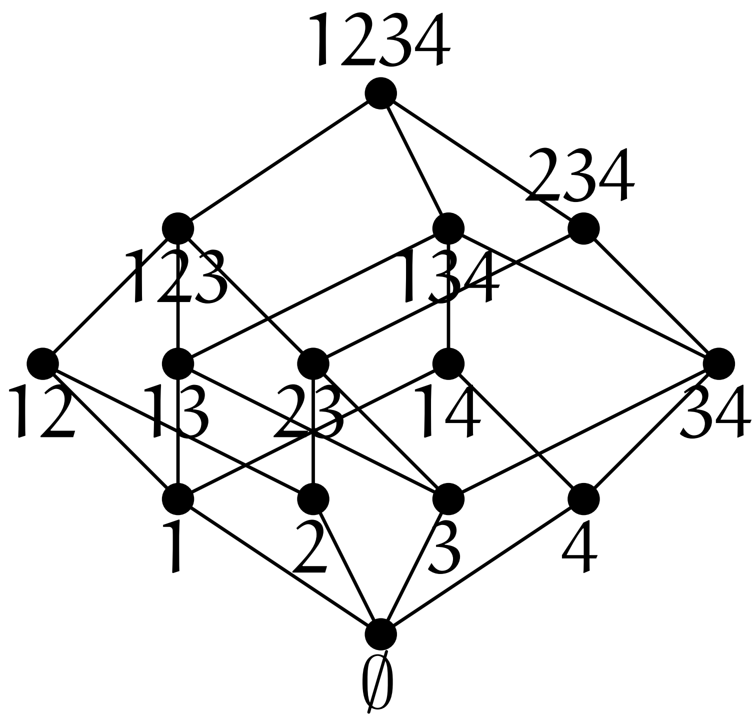
$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ $X \subseteq E$ is **free** in $\mathcal{L}$ if

- $X \in \mathcal{L}$ (convexity)
- the set of "extreme points" of $X = X$ (independence).

$\mathcal{L}$ a convex geometry on E, $X \in \mathcal{L}$ a convex set.

$\boxed{\text{Def.}}$ $e \in X$ is an **extreme point** of X if $X \setminus \{e\} \in \mathcal{L}$.



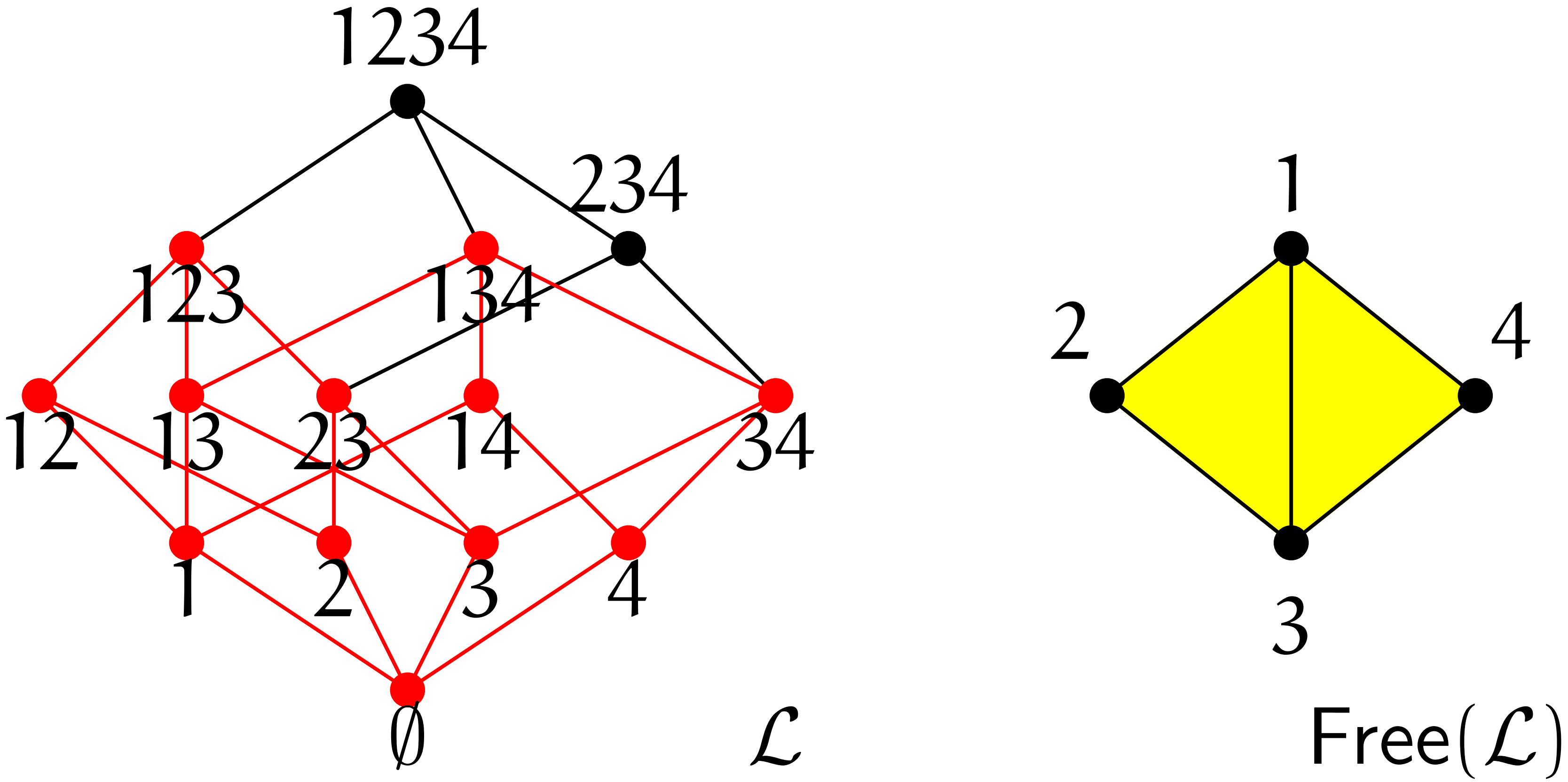
$X = \{2, 3, 4\} \in \mathcal{L}$

2	extreme
3	not extreme
4	extreme

X is **independent** if every $e \in X$ is extreme in X.

Def. The **free complex** Free($\mathcal{L}$) of $\mathcal{L}$ is the family of all free sets in $\mathcal{L}$,

Remark Free($\mathcal{L}$) is a simplicial complex.



Remark

$\mathcal{P}$ a point configuration in $\mathbb{R}^{\mathrm{d}}$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

Then

$$\mathsf{Free}(\mathcal{P}) = \mathsf{Free}(\mathcal{L}).$$

To generalize Edelman & Reiner's result,

We also need to generalize
"the boundary" and "the interior."

$\Longrightarrow$ a concept of "dependency sets"
(we omit the definition).

$\mathcal{P}$ a point configuration in $\mathbb{R}^d$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

$\mathrm{Dep}_{\mathcal{L}}(e)$ the dependency set of $e \in \mathcal{P}$ in $\mathcal{L}$.

Then

- e lies on the boundary of $\mathrm{conv}(\mathcal{P})$ $\iff \mathrm{Dep}_{\mathcal{L}}(e) \neq \mathcal{P}$.
- e lies in the interior of $\mathrm{conv}(\mathcal{P})$ $\iff \mathrm{Dep}_{\mathcal{L}}(e) = \mathcal{P}$.

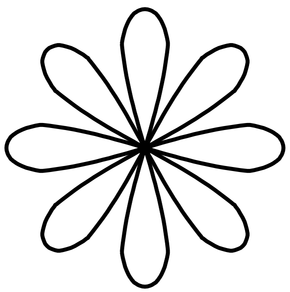
This leads to the following open problems.

Open Problems (Edelman & Reiner '00)

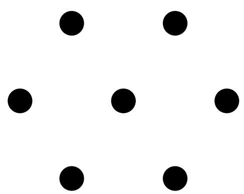
E a finite set; $\mathcal{L}$ a convex geometry on E.

(1) Is $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ contractible if $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$?

(2) Is $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ homotopy equivalent to a bouquet of equidimensional spheres if $\mathrm{Dep}_{\mathcal{L}}(x) = E$?



a bouquet of eight 1-dim spheres



a bouquet of six 0-dim spheres

a bouquet of zero sphere

Partial results in the literature

Both problems have been solved affirmatively for the following classes of convex geometries.

- Convex shellings of point configurations (Edelman & Reiner '00, Dong '02)
- Poset double shellings (Edelman & Reiner '00)
- Simplicial eliminations of chordal graphs (Edelman & Reiner '00)
- Conv shellings of acyclic oriented matroids (Edelman, Reiner & Welker '02)
- Poset shellings. (Easy)

Consider another class of convex geometries, the 2-dim separable generalized convex shellings.

(1) If $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$,
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible.

(2) If $\mathrm{Dep}_{\mathcal{L}}(x) = E$,
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is either contractible or homotopy equiv to a 0-dim sphere.

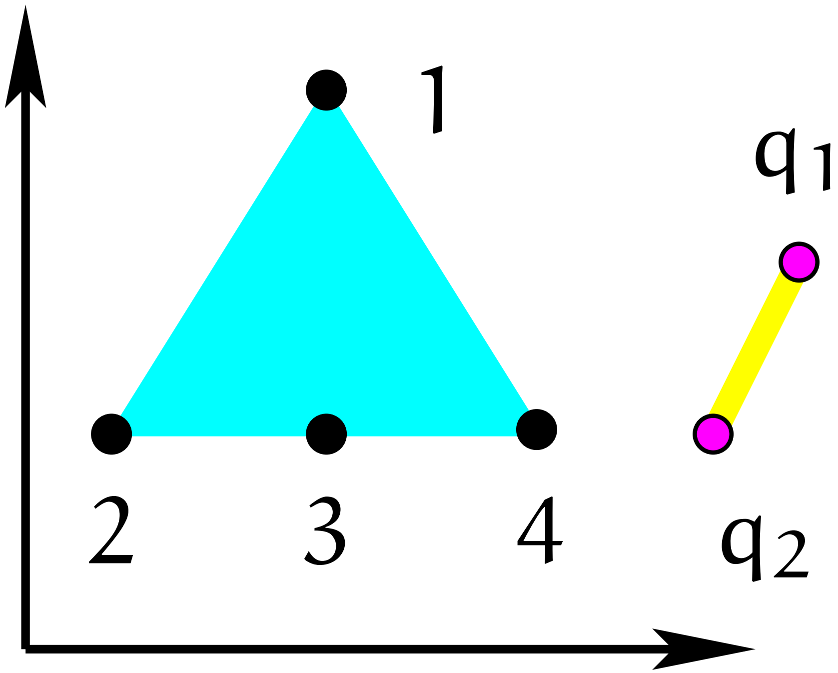
★ Verifies Open Problems for this special case!

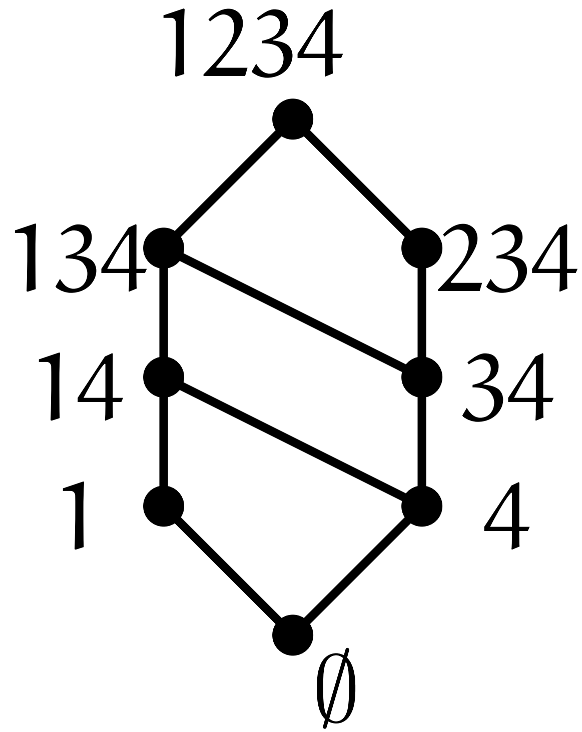
★ Gives the first example of $\mathcal{L}$ and x s.t. $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible & $\mathrm{Dep}_{\mathcal{L}}(x) = E$.

$\mathcal{P},\mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \mathrm{conv}(\mathcal{Q}) = \emptyset$.

Define:

$$\boldsymbol{\mathcal{L}} = \{\mathbf{X} \subseteq \boldsymbol{\mathcal{P}} : \mathbf{conv}(\mathbf{X} \cup \boldsymbol{\mathcal{Q}}) \cap \boldsymbol{\mathcal{P}} = \mathbf{X}\}.$$





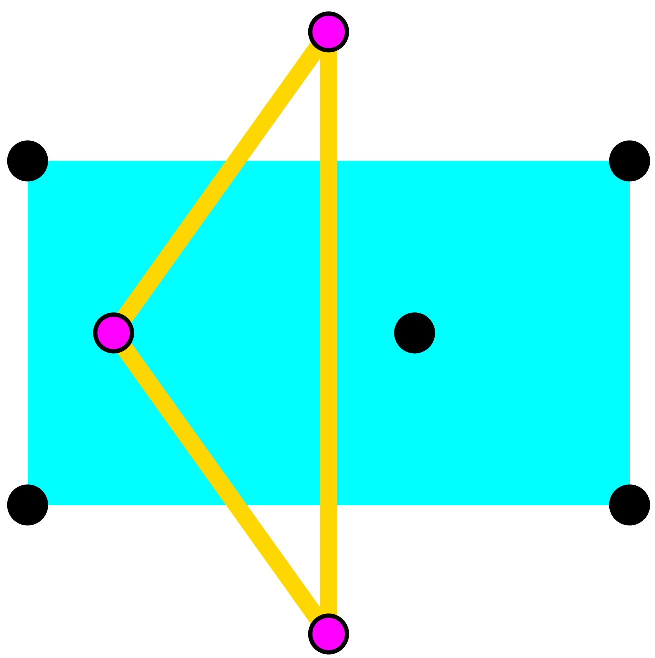
$\mathcal{L}$ is a convex geometry on $\mathcal{P}$ and called
the generalized conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

Technical terminology

$\mathcal{P}, \mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \text{conv}(\mathcal{Q}) = \emptyset$,

$\mathcal{L}$ the generalized convex shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

- $\mathcal{L}$ is **2-dimensional** if $d = 2$.
- $\mathcal{L}$ is **separable** if $\text{conv}(\mathcal{P}) \cap \text{conv}(\mathcal{Q}) = \emptyset$.



a non-separable case

Why separable generalized convex shellings?

$\boxed{\text{Thm}}$ (Kashiwabara, Nakamura & Okamoto, '03)

For every convex geometry $\mathcal{L}$, there exist point sets $\mathcal{P}$, $\mathcal{Q}$ with $\textbf{conv}(\mathcal{P}) \cap \textbf{conv}(\mathcal{Q}) = \emptyset$ s.t.

$\mathcal{L} \cong$ the gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

(Separable generalized convex shellings represent all convex geometries.)

$\longrightarrow\longrightarrow\longrightarrow$ The 2-dim case is a first step for resolution of Open Problems.

Our results (again)

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $x \in \mathcal{P}$.

(1) **If $\mathbf{Dep}_{\mathcal{L}}(x) \neq \mathcal{P}$,**
$\mathbf{del}_{\mathbf{Free}(\mathcal{L})}(x)$ is contractible.

(2) **If $\mathbf{Dep}_{\mathcal{L}}(x) = \mathcal{P}$,**
$\mathbf{del}_{\mathbf{Free}(\mathcal{L})}(x)$ is either contractible or homotopy equiv to a 0-dim sphere.

Problems are still open

- **We don't know yet**

  the problems are affirmative or negative in the general case!
- How about a 3-dim case??
- How about a non-separable 2-dim case??

Acknowledgements

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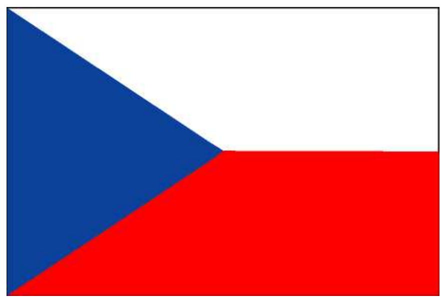
Děkuji vám mnohokrát.



Here are extra slides for possible questions from the audience.

$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ The **closure operator** $\tau_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$\tau_{\mathcal{L}}(A) = \bigcap \{X \in \mathcal{L} : A \subseteq X\}.$$

$\boxed{\text{Def.}}$ The **extreme point operator** $ex_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$ex_{\mathcal{L}}(A) = \{e \in A : e \notin \tau_{\mathcal{L}}(A \setminus \{e\})\}.$$

Dependency sets

$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ $A \subseteq E$ is **independent** if $\text{ex}_{\mathcal{L}}(A) = A$.

$\boxed{\text{Def.}}$ The **dependency set** of $e \in E$ in $\mathcal{L}$ is

$$\text{Dep}_{\mathcal{L}}(e) = \left\{ f \in E : \begin{array}{l} \exists \text{ independent } A \text{ s.t.} \\ f \in A, e \in \tau_{\mathcal{L}}(A), \\ e \notin \tau_{\mathcal{L}}(A \setminus \{f\}) \end{array} \right\}.$$

Outline of the proof

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $\mathcal{Q} \neq \emptyset$.

(1) Free$(\mathcal{L})$ is the clique complex of a graph G.

I.e., the family of all cliques of G.

(2) G is chordal & connected.

Chordal $\Leftrightarrow$ every ind. cycle is C_3.

(3) **(2)** $\Rightarrow$ Free$(\mathcal{L})$ contractible.

(4) $G - x$ has at most 2 connected components.

(5) x a cut-vertex of G $\Rightarrow$ $\mathrm{Dep}_{\mathcal{L}}(x) = \mathcal{P}$.