Fair cost allocations under conflicts
— a game-theoretic point of view —

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3rd CGC Annual Workshop

Park Hotel Fasanerie, Neustrelitz, Germany
September 29 – October 1, 2003

Supported by the Berlin-Zürich Joint Graduate Program

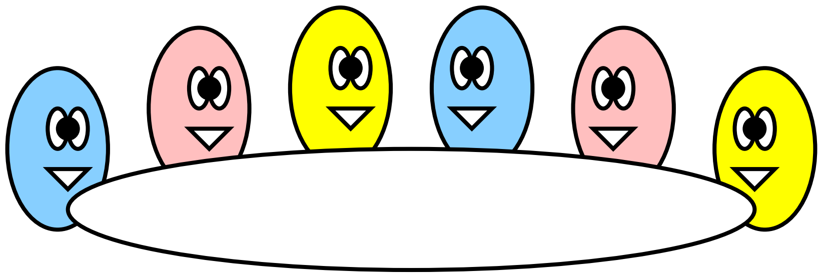
Combinatorics
Geometry
Computation

Framework: Several people are willing to work together...

- They want to have a largest possible benefit. . . . . optimization problem
- They want to allocate the benefit in a fair way. . . . . game-theoretic problem

Game Theory?

- Noncooperative Game Theory
- Cooperative Game Theory



Def.: A cooperative game (or a game) is a pair (N, γ) of

- a finite set N (set of players)
- a function $\gamma : 2^N \to \mathbb{R}$ with $\gamma(\emptyset) = 0$ (characteristic function).

Interpretation: For $S \subseteq N$,

$$\gamma(S) \text{ represents } \left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$$

when the players in S work in cooperation.

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$\gamma(S)$ represents $\left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$ when the players in S work in cooperation.

Goal: To allocate $\gamma(N)$ to each player in a "fair" way.

This work: study on "minimum coloring games."

$G = (V, E)$ an undirected graph

- A proper k-coloring of G

  is a surjective map $c : V \to \{1, \ldots, k\}$ s.t.
  if $\{u, v\} \in E$, then $c(u) \neq c(v)$.

- The chromatic number $\chi(G)$ of G

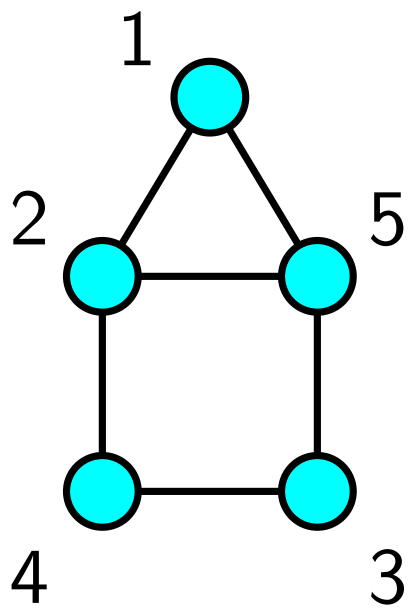
  $= \min\{\ k\ :\ $ a proper k-coloring of G exists $\}$.

- The minimum coloring game on G

  is a cooperative game (V, χ_G).

  $\chi_G : 2^V \to \mathbb{N}$ is defined as $\chi_G(S) = \chi(G[S])$,
  where $G[S]$ is the subgraph induced by $S \subseteq V$.

$\chi_G(S) = \chi(G[S])$ for $S \subseteq V$.

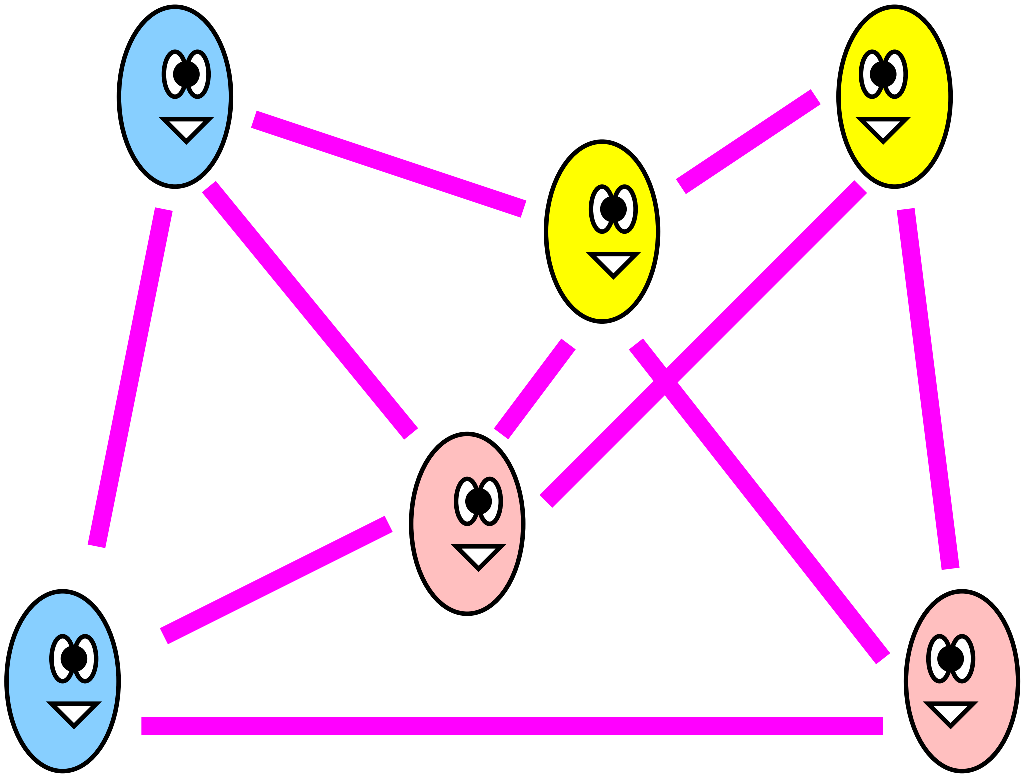


S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

Goal: To allocate $\chi(G)$ to each vertex in a fair way.

Conflict graph: a model of conflict

- the vertices = the agents, the principals...
- the edges = between two in conflict.



min. coloring game:

a simplest model of the
fair cost allocation problem
in conflict situations

We study minimum coloring games, and
investigate the following kinds of fairness concepts:

- Core (Gillies '53)
- Nucleolus (Schmeidler '69)
- τ-value (Tijs '81)
- Shapley value (Shapley '53).

Past works on minimum coloring games:

- Deng, Ibaraki & Nagamochi '99
- Deng, Ibaraki, Nagamochi & Zang '00
- Okamoto '03

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Why fair cost allocation problems??

Fair cost allocation problems are studied in OR community from the game-theoretic viewpoint.

- Megiddo '87

  First noticed the computational issue in fair cost allocation problems.

- So far, a lot of results have appeared in

  Mathematics of Operations Research,
  Mathematical Progrmming,
  Mathematical Methods of Operations Research,
  Discrete Applied Mathematics,
  International Journal of Game Theory,
  Games and Economic Behaviours,
  etc.

- They assume practical applications.

There are many kinds of "fairness" concepts
(called "solutions") in cooperative game theory.

$\boxed{\text{Thesis:}}$ Bounded Rationality (Simon '70s)

Decisions by realistic economic agents cannot
involve unbounded resources for reasoning.

$\boxed{\text{Thesis:}}$ (Deng & Papadimitriou '94)

[Economic concept]		[Algorithmic concept]
A solution follows Bounded Rationality	$\simeq$	Computation can be done in poly. time

$\Longrightarrow$ Algorithmic study of cooperative game theory

E a finite set

Def.: A set function on E is a function $f : 2^E \longrightarrow \mathbb{R}$.

Appearance:

- Cooperative game theory
- Combinatorial optimization
- Pseudo-boolean functions
- Nonadditive measure theory (fuzzy measure theory)
- ...

They study different aspects of set functions.

Focus on cores and nucleoli of minimum coloring games

- Def.: cost allocation
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Def.: A cost allocation for a game (N, γ) is a vector $z \in \mathbb{R}^N$ such that

$$\sum\{z[i] : i \in N\} = \gamma(N).$$

(Often in cooperative game theory, this is called a pre-imputation.)

. . . .

Q. What kinds of cost allocations are considered fair??

...... Core, Nucleolus, τ-value, Shapley value, etc.

<u>Let</u>

(N, γ)	a game
$z \in \mathbb{R}^N$	a cost allocation
$S \subseteq N$	(often called a coalition)

Def.: An excess $e(S, z)$ is defined as

$$e(S, z) := \sum_{i \in S} z[i] - \gamma(S).$$

Interpretation: The smaller $e(S, z)$, the happier S with z.

$\sum_{i \in S} z[i]$: cost owed to S when people in N work together

$\gamma(S)$: cost owed to S when people in S work together.

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

S	$\gamma(S)$	$e(S, z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
$\{1,2\}$	1	$1/2$
$\{1,3\}$	2	$-1/2$
$\{2,3\}$	2	-1
$\{1,2,3\}$	2	(0)

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.
- Arrange these excesses in non-increasing order to obtain $\theta_z \in \mathbb{R}^{2^{|N|}-2}$. ($\theta_z[i] \geq \theta_z[j]$ if $i \leq j$.)

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

$$\theta_z = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -1\right)^\top$$

S	$\gamma(S)$	$e(S,z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
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$\{1,3\}$	2	$-1/2$
$\{2,3\}$	2	-1
$\{1,2,3\}$	2	(0)

$\boxed{\text{Def.:}}$ The nucleolus of (N, γ) is defined as

$$\boldsymbol{\nu}(N, \gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^N : \begin{array}{l} \boldsymbol{z} \text{ lex-mins } \boldsymbol{\theta}_{\boldsymbol{z}} \text{ over all cost alloc's } \boldsymbol{y} \\ \text{s.t. } \boldsymbol{y}[i] \le \gamma(\{i\}) \quad \forall i \in N \end{array} \right\}.$$

$\boxed{\text{Interpretation:}}$ The smaller $e(S, \boldsymbol{z})$, the happier S with $\boldsymbol{z}$.

$\Rightarrow$ Want an allocation which minimizes max excess.

Def.: The nucleolus of (N, γ) is defined as

$$\nu(N,\gamma) = \left\{ z \in \mathbb{R}^N : \begin{array}{l} z \text{ lex-mins } \theta_z \text{ over all cost alloc's } y \\ \text{s.t. } y[i] \le \gamma(\{i\}) \quad \forall i \in N \end{array} \right\}.$$

Interpretation: The smaller $e(S, z)$, the happier S with z.

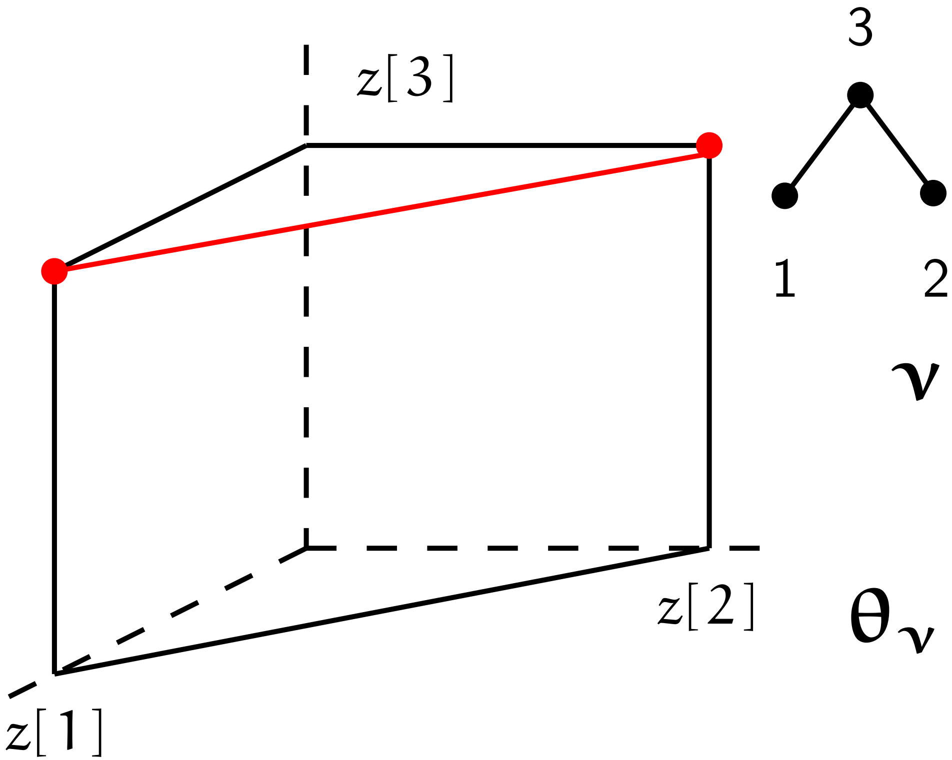
$\Rightarrow$ Want an allocation which minimizes max excess.

Thm. (Schmeidler '69)

The nucleolus consists of a single vector.

So we usually say $\nu(N, \gamma) = z$ instead of $\nu(N, \gamma) = \{z\}$.

Example: nucleolus

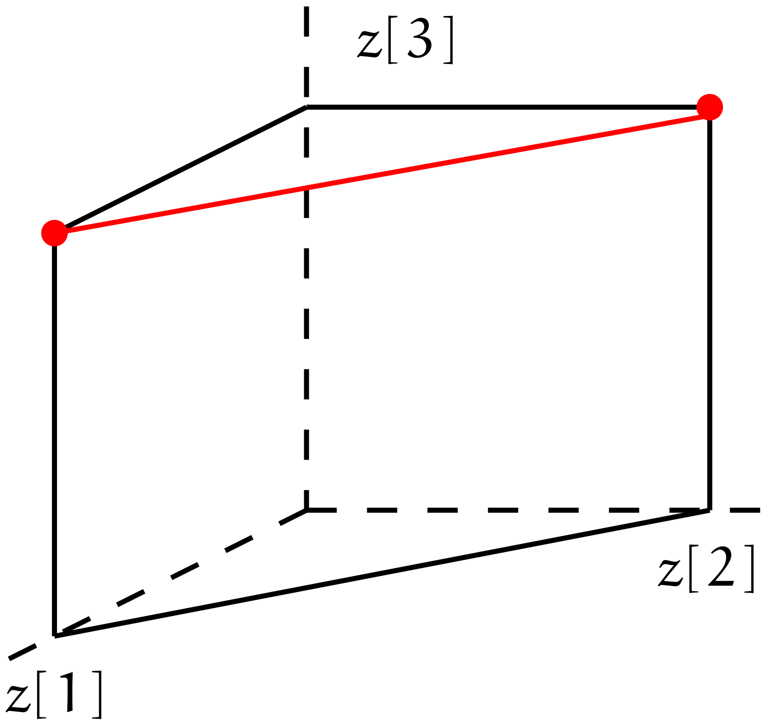


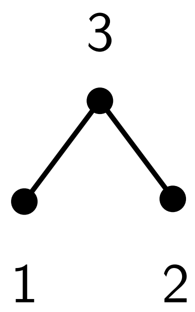
$$\nu = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$$ is the nucleolus.

$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\nu)$	$-1/2$	$-1/2$	0	0	$-1/2$	$-1/2$

$$e(S,\nu) := \sum_{i\in S} \nu[i] - \gamma(S).$$





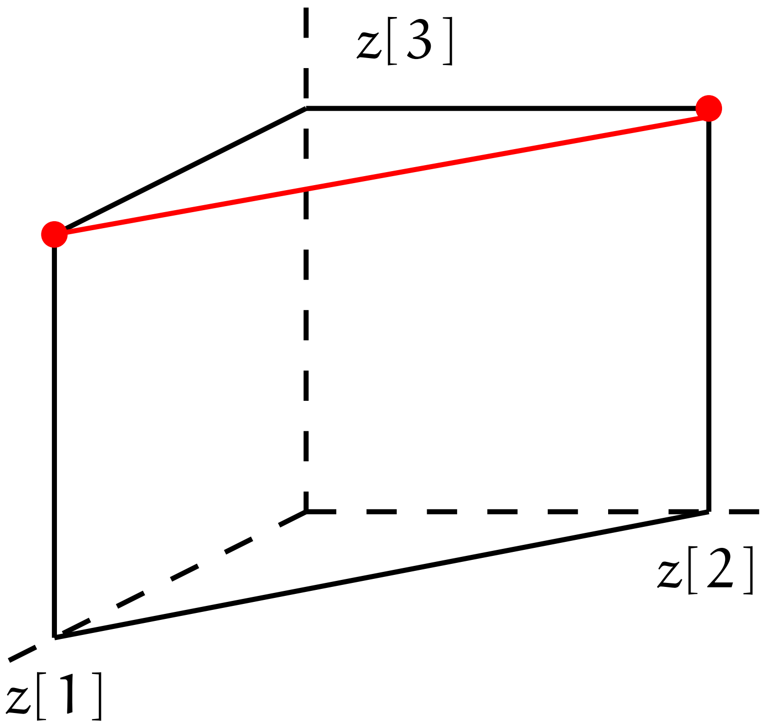
$$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$$ is the nucleolus.

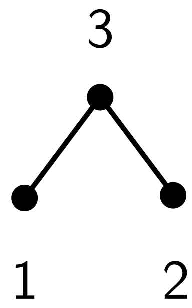
$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

Fact: the core is nonempty (i.e., the game is balanced)
$\Rightarrow$ the nucleolus $\in$ the core.

Def.: A cost allocation $\boldsymbol{z} \in$ the core of (N, γ)
if $e(S, \boldsymbol{z}) \leq 0$ $(\forall\, S \subseteq N)$.

Example: nucleolus





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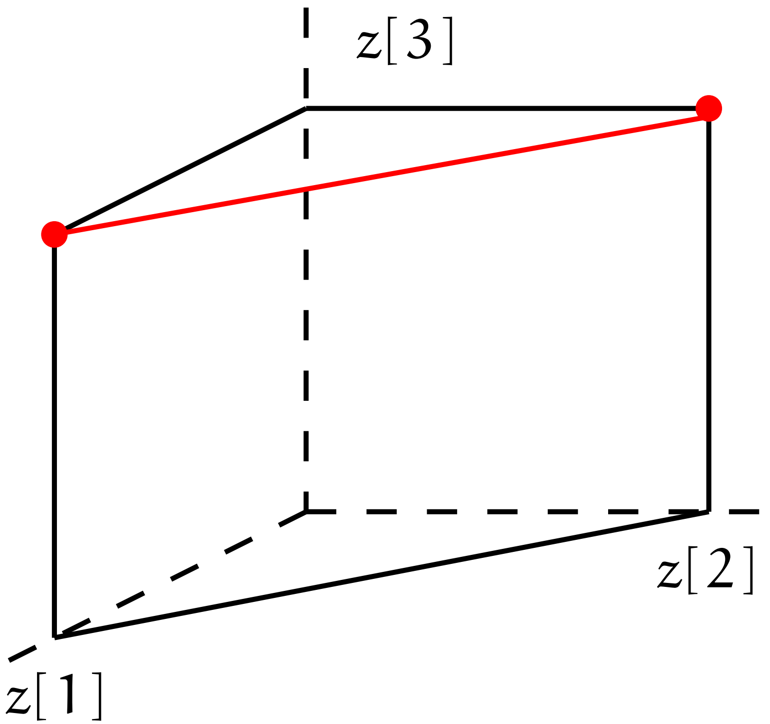
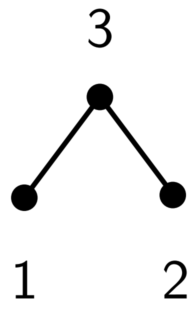
Let $\boldsymbol{z} = \lambda(1,0,1)^\top + (1-\lambda)(0,1,1)^\top \qquad (0 \le \lambda \le 1)$
$= (\lambda, 1-\lambda, 1)^\top.$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\boldsymbol{z})$	$\lambda-1$	$-\lambda$	0	0	$\lambda-1$	$-\lambda$





$$\nu = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$$ is the nucleolus.

$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

$$\theta_z = \begin{cases} (0, 0, -\lambda, -\lambda, \lambda - 1, \lambda - 1)^\top & \text{if } 0 \leq \lambda \leq 1/2 \\ (0, 0, \lambda - 1, \lambda - 1, -\lambda, -\lambda)^\top & \text{if } 1/2 \leq \lambda \leq 1 \end{cases}$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,z)$	$\lambda - 1$	$-\lambda$	0	0	$\lambda - 1$	$-\lambda$

Thm. (Kuipers '96, Faigle, Kern & Kuipers '01)

The nucleolus can be computed in polynomial time for submodular games.

Thm. (Okamoto '03)

χ_G is submodular $\Leftrightarrow$ G is complete multipartite.

Cor.

G complete multipartite
$\Rightarrow$ the nucleolus of χ_G computed in poly. time.

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\vert$		
zero duality gap		???
$\cup\vert$		
perfect		???
$\cup\vert$		
complete multipartite		Poly

Obs. The computation of the nucleolus of a min coloring game is NP-hard.

Proof Suppose we get the nucleolus $\boldsymbol{\nu}$ in poly time.

$\Rightarrow$ Compute $\sum_{i \in V} \boldsymbol{\nu}[i] = \chi(G)$.

$\Rightarrow$ We have obtained $\chi(G)$ in poly time.

$\Rightarrow$ P $=$ NP. [qed]

On the computation of the nucleolus of a min coloring game

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$\cup\!\mid$		
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On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\vert$		
zero duality gap		???
$\cup\vert$		
perfect		???
$\cup\vert$		
O-good		characterization
$\cup\vert$		
complete multipartite		Poly

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the characteristic vectors of the maximum cliques of G.

Namely,

$$\boldsymbol{\nu}[i] = \frac{\#\text{ of maximum cliques containing } i}{\#\text{ of maximum cliques}}.$$

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the characteristic vectors of the maximum cliques of G.

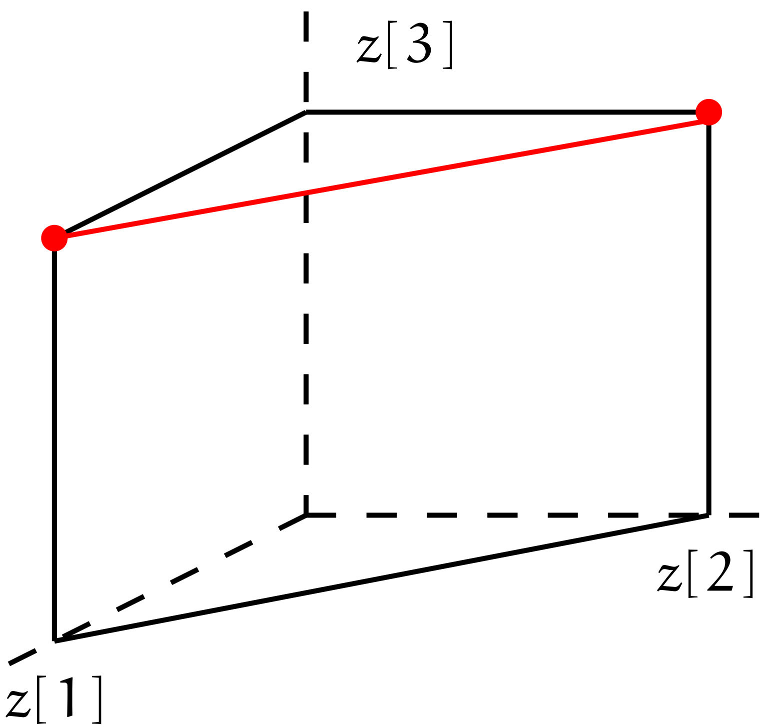
Remark:

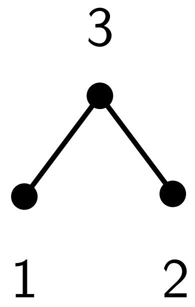
(1) We omit the def. of O-good perfect graphs.

(2) The class of O-good perfect graphs contains

- the graphs with unique maximum cliques
- the complete multipartite graphs
- the chordal graphs (especially the forests).

(3) A graph is chordal if every induced cycle is of length 3.





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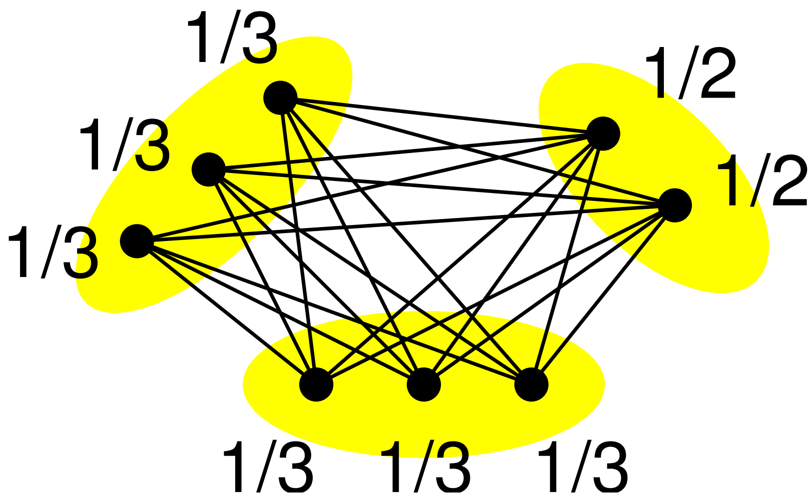
$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^{\top}.$$

Indeed, this graph is $\begin{cases}\text{a forest}\\ \text{complete multipartite.}\end{cases}$

Consider a complete multipartite graph.

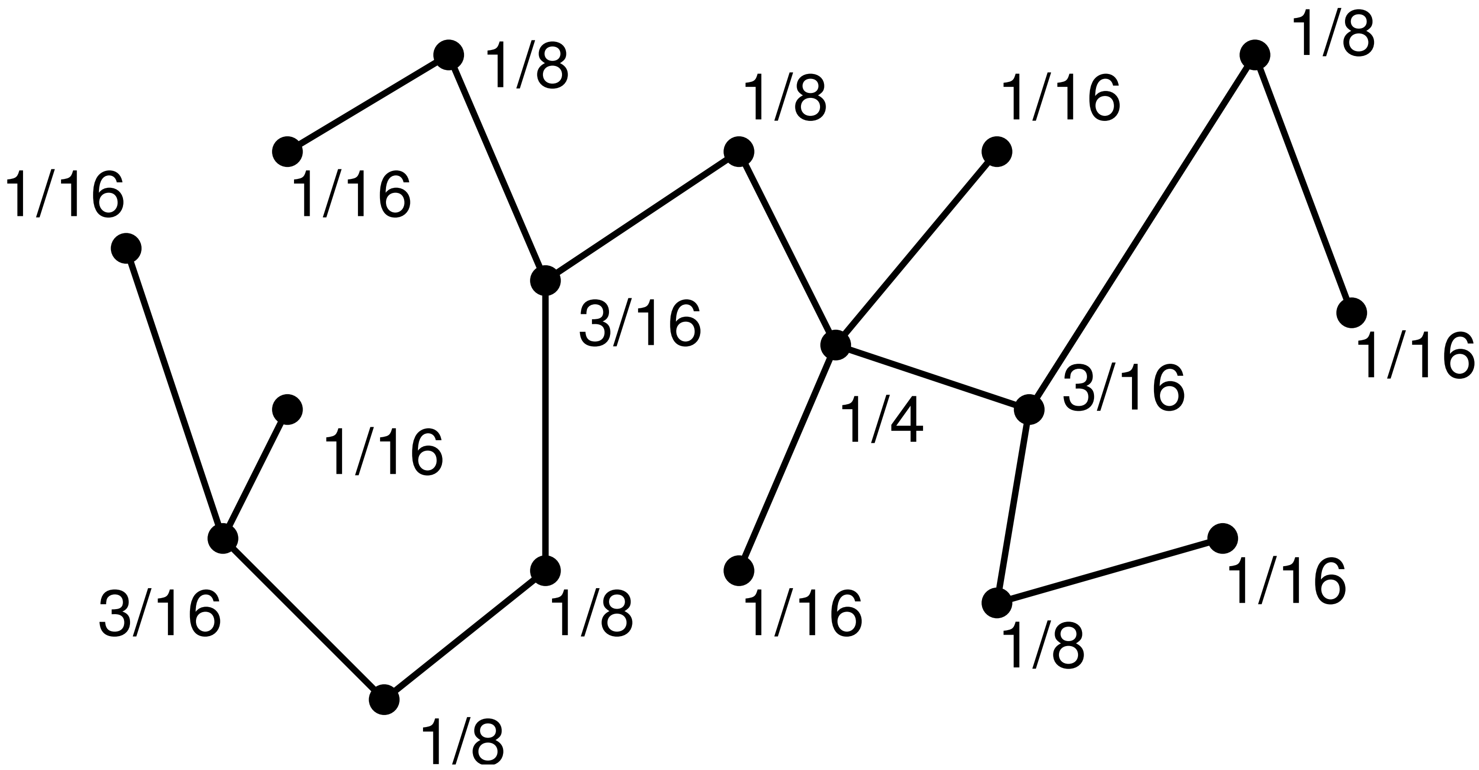


We have

$$\nu[i] = \frac{1}{n_i},$$

where n_i is $\#$ of vertices of the class to which i belongs.

Consider a forest.



We have

$$\nu[i] = \frac{\deg(i)}{|E|},$$

where $\deg(i)$ is # of edges incident to i.

For chordal graphs, we can use the following theorem to compute the nucleoli.

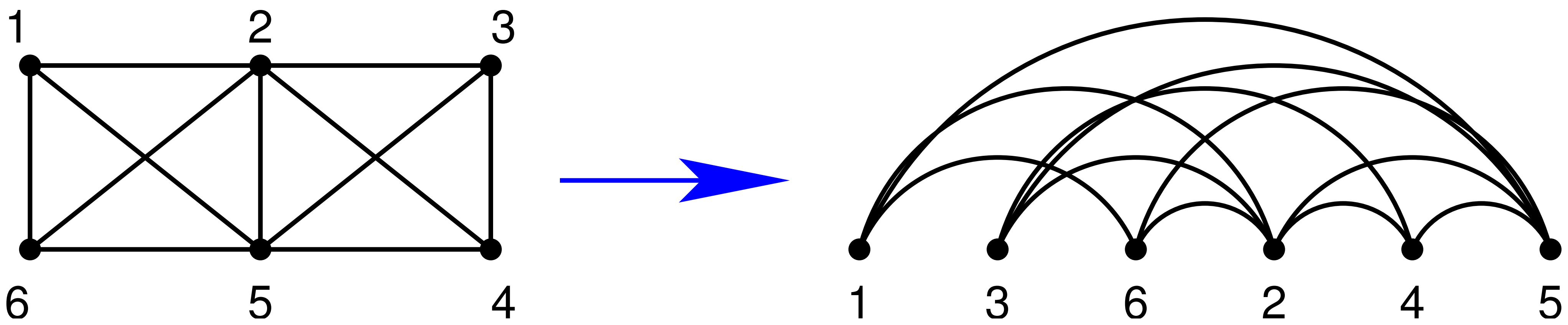
Thm. (Fulkerson & Gross '65)

In a chordal graph,

$\#$ of maximal cliques $\leq$ $\#$ of vertices.

We can enumerate them in poly time.

Proof Sketch



(1) Use the LP formulation for the nucleolus computation.

(Peleg)

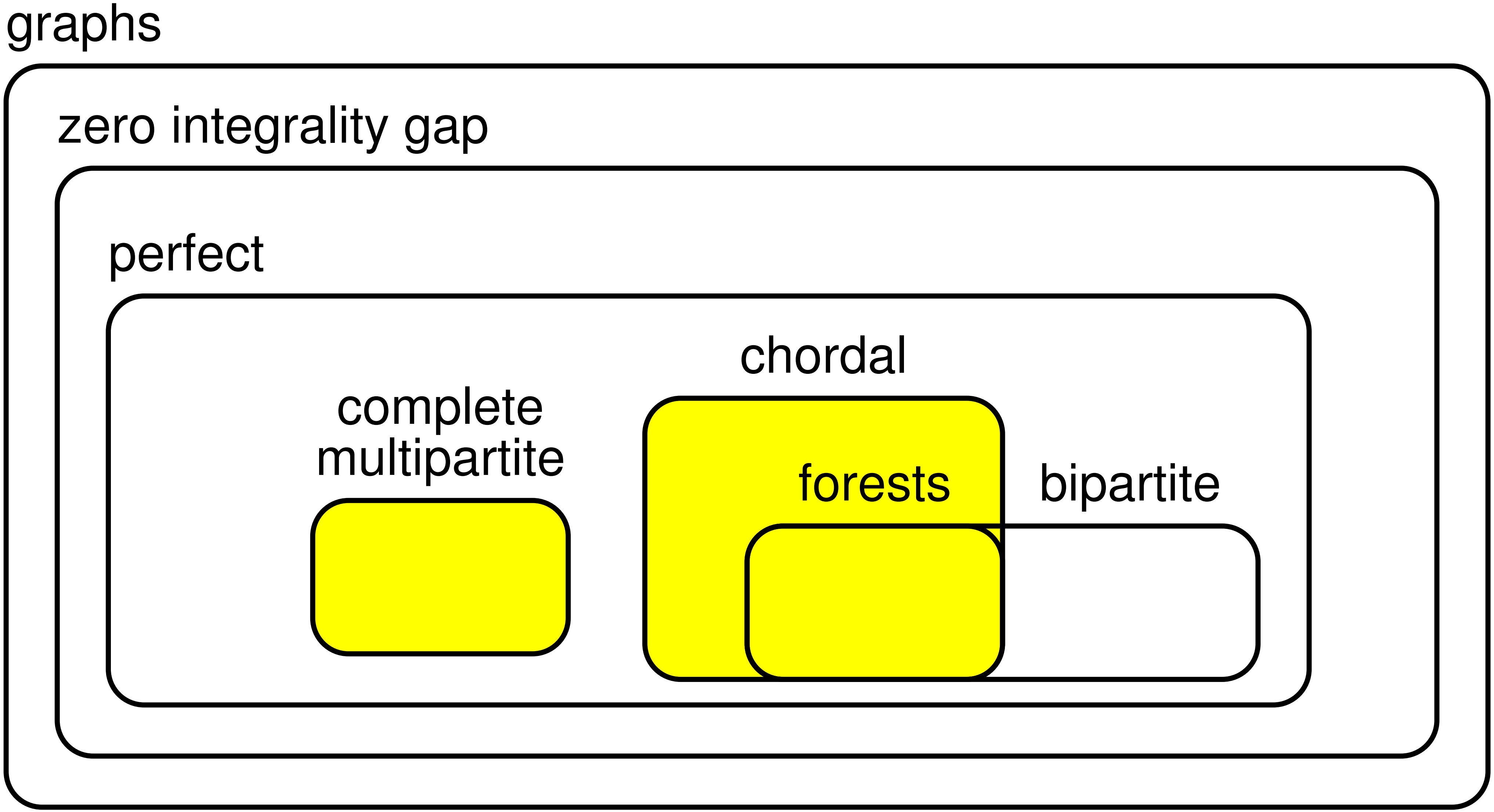
By solving a sequence of LP problems, we can obtain the nucleolus (not in poly time).

(2) Identify the essential coalitions. (Huberman '80)

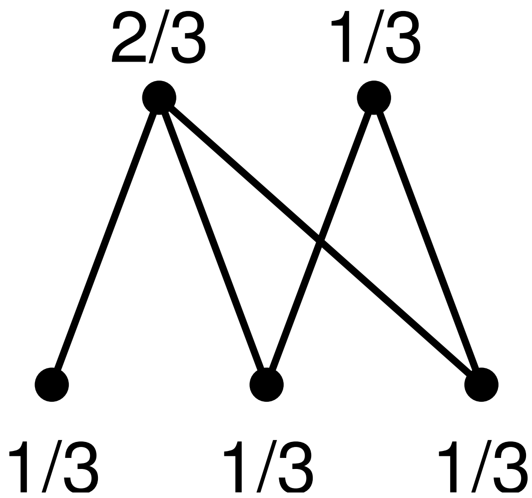
The essential coalitions reduce the work load.
$S \subseteq V$ is essential $\Leftrightarrow$ S is an independent set.

(3) Analyze the LP.

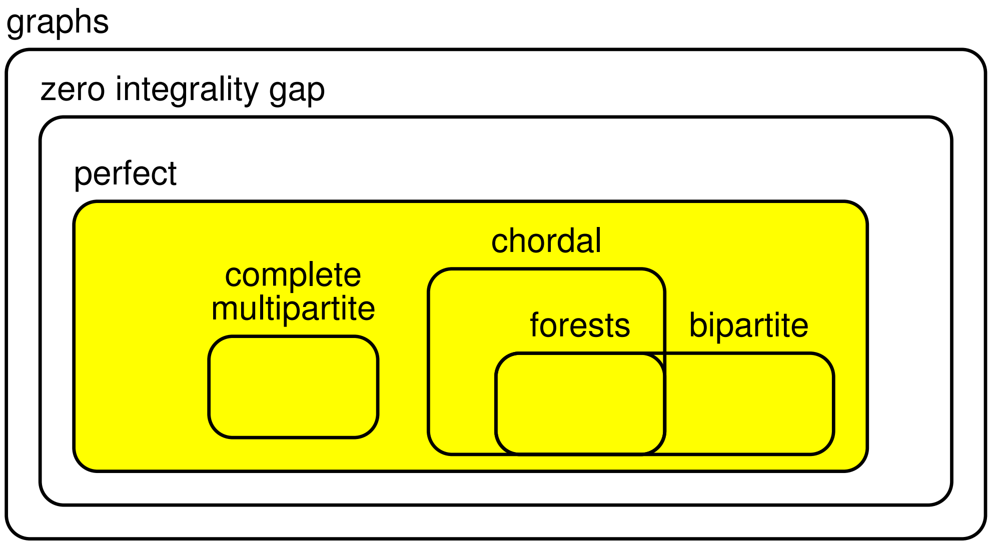
For O-good perfect graphs, we can precisely tell what are the optimal solutions in the LP problems with help of the characterization of the extreme points of the core.


graphs
zero integrality gap
perfect
chordal
complete
multipartite
forests
bipartite

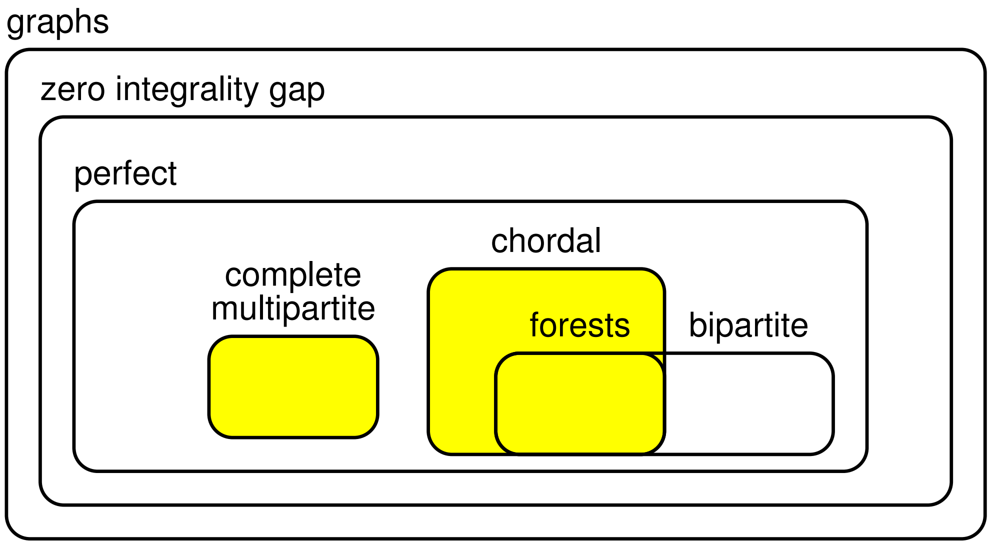
There is a bipartite graph for which the nucleolus is not the barycenter of the char. vectors of the maximum cliques.



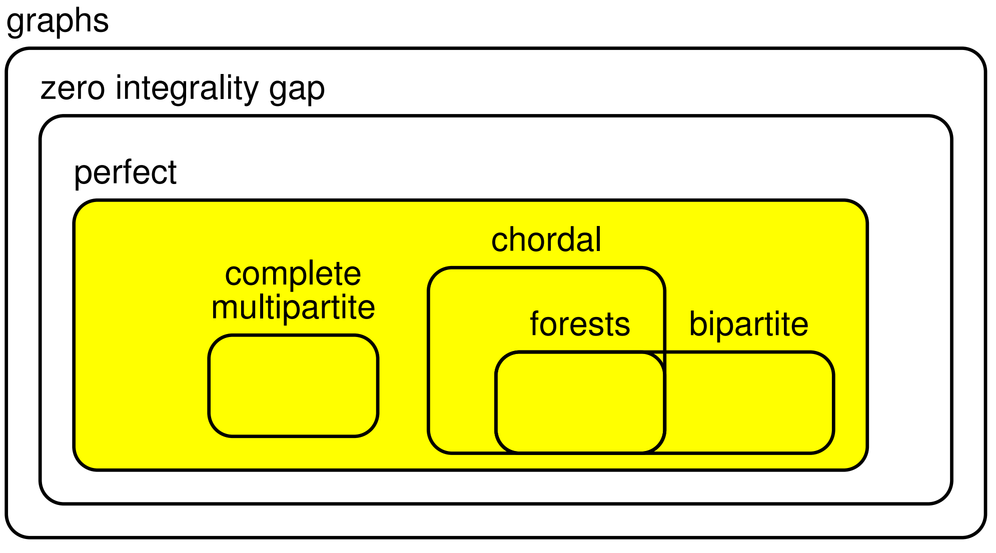
Core



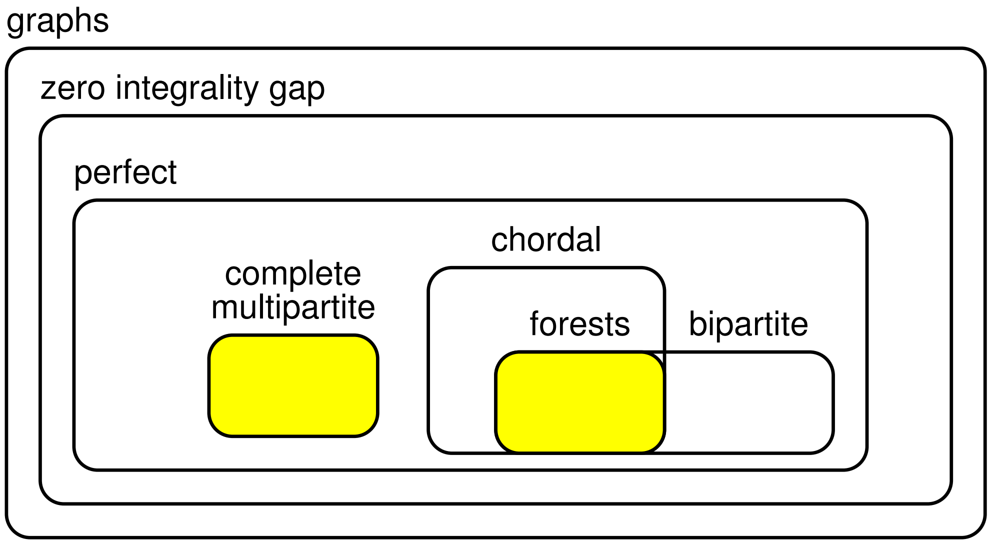
Nucleolus



τ-value



Shapley value



Computation of the nucleoli for

- perfect graphs ??
- bipartite graphs ??
- outerplaner graphs ??
- cographs ??

The list of what we discussed

- Def.: minimum coloring game
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  (optimization theory)
- They want to allocate the benefit in a fair way.
  (cooperative game theory)

Status of algorithmic problems on cooperative games

- As many cooperative games as optimization problems!!
- Many algorithmic problems remain unsolved!!

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Status of algorithmic problems on cooperative games

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$\Longrightarrow$ Why not work on them??

[End of the talk]