Local topology of the free complex of a two-dimensional generalized convex shelling

Yoshio Okamoto

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Mittagsseminar

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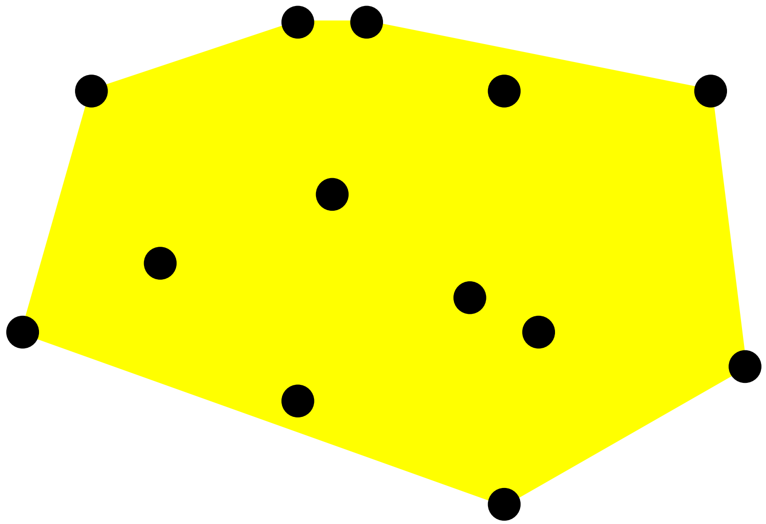
CGC
Combinatorics
Geometry
Computation

How many interior points are there in a finite point configuration $\mathcal{P}$?



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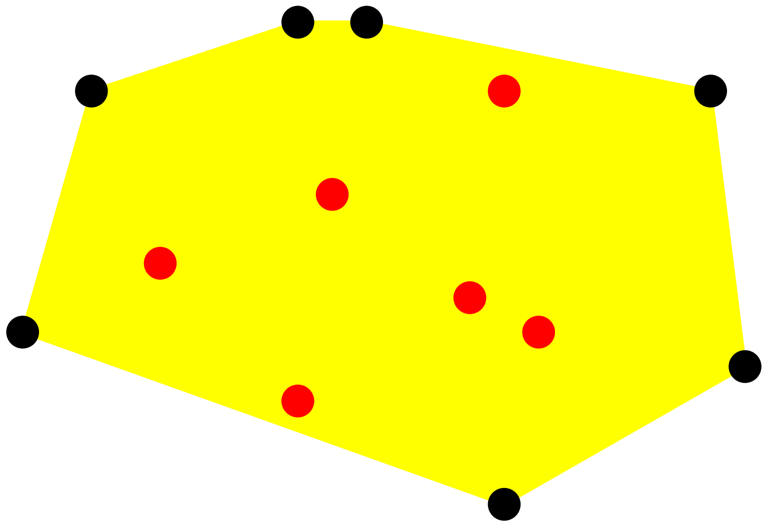

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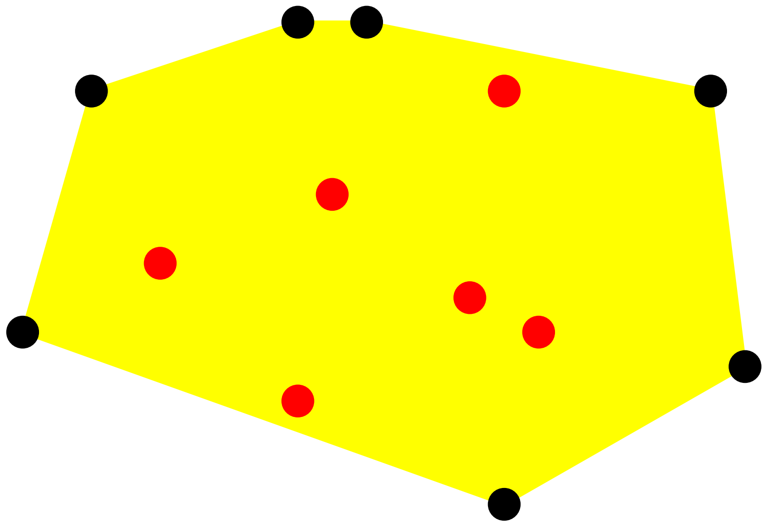

An Euler-type formula:

$$\#\text{ of int. pts in } \mathcal{P} = (-1)^{d-1} \sum_{\text{free } A \subseteq \mathcal{P}} (-1)^{|A|} |A|.$$

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Proved by:

- Ahrens, Gordon & McMahon (DCG '99)
  for $d = 2$, geometric proof
  Conj.: This formula holds for general d.

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Proved by:

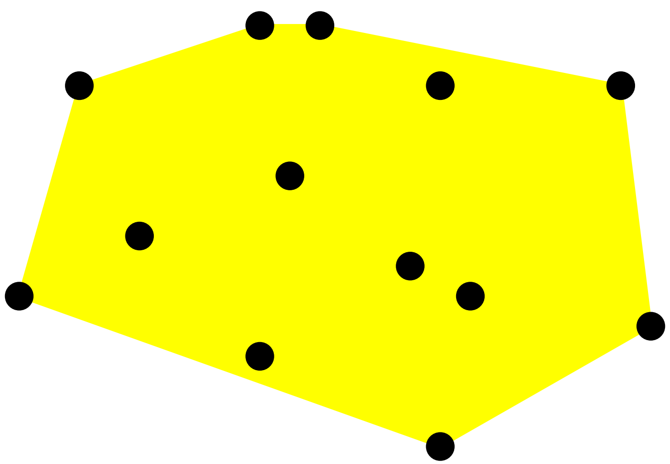
- Ahrens, Gordon & McMahon (DCG '99)
  for $\mathrm{d} = 2$, geometric proof
- Klain (Adv Math '99)
  for general d, using a valuation
- **Edelman & Reiner** (DCG '00)
  for general d, topological proof
  $\longrightarrow\longrightarrow\longrightarrow$ making use of **free complexes**

Free sets in a point configuration

$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$.

Def. $X \subseteq \mathcal{P}$ is **free** if

- $\text{conv}(X) \cap \mathcal{P} = X$ (convexity)
- the extreme points of $\text{conv}(X) = X$ (the points of X lie in convex position) (independence).



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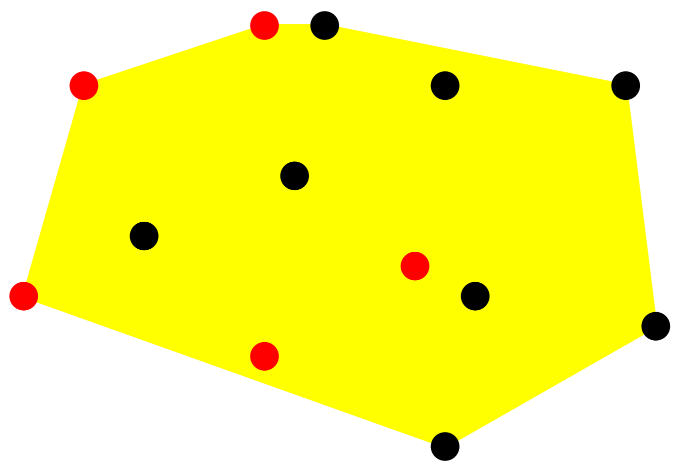

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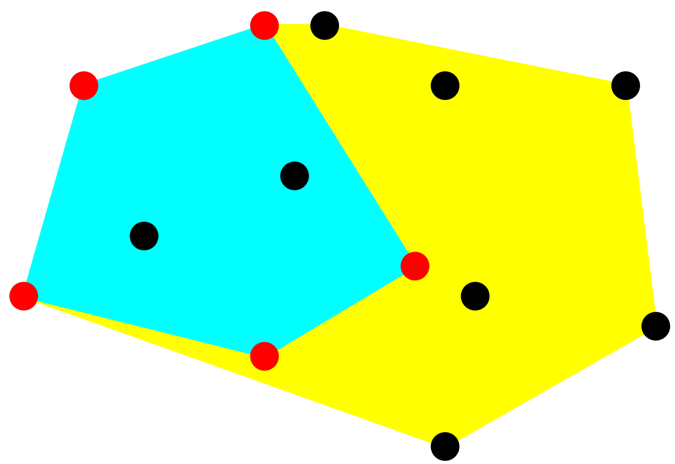

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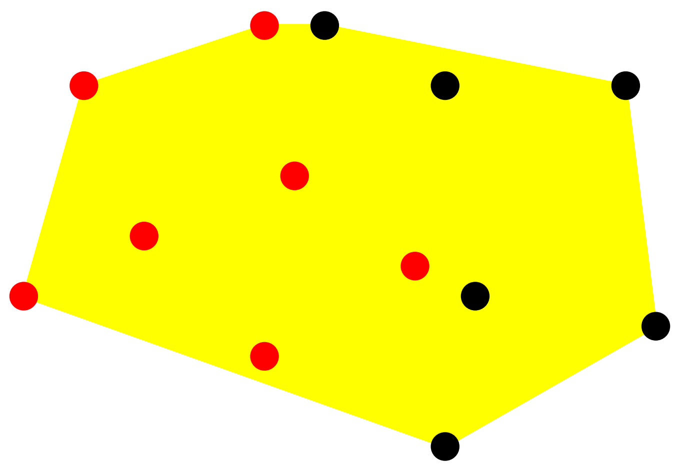

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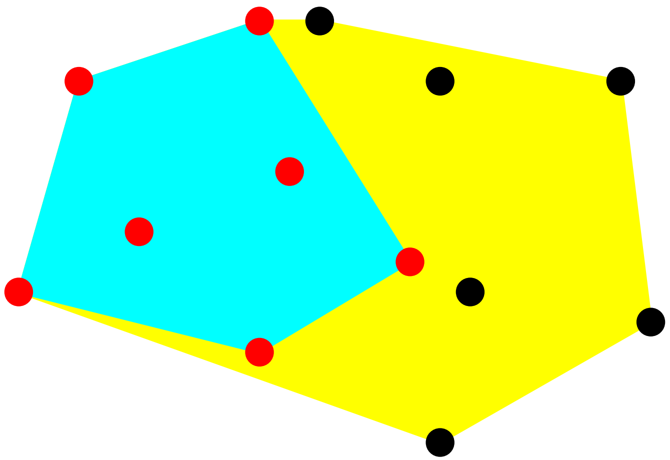

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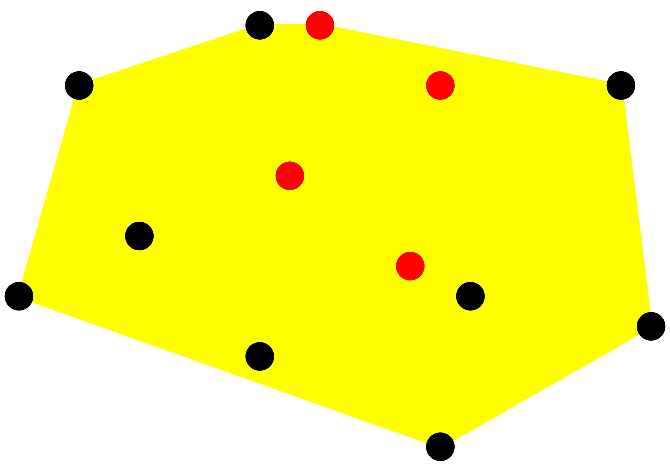

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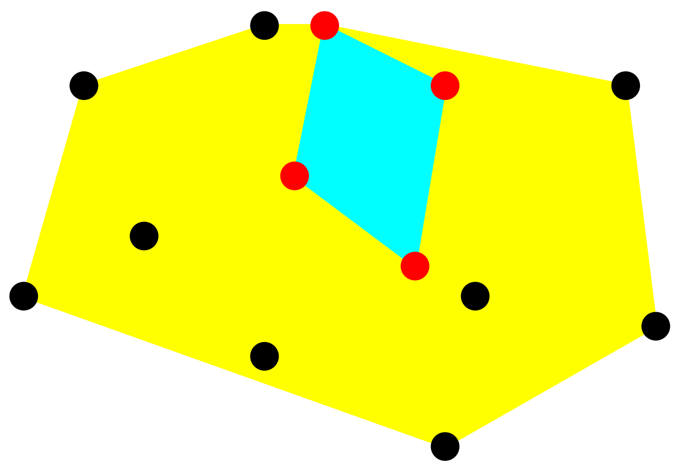
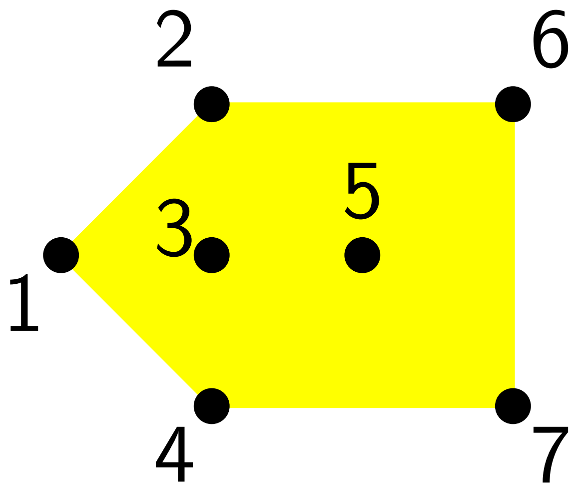
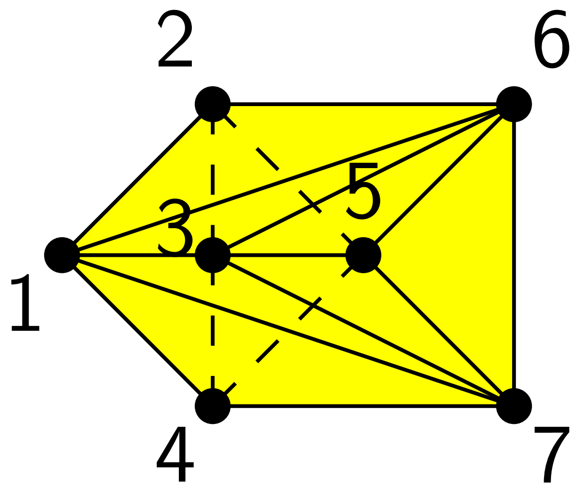


Def. The **free complex** of $\mathcal{P}$ is the family of all free sets in $\mathcal{P}$, denoted by $\mathrm{Free}(\mathcal{P})$.

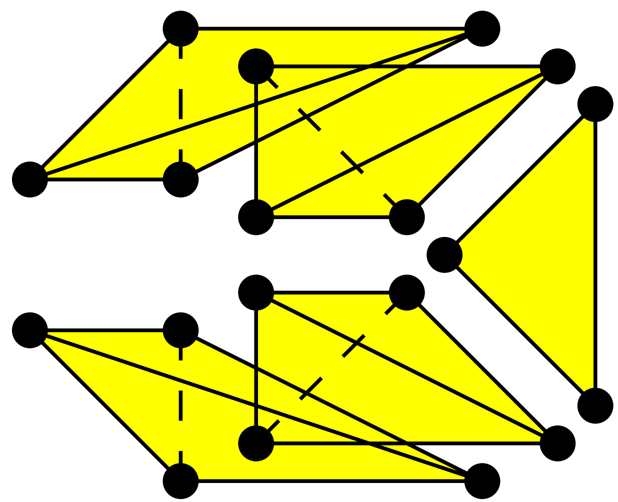
Remark $\mathrm{Free}(\mathcal{P})$ is a simplicial complex.



$\mathcal{P}$



$\mathrm{Free}(\mathcal{P})$



the facets of $\mathrm{Free}(\mathcal{P})$

Proof by Edelman & Reiner

$\mathcal{P}$ a finite point configuration in $\mathbb{R}^{\mathrm{d}}$

Consider the free complex $\mathsf{Free}(\mathcal{P})$ of $\mathcal{P}$.

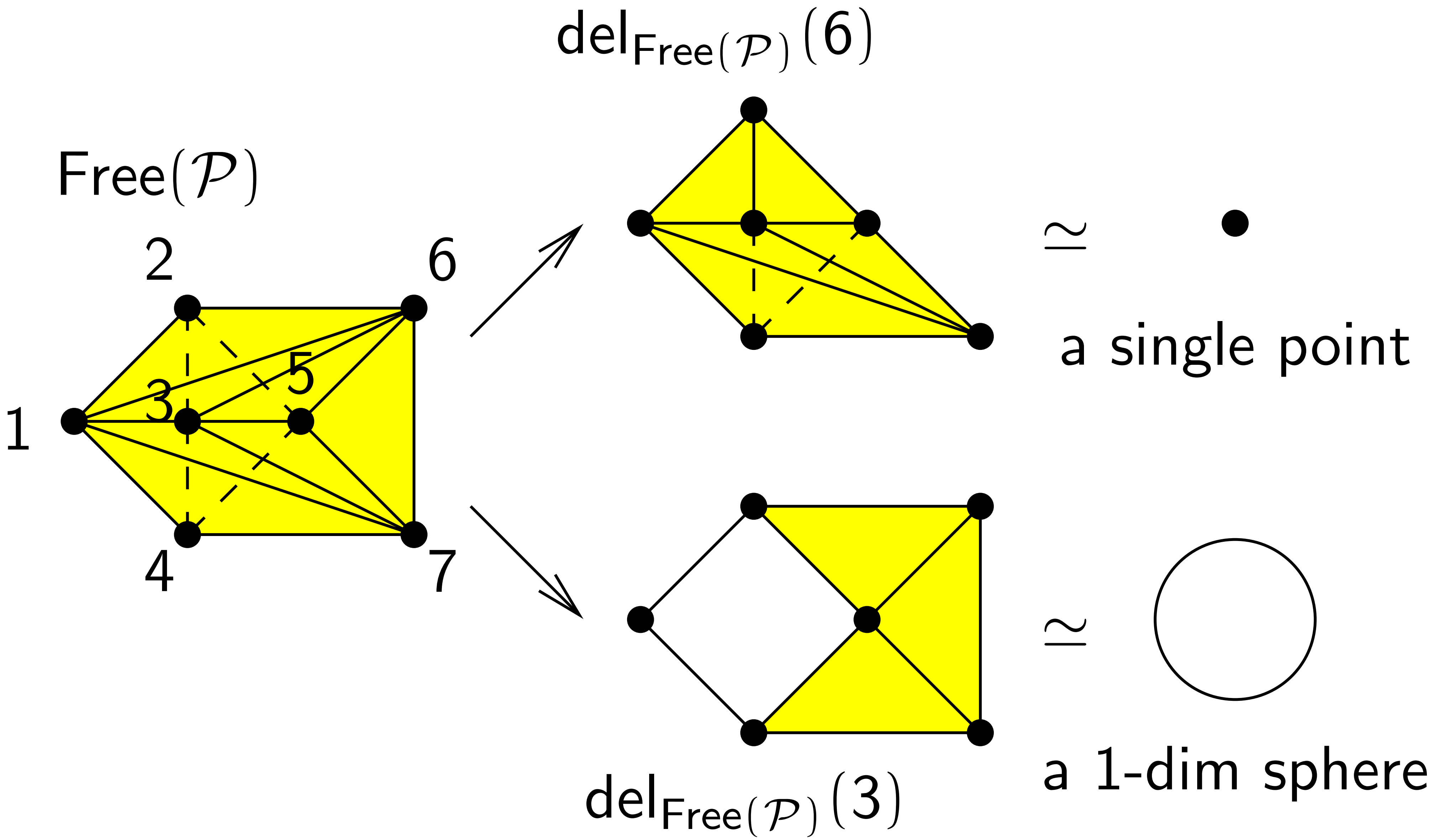
$\mathcal{P}$ a finite point configuration in $\mathbb{R}^d$

Consider the free complex $\mathrm{Free}(\mathcal{P})$ of $\mathcal{P}$.

In Edelman & Reiner's proof, it was a key that

- **$\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)$ is contractible if $x \in \mathcal{P}$ lies on the bd of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)) = 0$),
- **$\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)$ has the integral homology of a $(d-1)$-dim sphere if $x \in \mathcal{P}$ lies in the interior of $\mathrm{conv}(\mathcal{P})$**
  (implying $\tilde{\chi}(\mathrm{del}_{\mathrm{Free}(\mathcal{P})}(x)) = (-1)^{d-1}$).

Proof by Edelman & Reiner: Example



Q.

How about a generalization to abstract convex geometries??

This work

- Study on their problems for a special case (2-dim generalized convex shellings).
- Result for this special case.

(1) (Abstract) convex geometries and Free complexes

(2) Questions by Edelman & Reiner for (abstract) convex geometries

(3) 2-dim generalized convex shellings

(4) Results

Convex geometries (Edelman & Jamison '85)

E a nonempty finite set,
$\mathcal{L} \subseteq 2^{E}$ a family of subsets of E.

$\boxed{\text{Def.}}$ $\mathcal{L}$ is called a **convex geometry** on E if $\mathcal{L}$ satisfies the following conditions.

(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$.

(2) $X, Y \in \mathcal{L} \Rightarrow X \cap Y \in \mathcal{L}$.

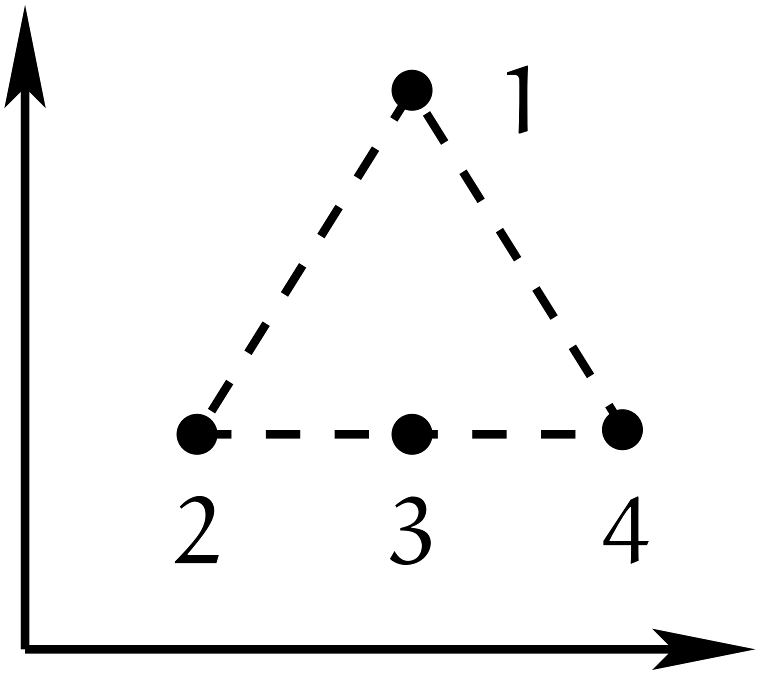
(3) $X \in \mathcal{L} \setminus \{E\} \Rightarrow \exists\, e \in E \setminus X$: $X \cup \{e\} \in \mathcal{L}$.

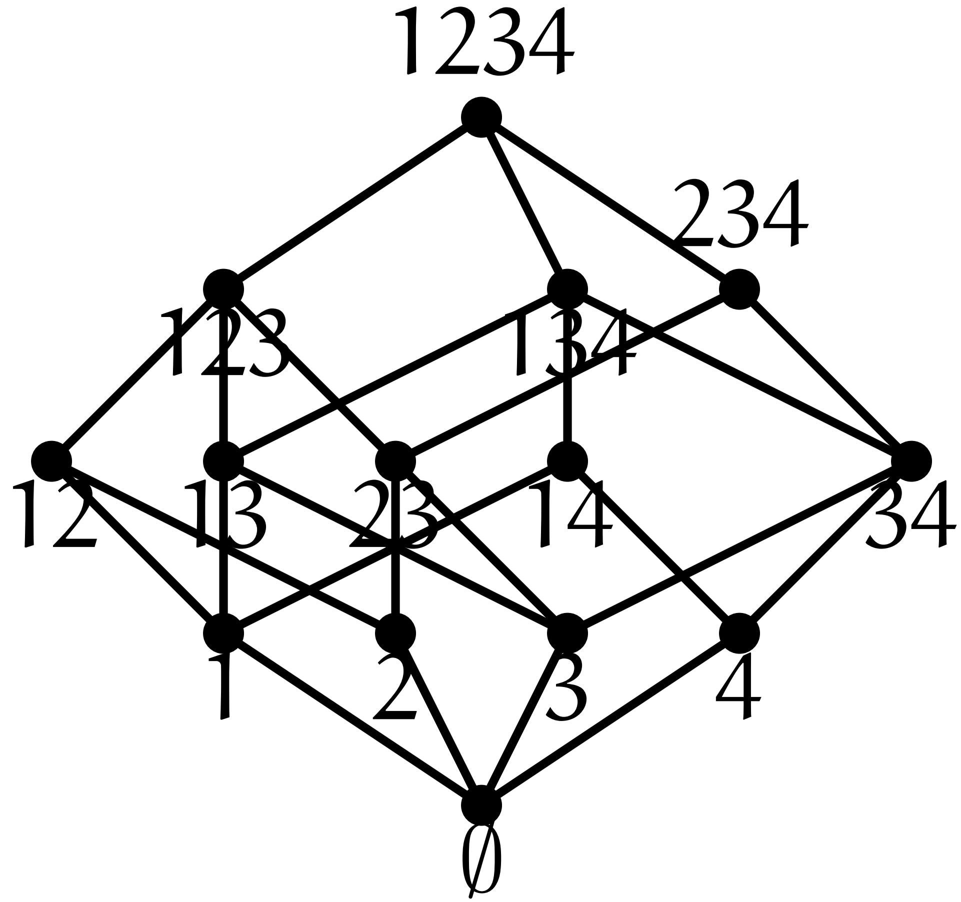
$X \subseteq E$ is called **convex** if $X \in \mathcal{L}$.

Example 1: convex shelling

$\mathcal{P}$ a finite point set in $\mathbb{R}^d$.

Define: $\boldsymbol{\mathcal{L}} = \{\mathbf{X} \subseteq \boldsymbol{\mathcal{P}} : \mathbf{conv}(\mathbf{X}) \cap \boldsymbol{\mathcal{P}} = \mathbf{X}\}$.





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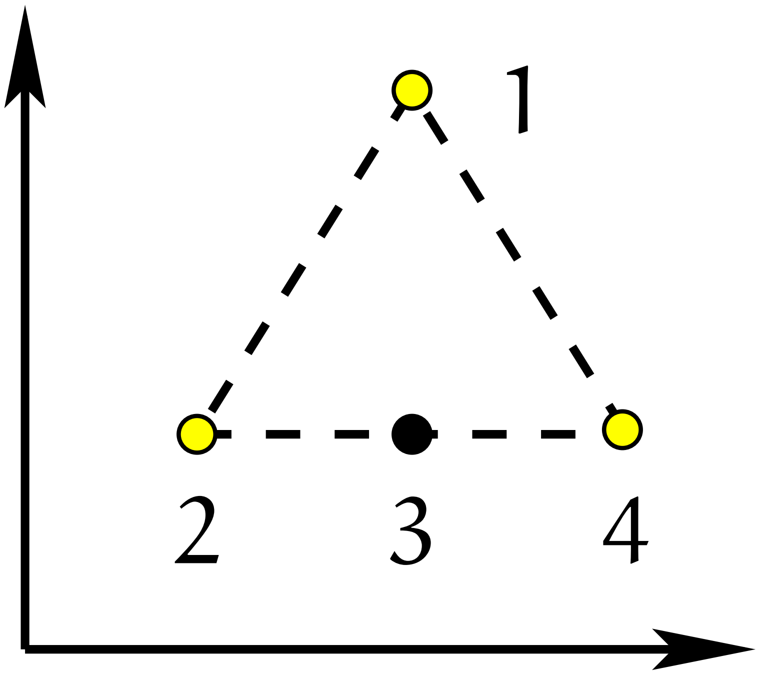

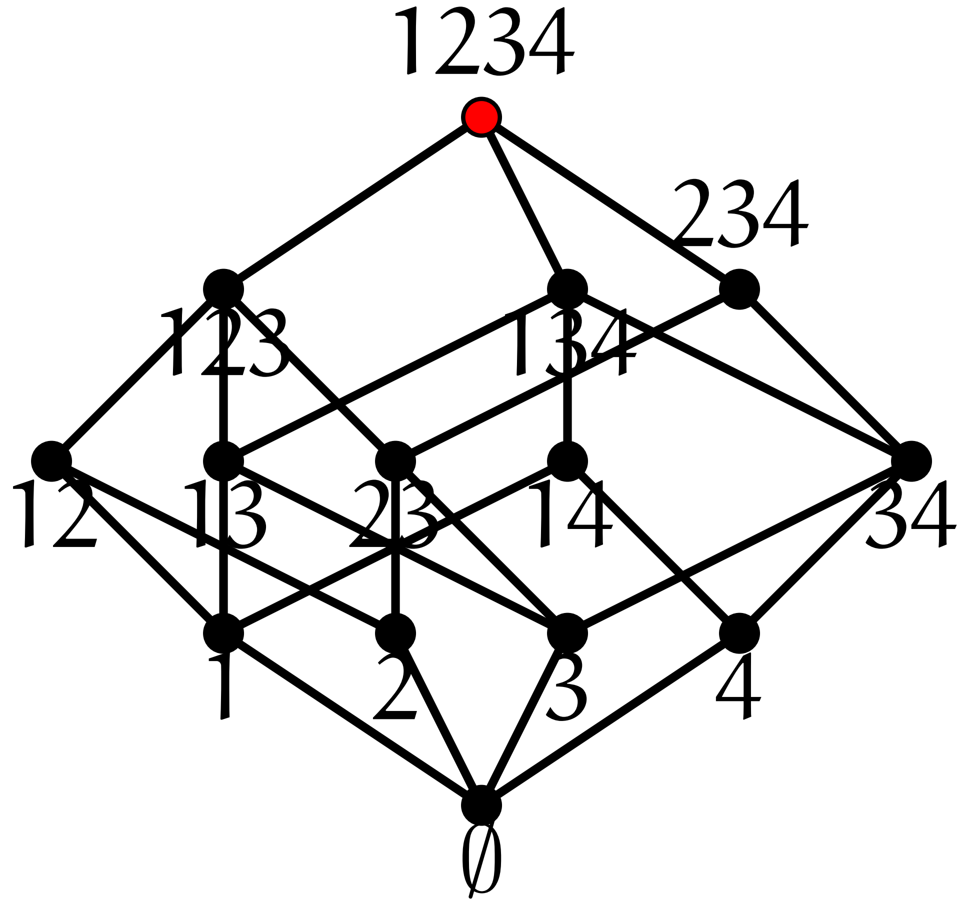

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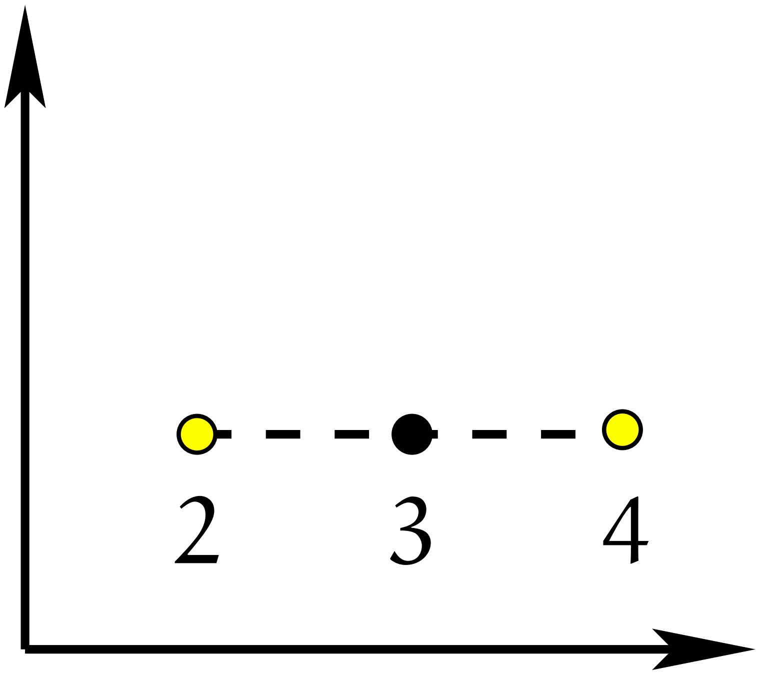

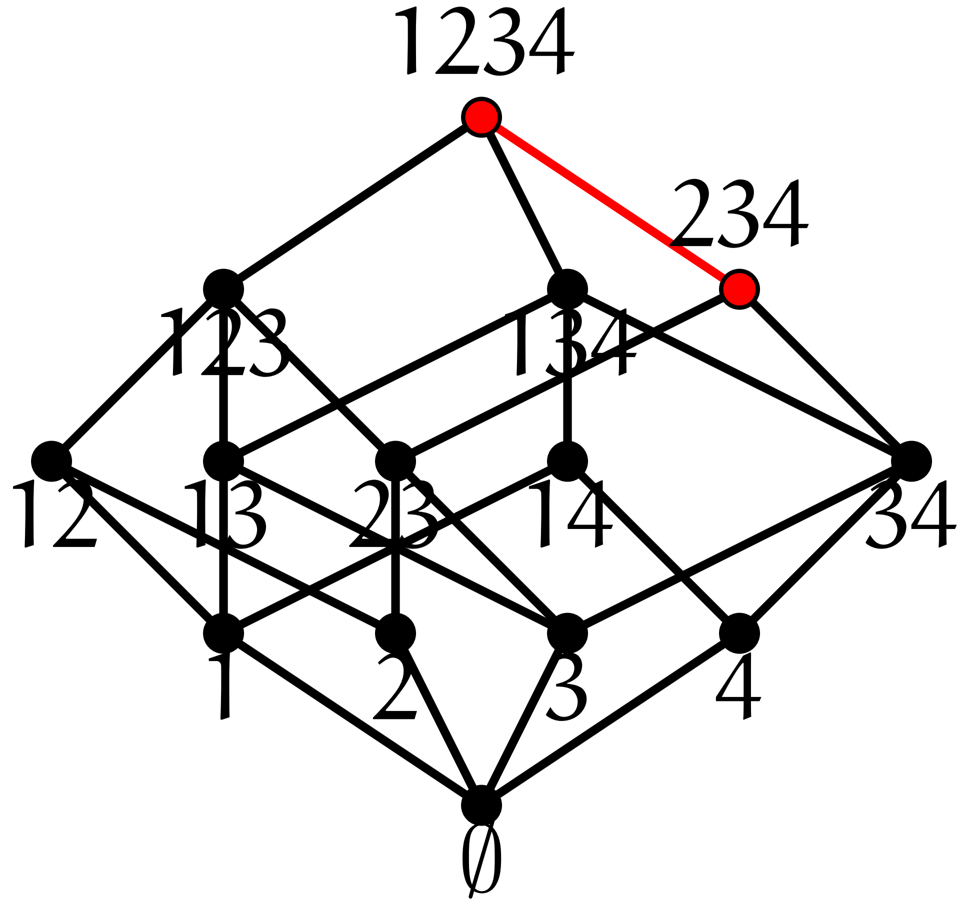

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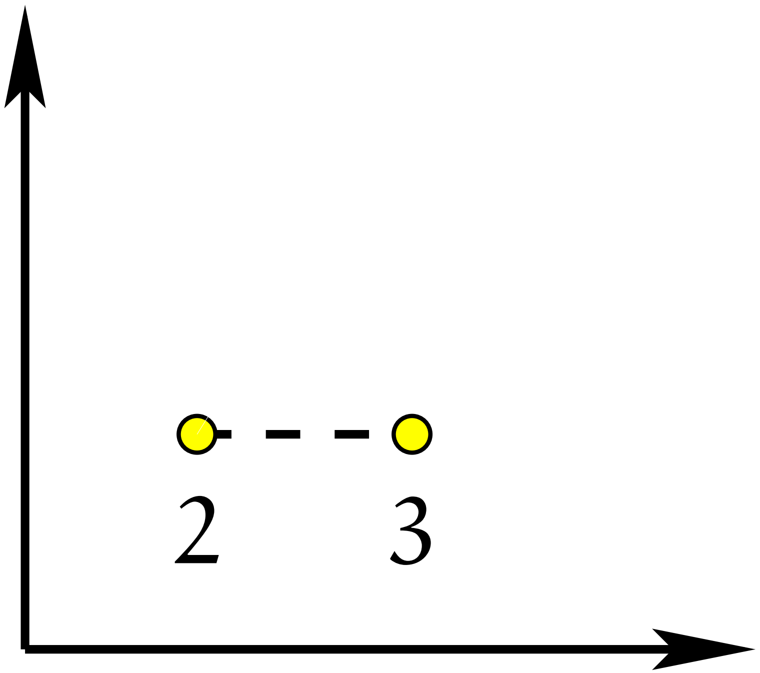

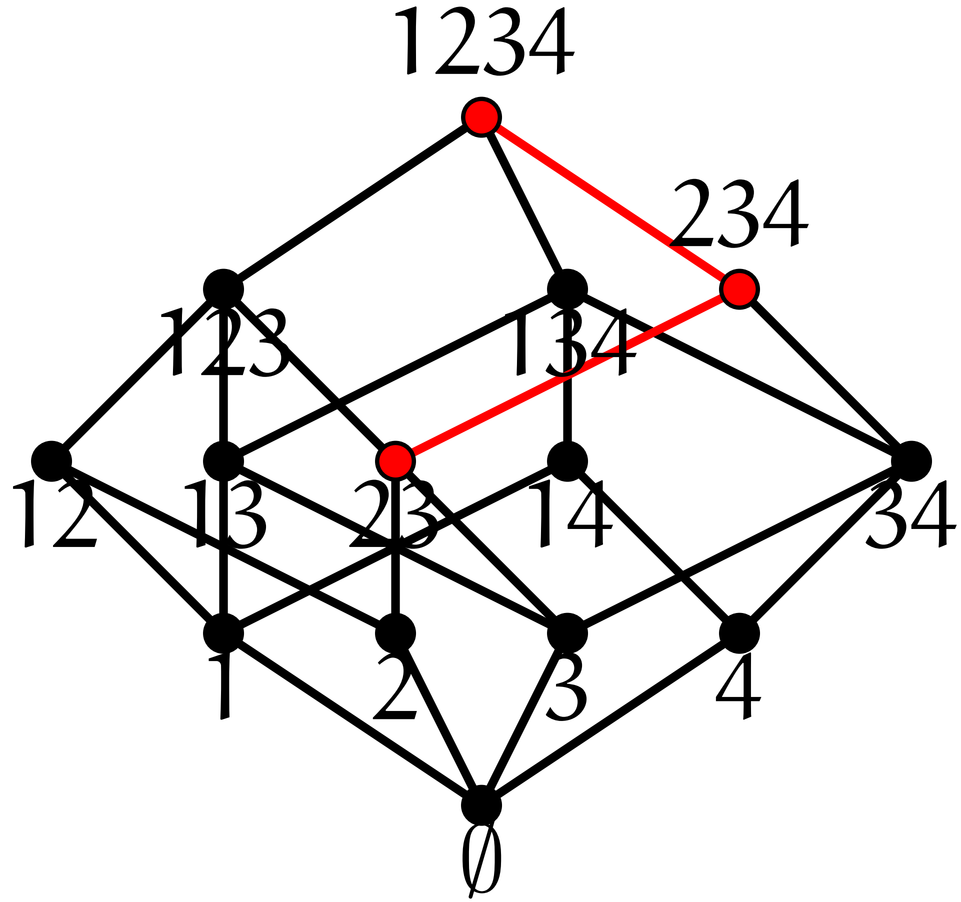

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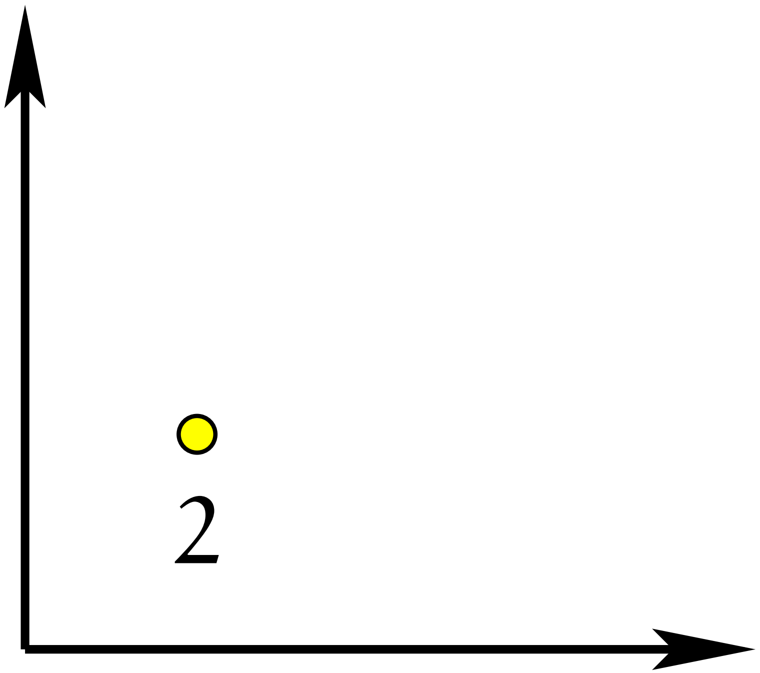

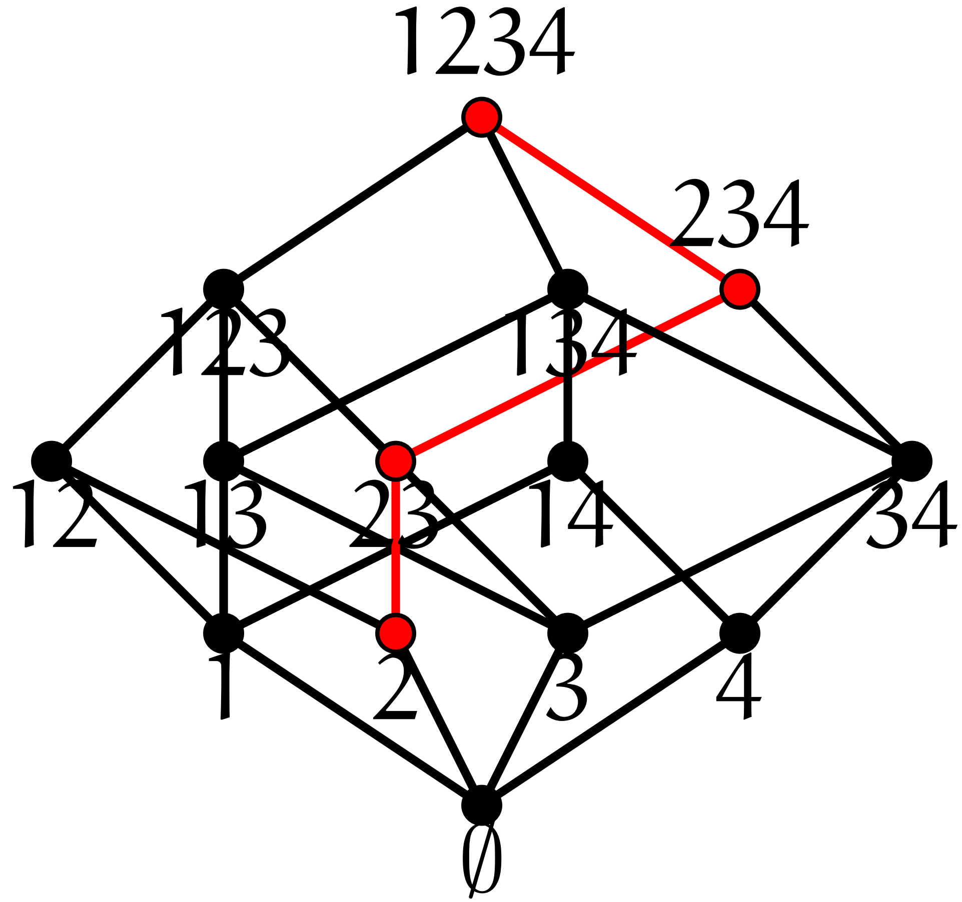

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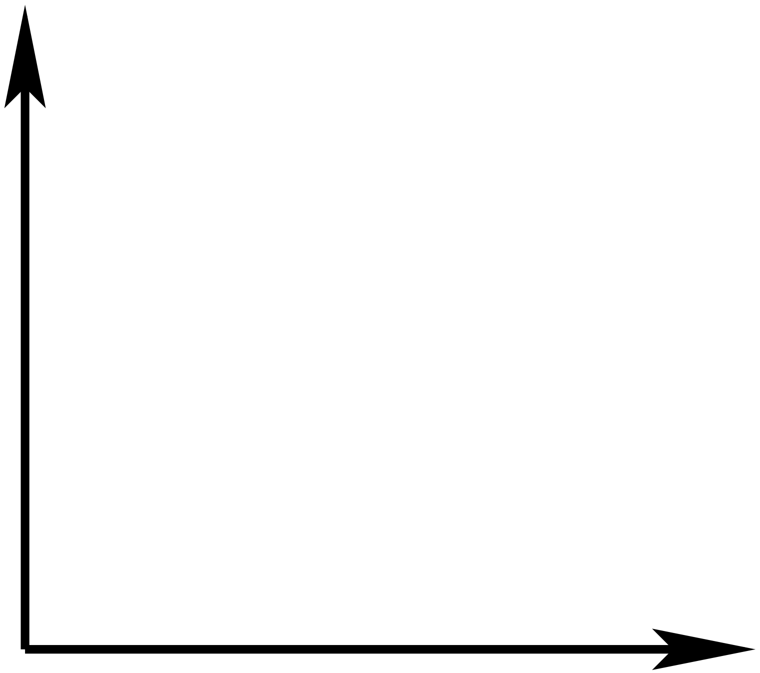

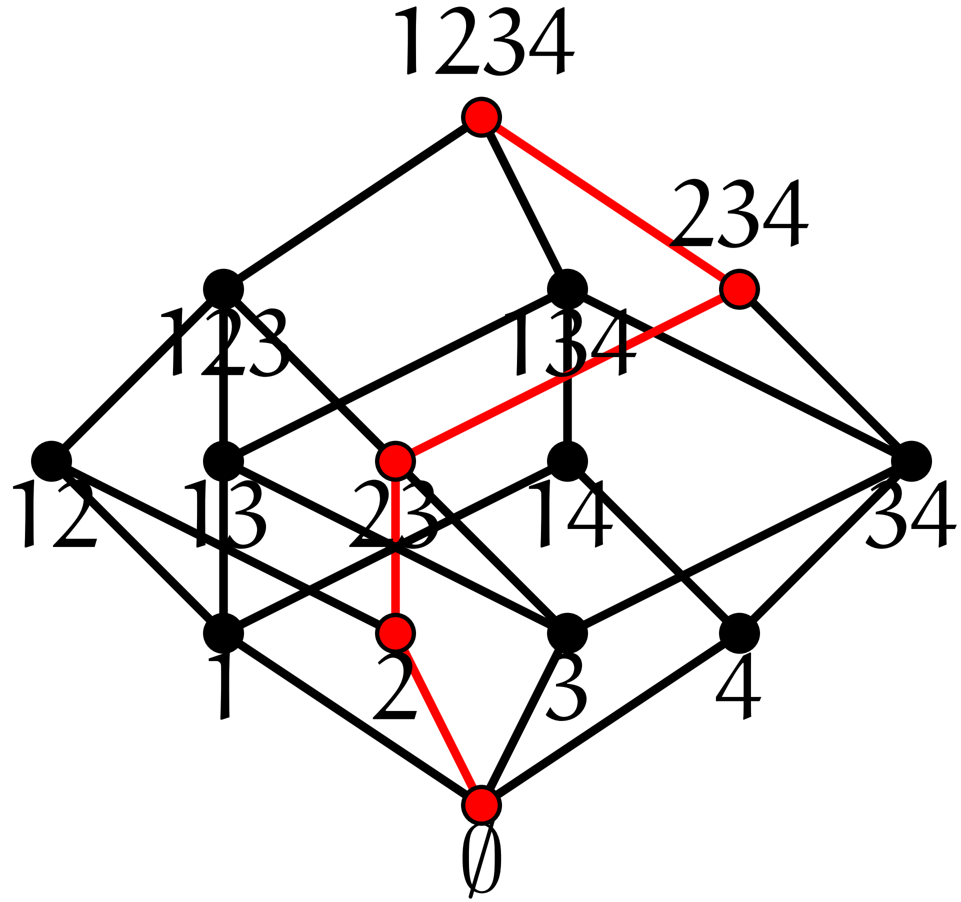

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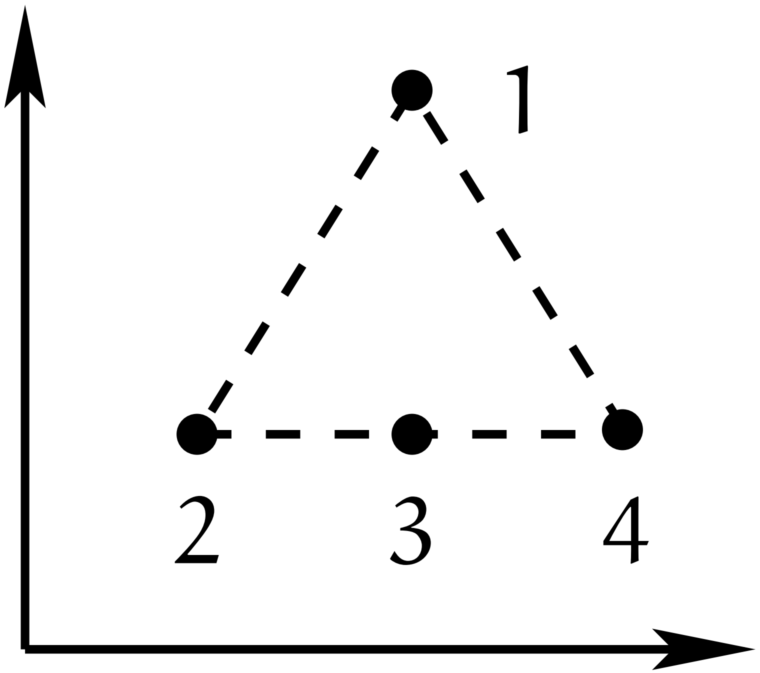

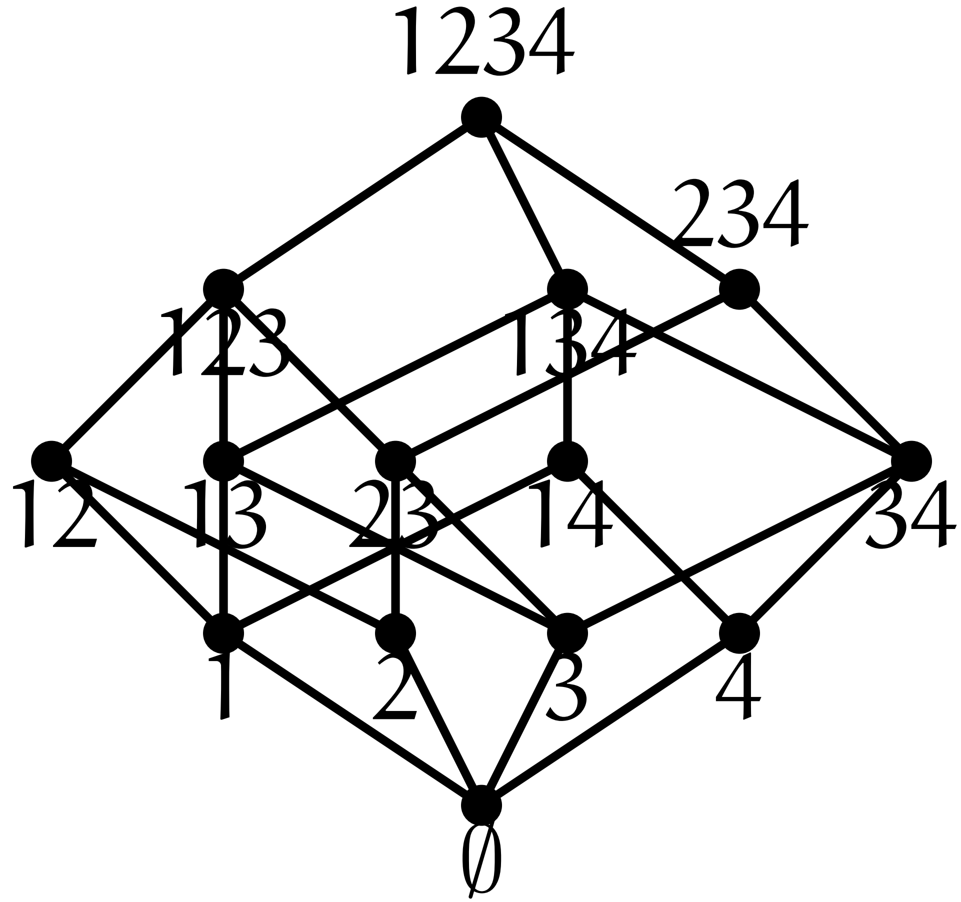

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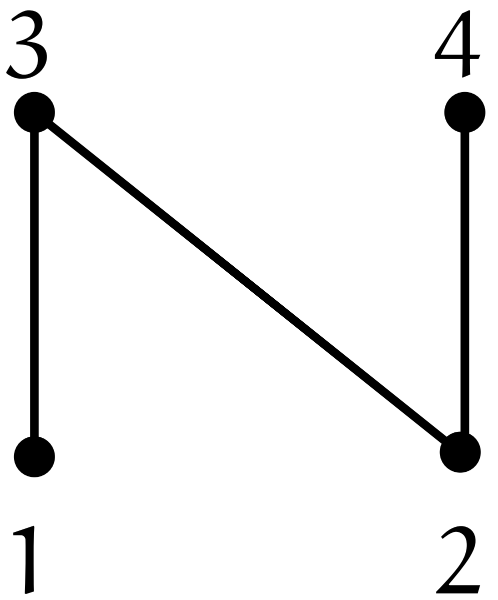

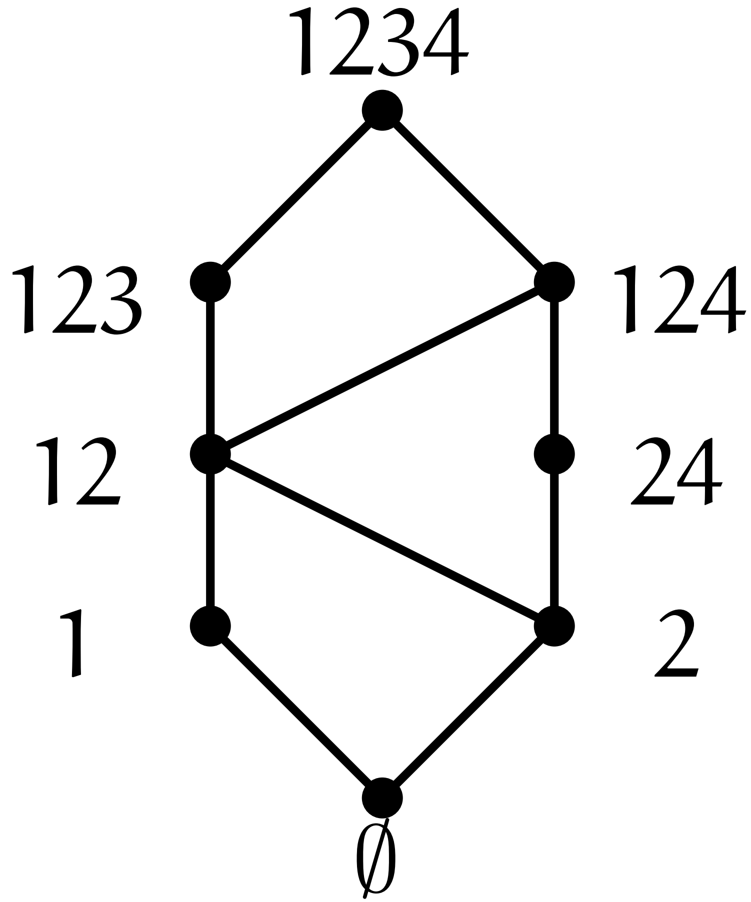

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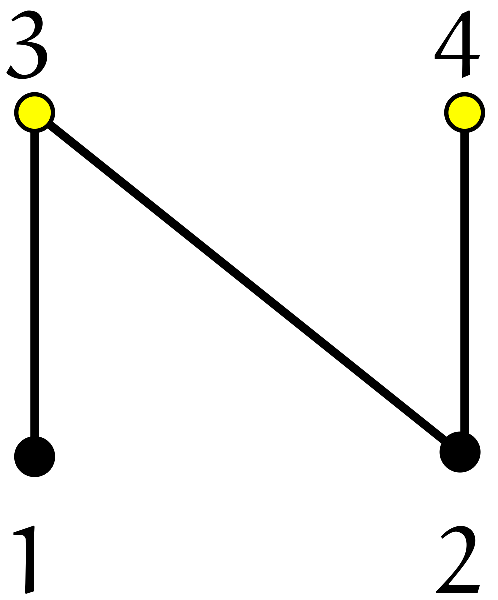

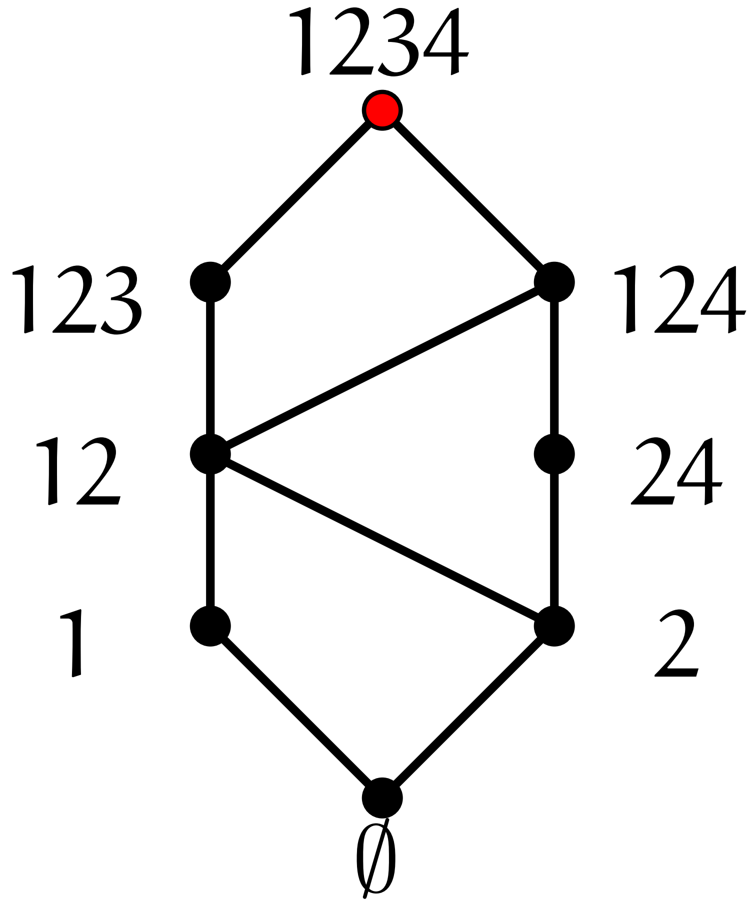

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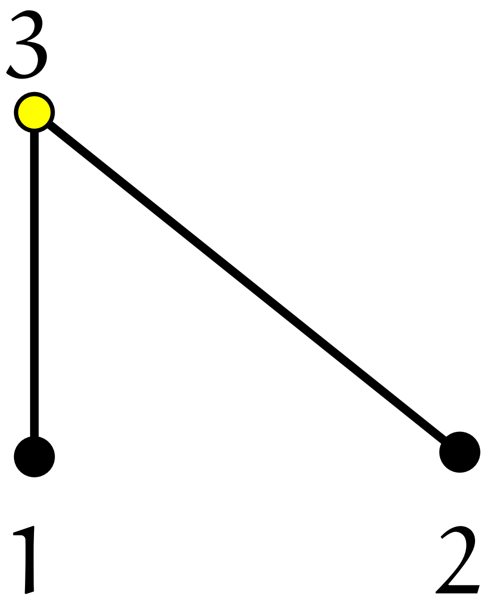

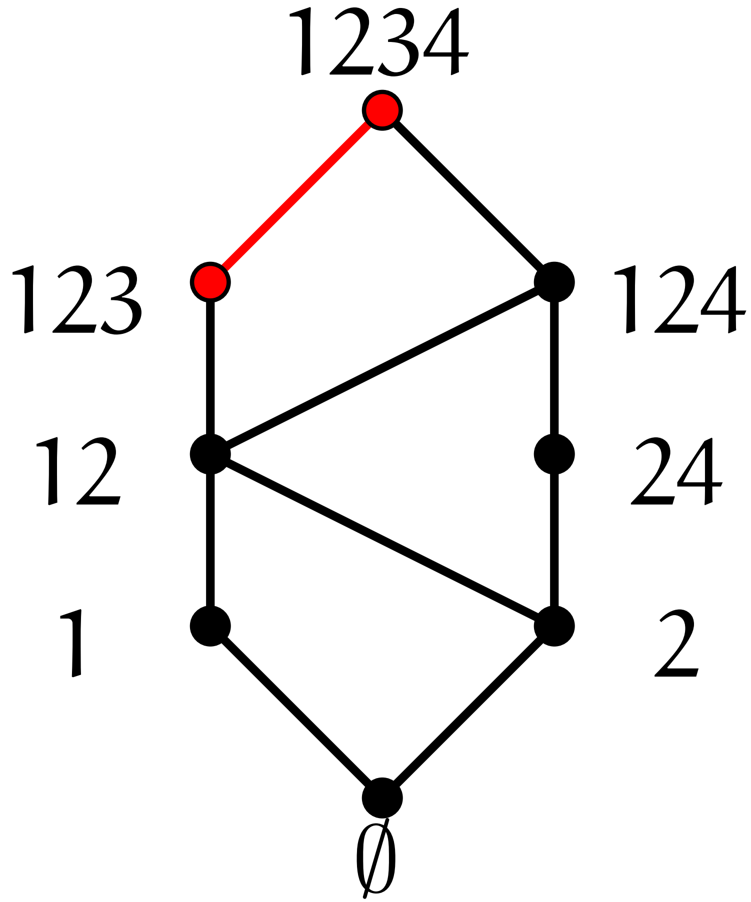

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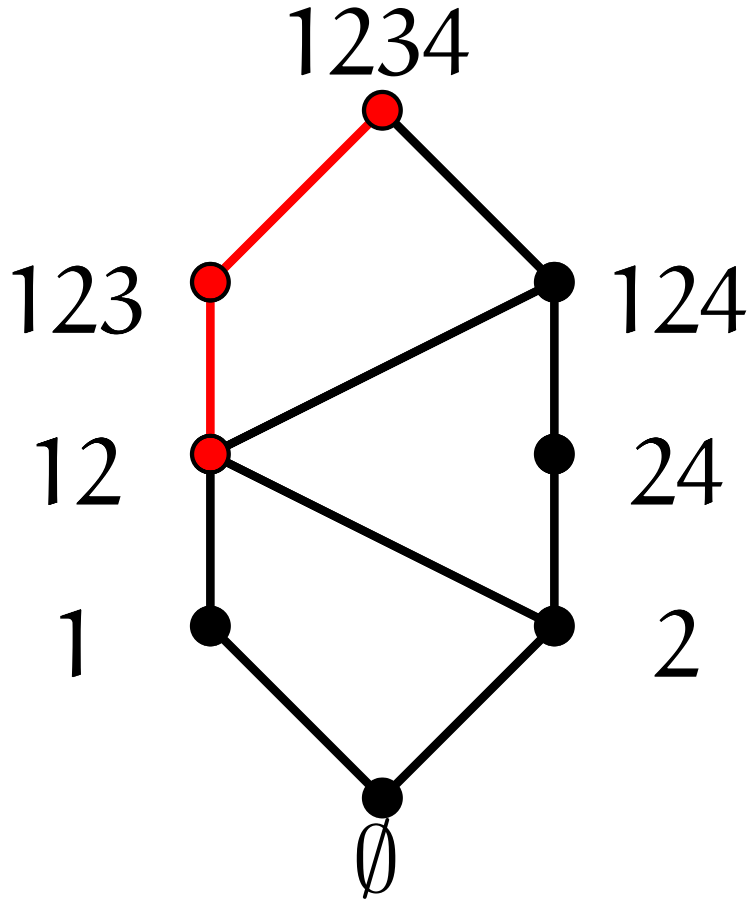

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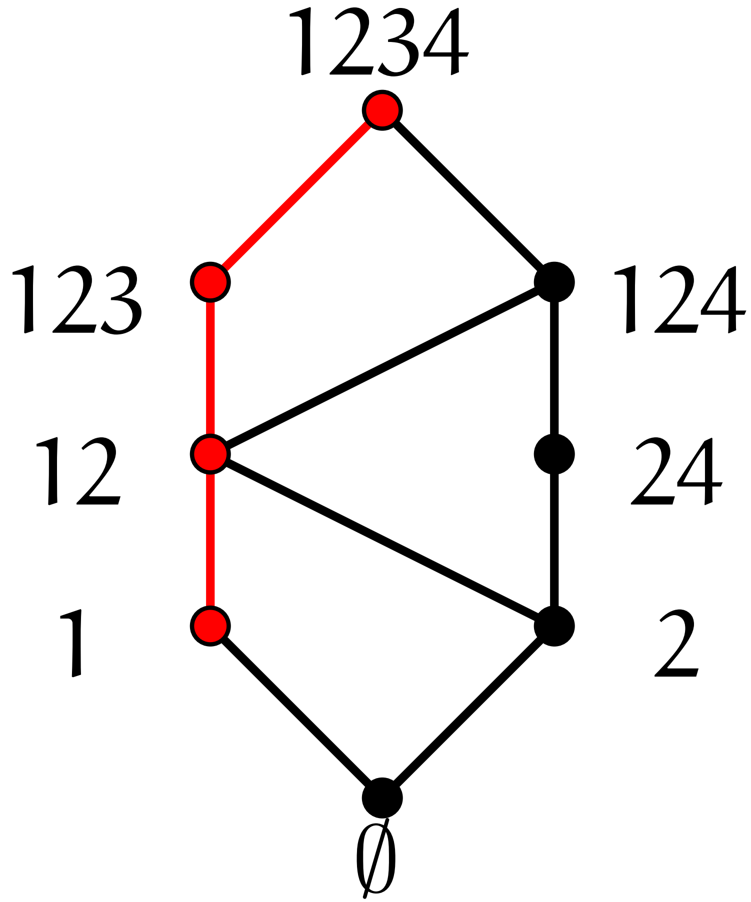

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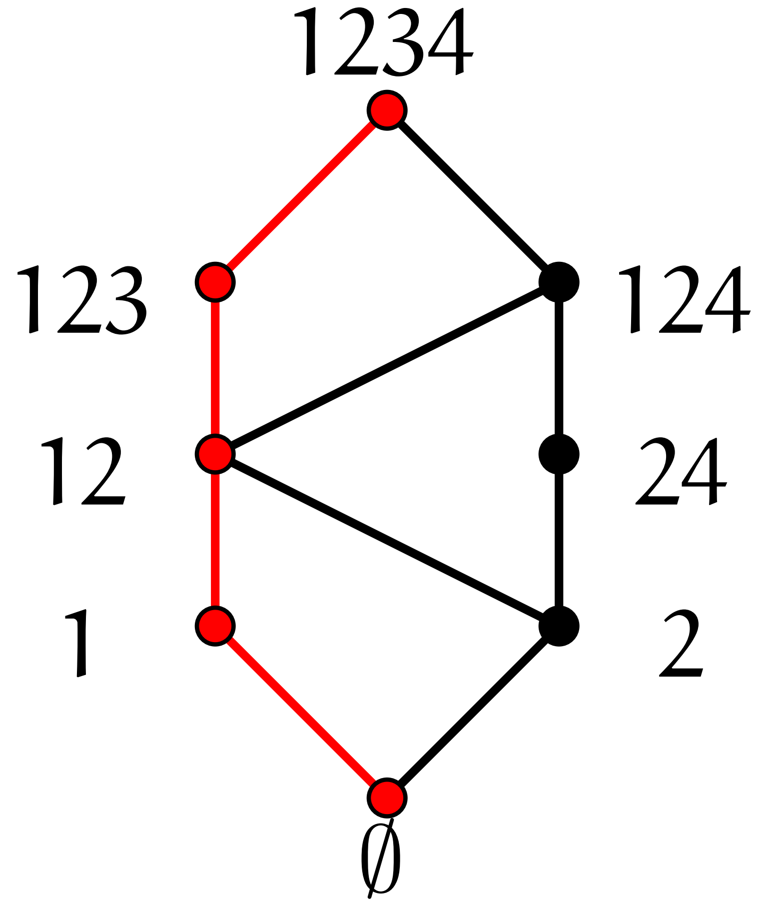

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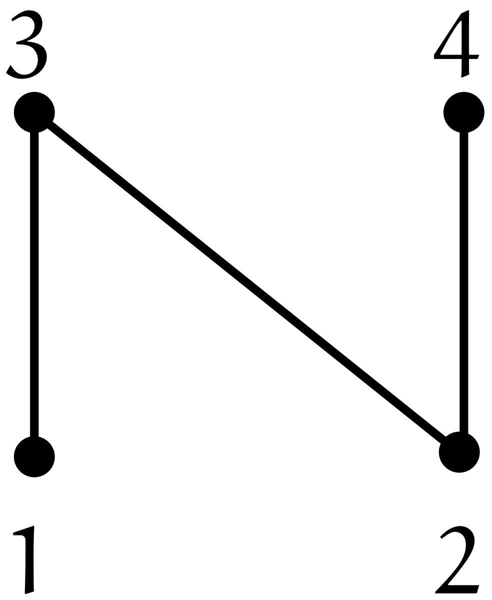

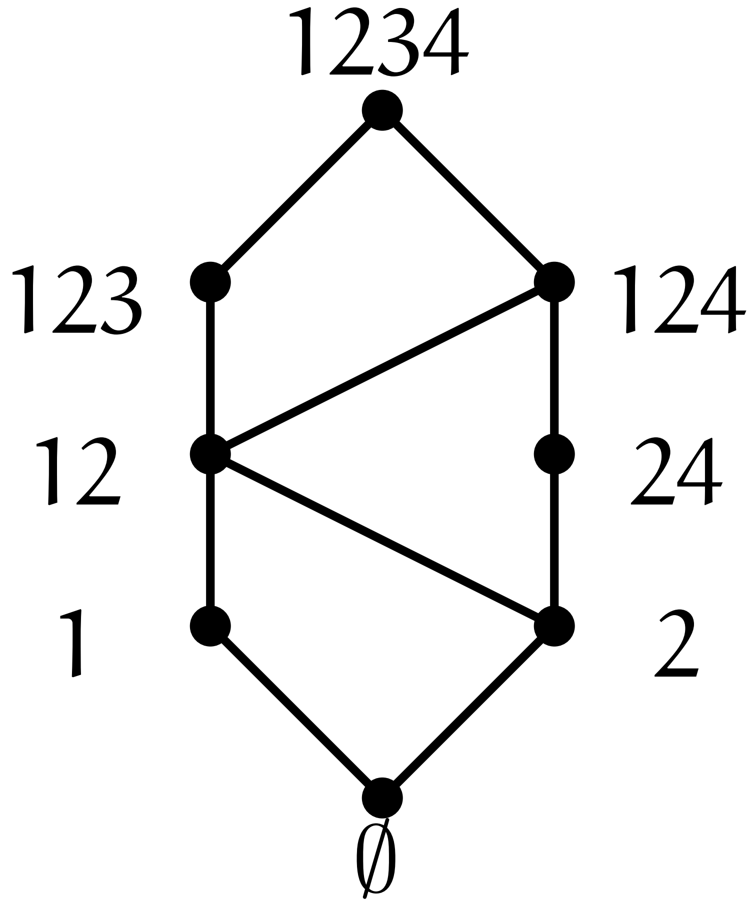

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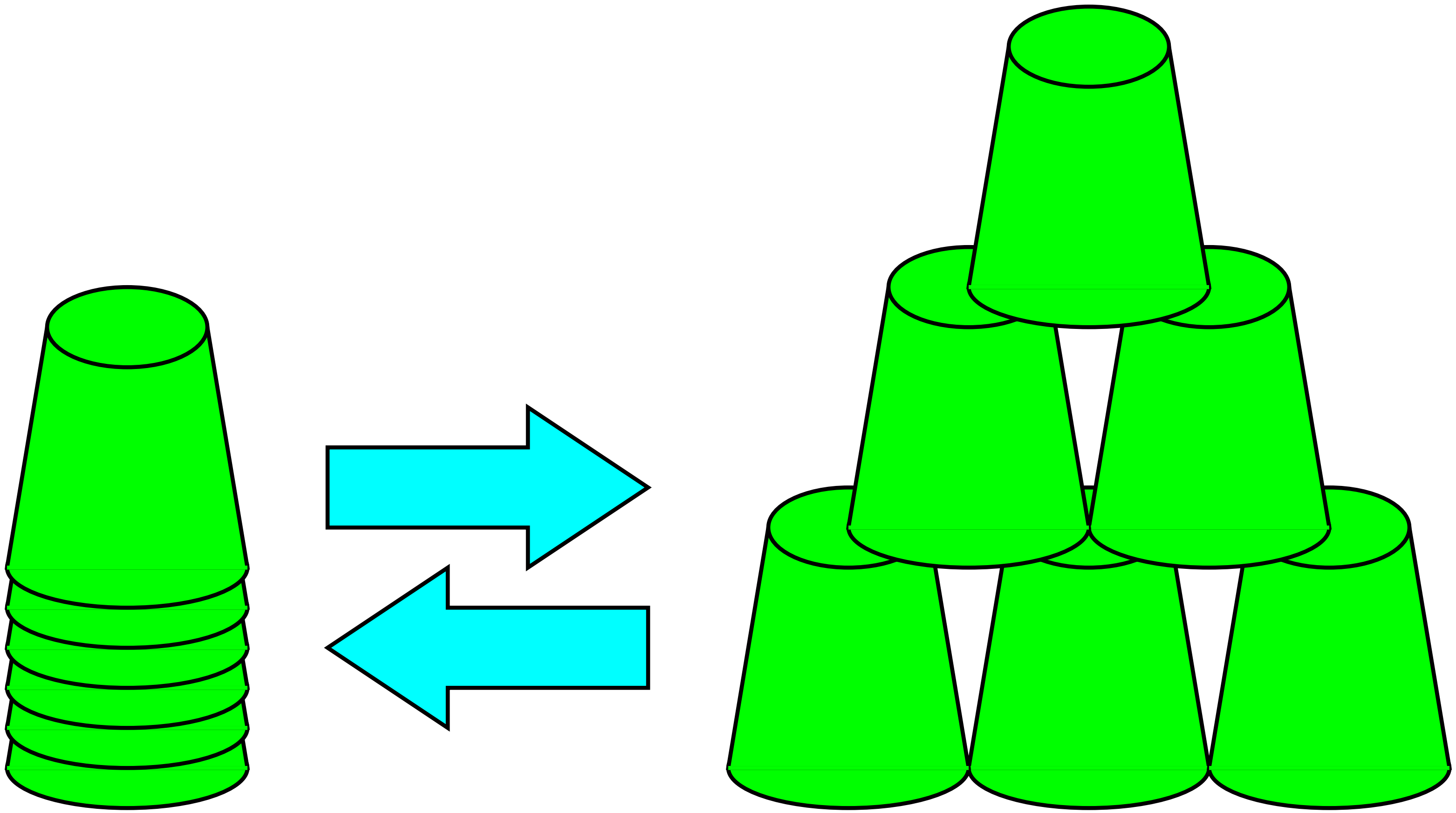
Convex geometries arise from various objects.

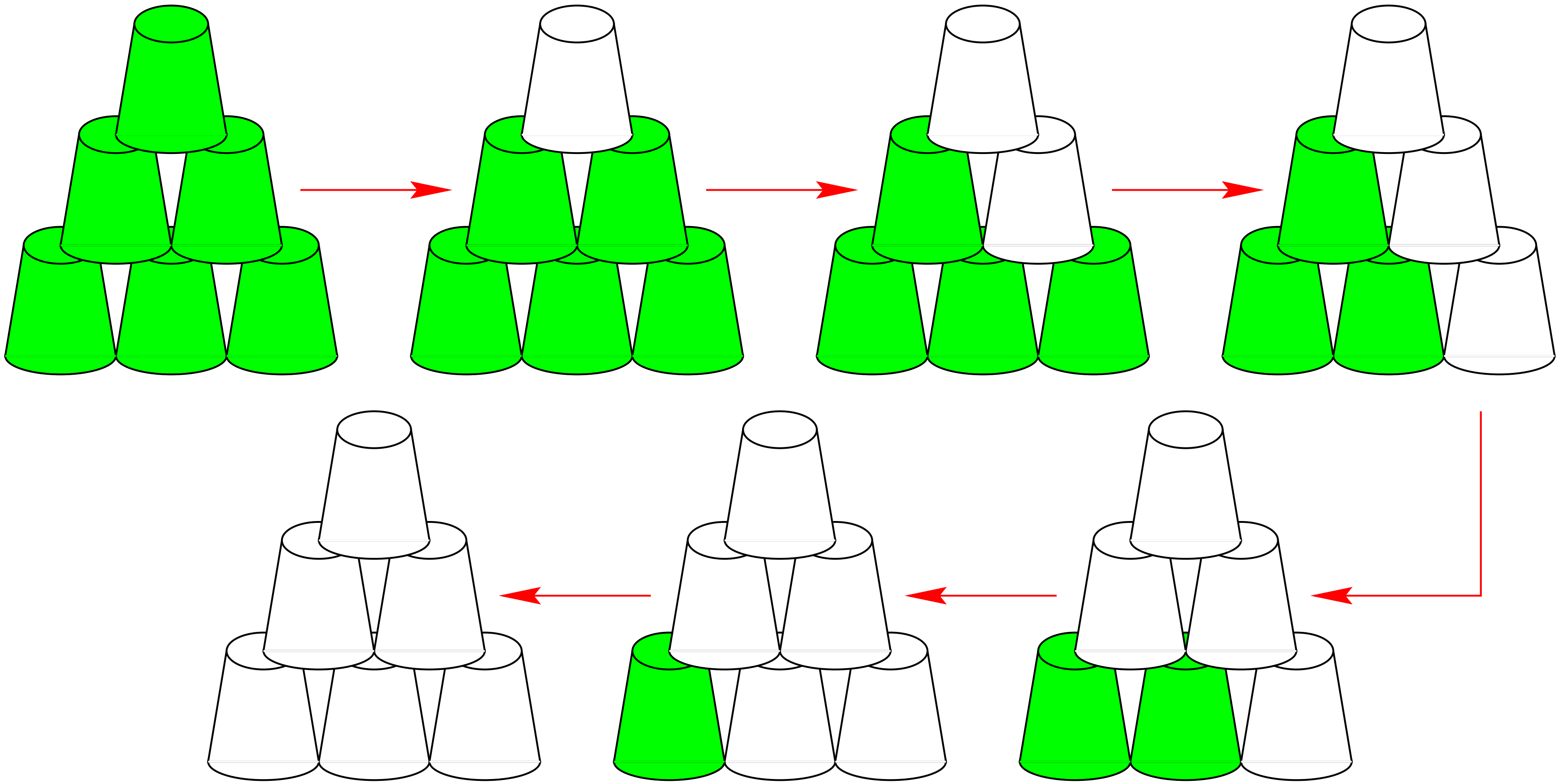
- From graphs
  - Tree shellings
  - Graph searches
  - Simplicial elimination of chordal graphs
- From partially ordered sets
  - Poset double shellings
  - k-chains
- From finite point sets in $\mathbb{R}^d$
  - Lower convex shellings
- From oriented matroids
  - Convex shellings of acyclic OMs
- ...

What is "cupstacks"?

Construct the tower from the pile
and get it back as quickly as possible.



A sequence in collapsing



$\mathcal{L}$ a convex geometry on E.

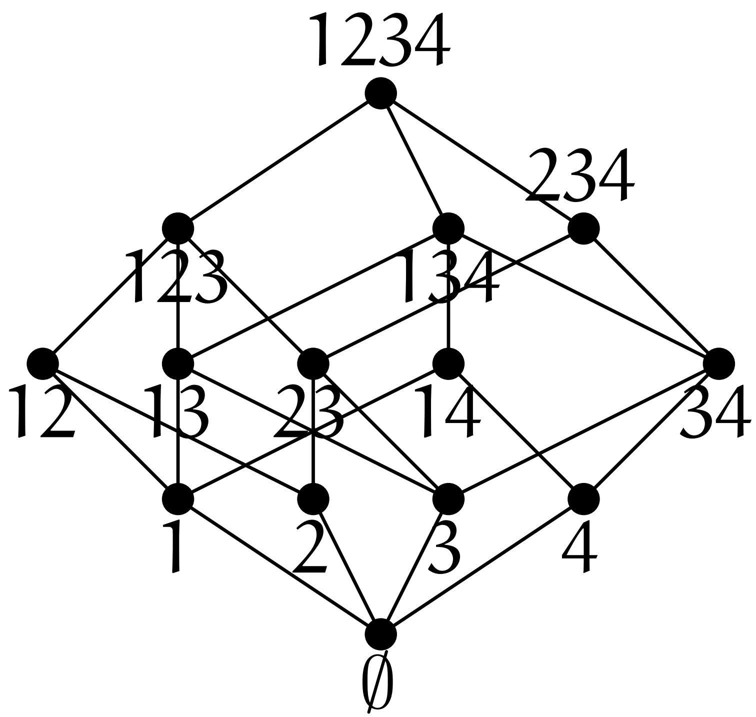
Def. $X \subseteq E$ is **free** in $\mathcal{L}$ if

- $X \in \mathcal{L}$ (convexity)
- the set of "extreme points" of $X = X$ (independence).

Extreme points

$\mathcal{L}$ a convex geometry on E, $X \in \mathcal{L}$ a convex set.

Def. $e \in X$ is an **extreme point** of X if $X \setminus \{e\} \in \mathcal{L}$.



$X = \{2, 3, 4\} \in \mathcal{L}$

2	extreme
3	not extreme
4	extreme

X is **independent** if every $e \in X$ is extreme in X.

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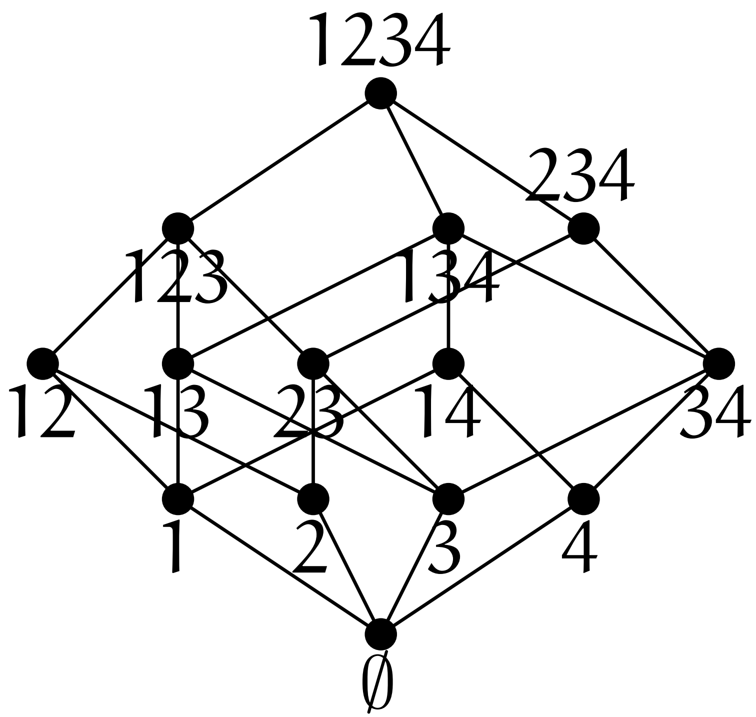

$X = \{1, 3, 4\} \in \mathcal{L}$

1 extreme
3 extreme
4 extreme

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Free complex of a convex geometry

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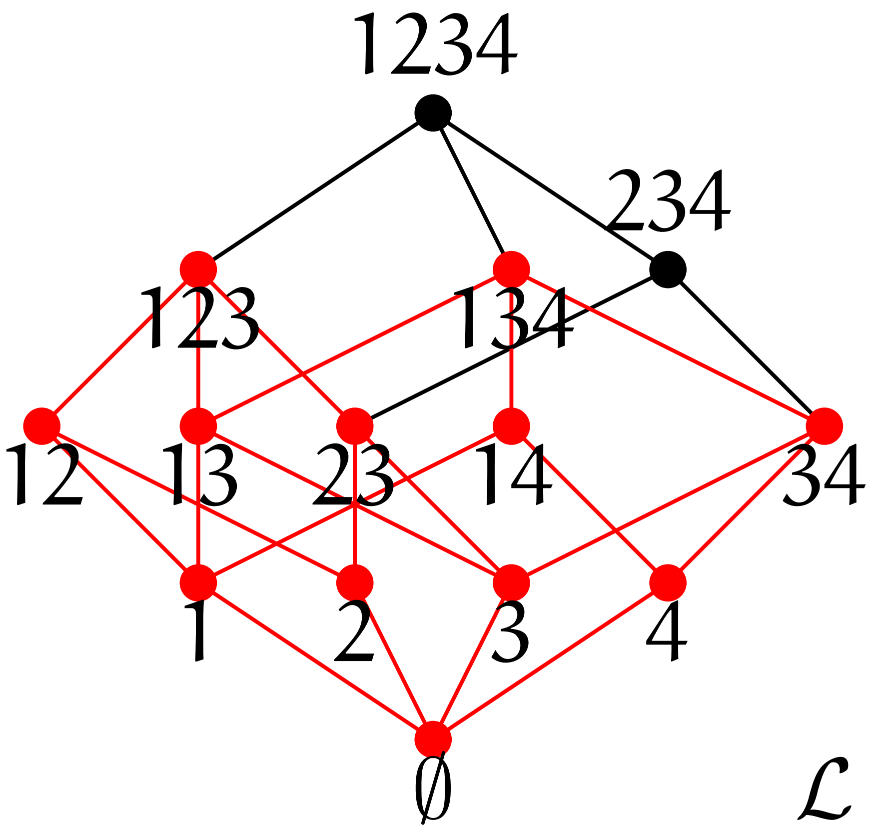
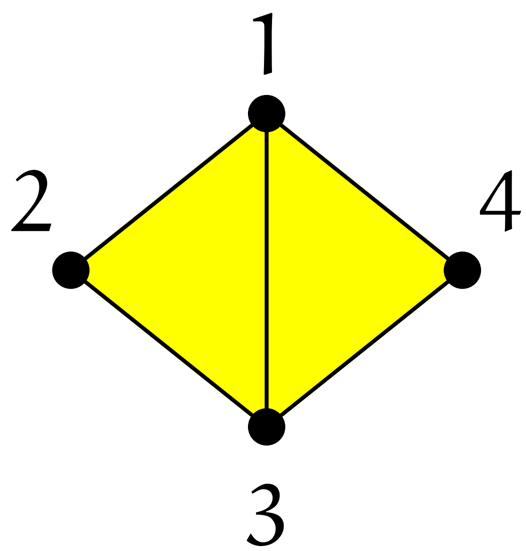


$\mathcal{L}$



$\mathrm{Free}(\mathcal{L})$

Remark

$\mathcal{P}$ a point configuration in $\mathbb{R}^{\mathrm{d}}$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

Then

$$\mathsf{Free}(\mathcal{P}) = \mathsf{Free}(\mathcal{L}).$$

Remark

$\mathcal{P}$ a point configuration in $\mathbb{R}^{\mathrm{d}}$,
$\mathcal{L}$ the convex shelling of $\mathcal{P}$.

Then

$$\mathsf{Free}(\mathcal{P}) = \mathsf{Free}(\mathcal{L}).$$

To generalize Edelman & Reiner's result,

We also need to generalize
"the boundary" and "the interior."

$\Longrightarrow$ a concept of "dependency sets"
(we omit the definition).

point config $\mathcal{P}$	conv geometry $\mathcal{L}$ on E
$\mathrm{Free}(\mathcal{P})$	$\mathrm{Free}(\mathcal{L})$
$e \in \mathrm{bd}(\mathrm{conv}(\mathcal{P}))$	$\mathrm{Dep}_{\mathcal{L}}(e) \neq E$
$e \in \mathrm{int}(\mathrm{conv}(\mathcal{P}))$	$\mathrm{Dep}_{\mathcal{L}}(e) = E$

$$\Downarrow$$

This leads to the following open problems.

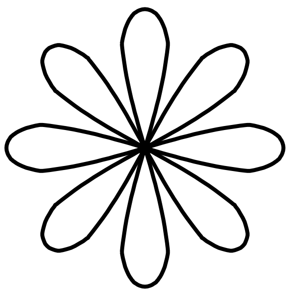
Open Problems

(Edelman & Reiner '00)

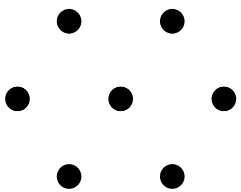
E a finite set, $\mathcal{L}$ a convex geometry on E.

(1) **Is $\mathrm{del}_{\mathsf{Free}(\mathcal{L})}(x)$ contractible if $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$?**

(2) **Is $\mathrm{del}_{\mathsf{Free}(\mathcal{L})}(x)$ homotopy equivalent to a bouquet of equidimensional spheres if $\mathrm{Dep}_{\mathcal{L}}(x) = E$?**



a bouquet of eight 1-dim spheres



a bouquet of six 0-dim spheres

a bouquet of zero sphere

Partial results in the literature

Both problems have been solved affirmatively for the following classes of convex geometries.

- Convex shellings of point configurations (Edelman & Reiner '00, Dong '02)
- Poset double shellings (Edelman & Reiner '00)
- Simplicial eliminations of chordal graphs (Edelman & Reiner '00)
- Conv shellings of acyclic oriented matroids (Edelman, Reiner & Welker '02)
- Poset shellings. (Easy)

Consider another class of convex geometries,
2-dim separable generalized convex shellings.

(1) **If $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$,**
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible.

(2) **If $\mathrm{Dep}_{\mathcal{L}}(x) = E$,**
$\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is either contractible or homotopy equiv to a 0-dim sphere.

What does it mean?

- ★ Verifies Open Problems for this special case.
- ★ Gives the first example of $\mathcal{L}$ and x s.t. $\mathrm{del}_{\mathsf{Free}(\mathcal{L})}(x)$ is contractible & $\mathrm{Dep}_{\mathcal{L}}(x) = \mathrm{E}$.
- ★ Actually, it's not just a special case...

Why separable generalized convex shellings?

$\boxed{\text{Thm}}$ (Kashiwabara, Nakamura & Okamoto, '03)

For every convex geometry $\mathcal{L}$, there exist point sets $\mathcal{P}$, $\mathcal{Q}$ with conv$(\mathcal{P}) \cap$ conv$(\mathcal{Q}) = \emptyset$ s.t.

$\mathcal{L} \cong$ the gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

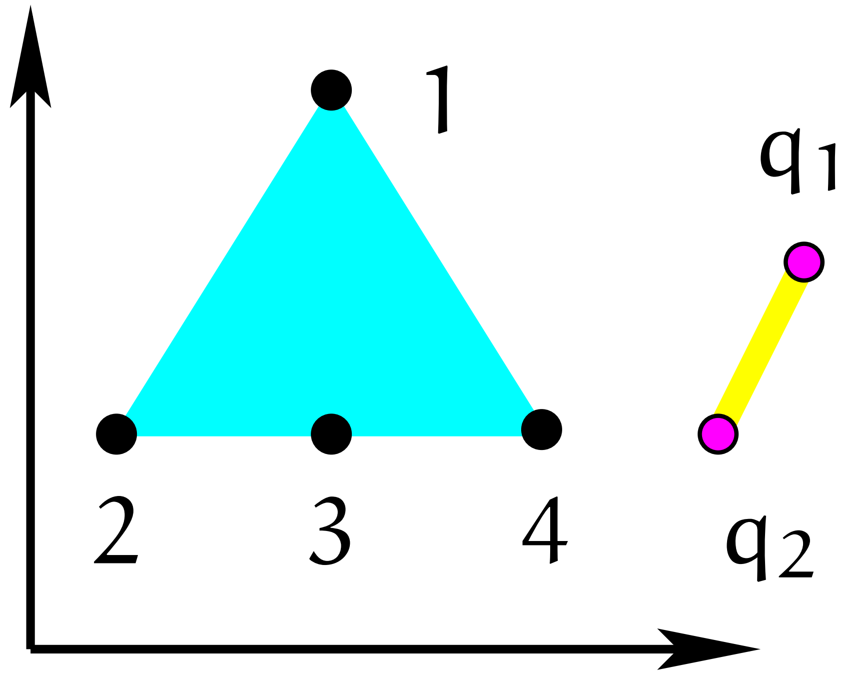
(Separable generalized convex shellings represent all convex geometries.)

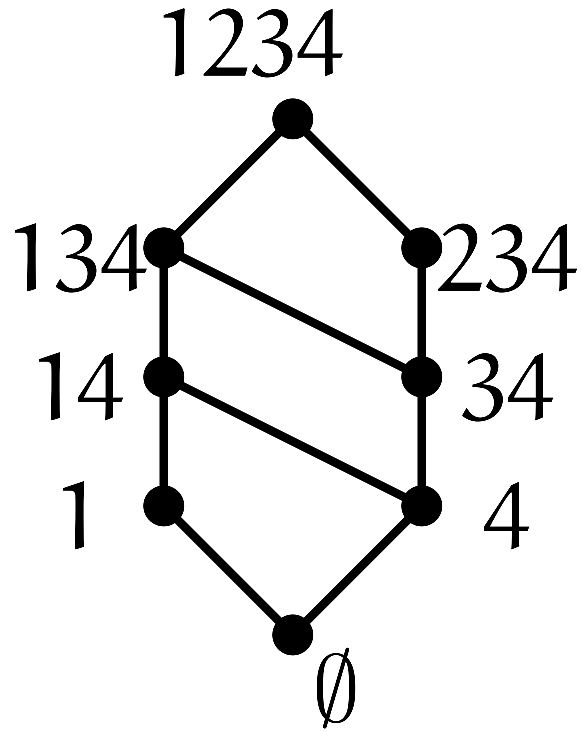
$\longrightarrow\longrightarrow\longrightarrow$ The 2-dim case is a first step for resolution of Open Problems.

$\mathcal{P}, \mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \mathrm{conv}(\mathcal{Q}) = \emptyset$.

Define:

$$\boldsymbol{\mathcal{L} = \{X \subseteq \mathcal{P} : \mathrm{conv}(X \cup \mathcal{Q}) \cap \mathcal{P} = X\}}.$$





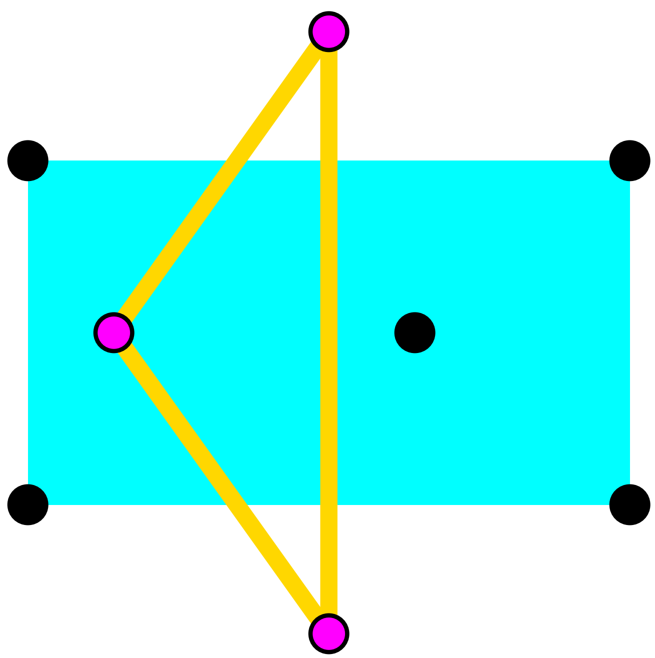
$\mathcal{L}$ is a convex geometry on $\mathcal{P}$ and called
the generalized conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

Technical terminology

$\mathcal{P}$,$\mathcal{Q}$ point sets in $\mathbb{R}^d$ with $\mathcal{P} \cap \text{conv}(\mathcal{Q}) = \emptyset$,

$\mathcal{L}$ the generalized convex shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$.

- $\mathcal{L}$ is **2-dimensional** if $d = 2$.
- $\mathcal{L}$ is **separable** if $\text{conv}(\mathcal{P}) \cap \text{conv}(\mathcal{Q}) = \emptyset$.



a non-separable case

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $\mathcal{Q} \neq \emptyset$.

(1) Free$(\mathcal{L})$ is the clique complex of a graph G.

I.e., the family of all cliques of G.

(2) G is chordal & connected.

Chordal $\Leftrightarrow$ every ind. cycle is C_3.

(3) **(2)** $\Rightarrow$ Free$(\mathcal{L})$ contractible.

(4) $G - x$ has at most 2 connected components.

(5) x a cut-vertex of G $\Rightarrow$ $\text{Dep}_{\mathcal{L}}(x) = \mathcal{P}$.

Our results (again)

$\mathcal{L}$ the 2-dim sep gen conv shelling on $\mathcal{P}$ w.r.t. $\mathcal{Q}$, $x \in \mathcal{P}$.

(1) **If** $\mathbf{Dep}_{\mathcal{L}}(x) \neq \mathcal{P}$**,**
del$_{\mathbf{Free}(\mathcal{L})}(x)$ **is contractible.**

(2) **If** $\mathbf{Dep}_{\mathcal{L}}(x) = \mathcal{P}$**,**
del$_{\mathbf{Free}(\mathcal{L})}(x)$ **is either contractible or homotopy equiv to a 0-dim sphere.**

Problems are still open

- **We don't know yet**
  the problems are affirmative or negative in the general case!
- How about a 3-dim case??
- How about a non-separable 2-dim case??

[End of Talk]

Here are extra slides for possible questions from the audience.

Closure operators, extreme point operators

$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ The **closure operator**

$\tau_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$\tau_{\mathcal{L}}(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}.$$

$\boxed{\text{Def.}}$ The **extreme point operator**

$\mathrm{ex}_{\mathcal{L}} : 2^E \to 2^E$ is defined as

$$\mathrm{ex}_{\mathcal{L}}(A) = \{e \in A : e \notin \tau_{\mathcal{L}}(A \setminus \{e\})\}.$$

$\mathcal{L}$ a convex geometry on E.

$\boxed{\text{Def.}}$ $A \subseteq E$ is **independent** if $\mathrm{ex}_{\mathcal{L}}(A) = A$.

$\boxed{\text{Def.}}$ The **dependency set** of $e \in E$ in $\mathcal{L}$ is

$$\mathrm{Dep}_{\mathcal{L}}(e) = \left\{ f \in E : \begin{array}{l} \exists \text{ independent } A \text{ s.t.} \\ f \in A, e \in \tau_{\mathcal{L}}(A), \\ e \notin \tau_{\mathcal{L}}(A \setminus \{f\}) \end{array} \right\}.$$