Fair cost allocations under conflicts
— a game-theoretic point of view —

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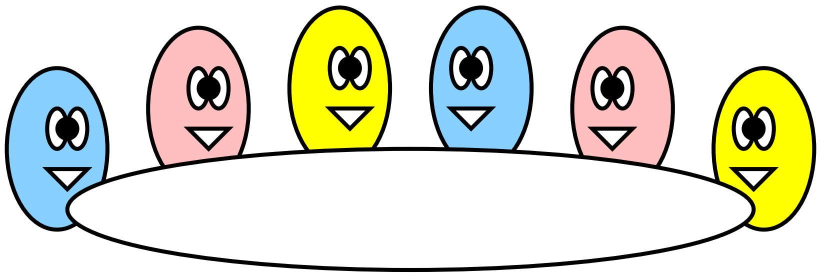
Combinatorics
Geometry
Computation

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  .... optimization problem
- They want to allocate the benefit in a fair way.
  .... game-theoretic problem

Game Theory?

- Noncooperative Game Theory
- Cooperative Game Theory



Def. A cooperative game (or a game) is a pair (N, γ) of

- a finite set N (set of players)
- a function $\gamma : 2^{\mathrm{N}} \to \mathbb{R}$ with $\gamma(\emptyset) = 0$ (characteristic function).

Interpretation: For $S \subseteq \mathrm{N}$,

$$\gamma(S) \text{ represents } \left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$$

when the players in S work in cooperation.

Goal: To allocate $\gamma(\mathrm{N})$ to each player in a "fair" way.

The paper

Y. Okamoto: Fair cost allocations under conflicts.
To appear in ISAAC 2003.

studies minimum coloring games, and
investigates the following kinds of fairness concepts:

- Core (Gillies '53)
- Nucleolus (Schmeidler '69)
- τ-value (Tijs '81)
- Shapley value (Shapley '53).

Past works on minimum coloring games:

- Deng, Ibaraki & Nagamochi '99
- Deng, Ibaraki, Nagamochi & Zang '00
- Okamoto '03

Focus on cores and nucleoli of minimum coloring games

- Def.: minimum coloring game
- Def.: core
- Properties:
  Balancedness, Total balancedness, Submodularity
- Characterization: the core for a perfect graph
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

$G = (V, E)$ an undirected graph

- A proper k-coloring of G

  is a surjective map $c : V \to \{1, \ldots, k\}$ s.t.
  if $\{u, v\} \in E$, then $c(u) \neq c(v)$.

- The chromatic number $\chi(G)$ of G

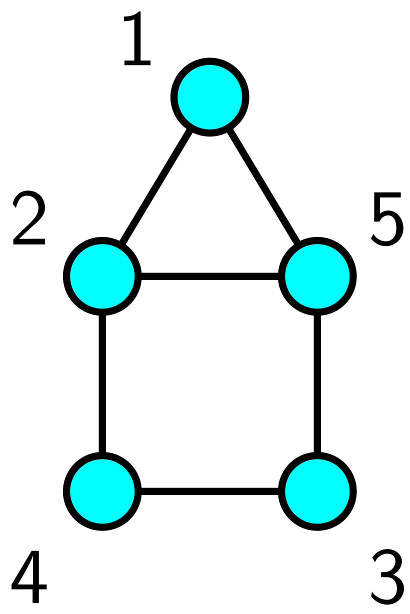
  $= \min\{\ k\ :\ $ a proper k-coloring of G exists $\}$.

- The minimum coloring game on G

  is a cooperative game (V, χ_G).

  $\chi_G : 2^V \to \mathbb{N}$ is defined as $\chi_G(S) = \chi(G[S])$,
  where $G[S]$ is the subgraph induced by $S \subseteq V$.

$\chi_G(S) = \chi(G[S])$ for $S \subseteq V$.



S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

$\boxed{\text{Goal:}}$ To allocate $\chi(G)$ to each vertex in a fair way.

Def.: A cost allocation for a game (N, γ) is a vector $z \in \mathbb{R}^N$ such that

$$\sum \{z[i] : i \in N\} = \gamma(N).$$

(Often in cooperative game theory, this is called a pre-imputation.)

. . . .

Q. What kinds of cost allocations are considered fair??

...... Core, Nucleolus, τ-value, Shapley value, etc.

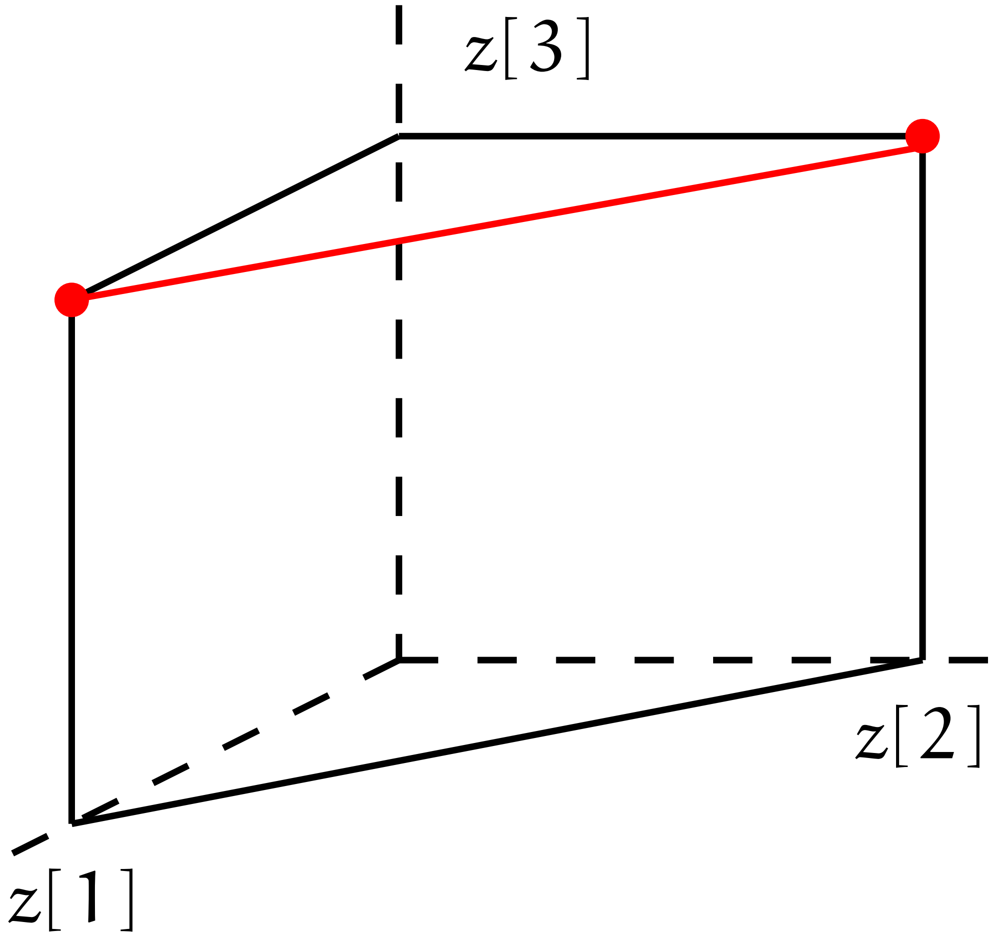
Def.: A cost allocation $z \in \mathbb{R}^N$ for (N, γ) is a core allocation if

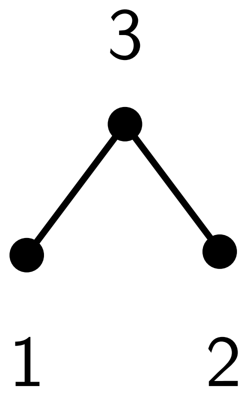
$$\sum\{z[i] : i \in S\} \leq \gamma(S) \qquad \text{for all } S \subseteq N$$

(each subset $S \subseteq N$ is satisfied with z).

The core of (N, γ) is the set of all core allocations.

Remark The core is a bounded polyhedron, possibly empty.

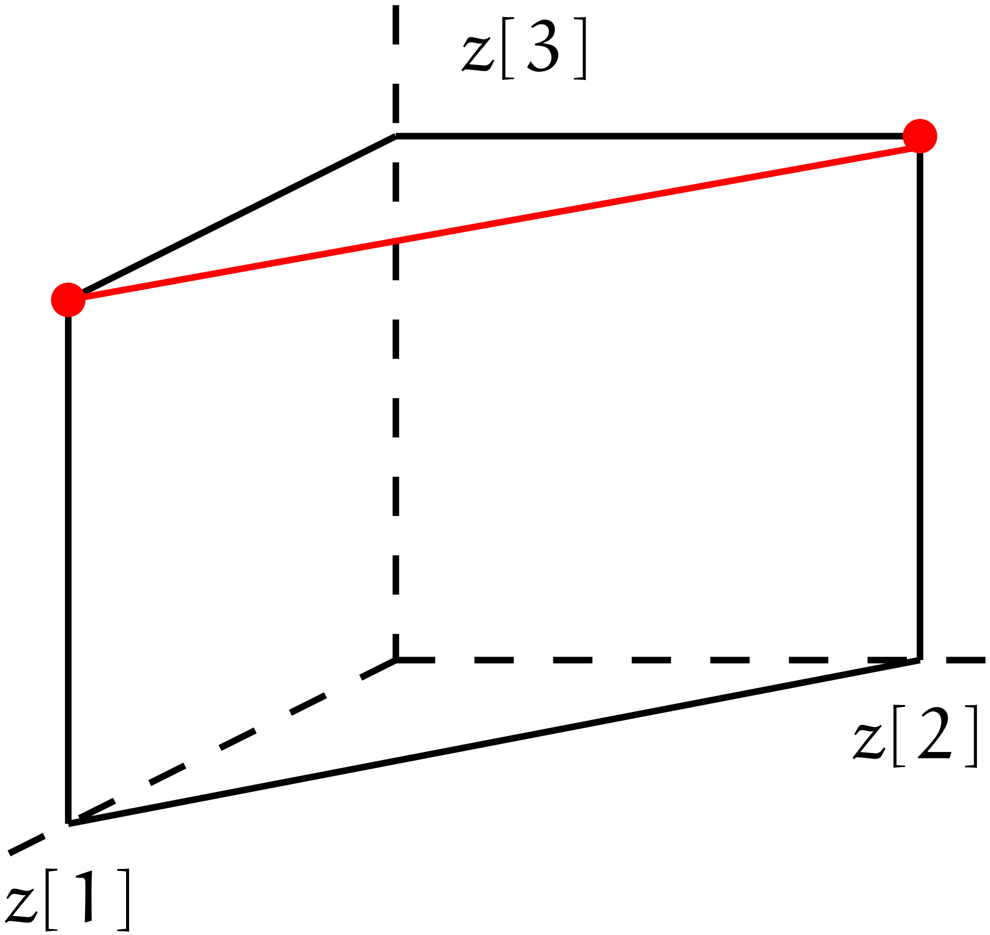


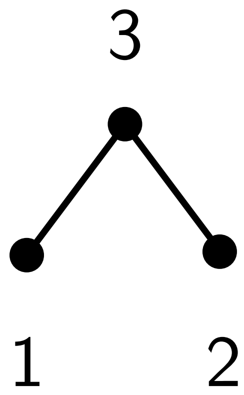


$V = \{1, 2, 3\}$

$\chi_G(\emptyset)$	0
$\chi_G(\{1\})$	1
$\chi_G(\{2\})$	1
$\chi_G(\{3\})$	1
$\chi_G(\{1,2\})$	1
$\chi_G(\{1,3\})$	2
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$$\text{Core} = \left\{ z \in \mathbb{R}^3 \;:\; \begin{array}{l} z[1] \le 1,\; z[2] \le 1,\; z[3] \le 1, \\ z[1] + z[2] \le 1,\; z[1] + z[3] \le 2, \\ z[2] + z[3] \le 2,\; z[1] + z[2] + z[3] = 2 \end{array} \right\}$$





$$V = \{1, 2, 3\}$$

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$\chi_G(\{1,2\})$	1
$\chi_G(\{1,3\})$	2
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$$\mathsf{Core} = \mathsf{conv}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

We want to know:

(1) When is the core empty??

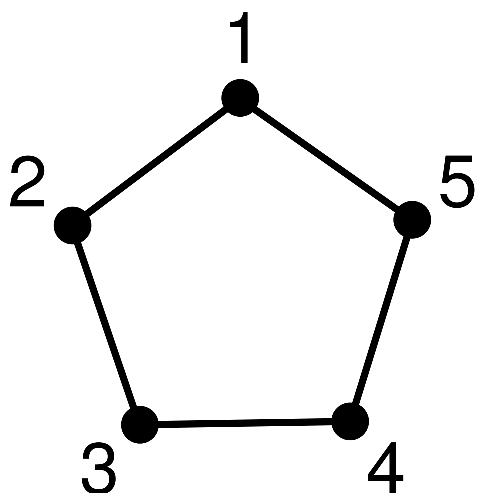
(2) If the core is nonempty,

- **(a)** What are the extreme points of the core??
  (what is the $\mathcal{V}$-representation of the core??)
- **(b)** How can we compute a core allocation??

. . . .

$\boxed{\text{Def.}}$ A game (N, γ) is balanced if the core is nonempty.

The min col. game on a cycle with 5 vert. has an empty core.



Suppose $\exists\, z \in$ core.
Then

$$\sum z[i] = \chi_G(V) = 3,$$

and

$$z[1] + z[3] \le \chi_G(\{1,3\}) = 1,$$
$$z[2] + z[4] \le \chi_G(\{2,4\}) = 1,$$
$$z[3] + z[5] \le \chi_G(\{3,5\}) = 1,$$
$$z[4] + z[1] \le \chi_G(\{4,1\}) = 1,$$
$$z[5] + z[2] \le \chi_G(\{5,2\}) = 1.$$

A contradiction.

The min coloring problem can be formulated as the min set cover problem.

$\boxed{\text{Thm.}}$ (Deng, Ibaraki & Nagamochi '99)

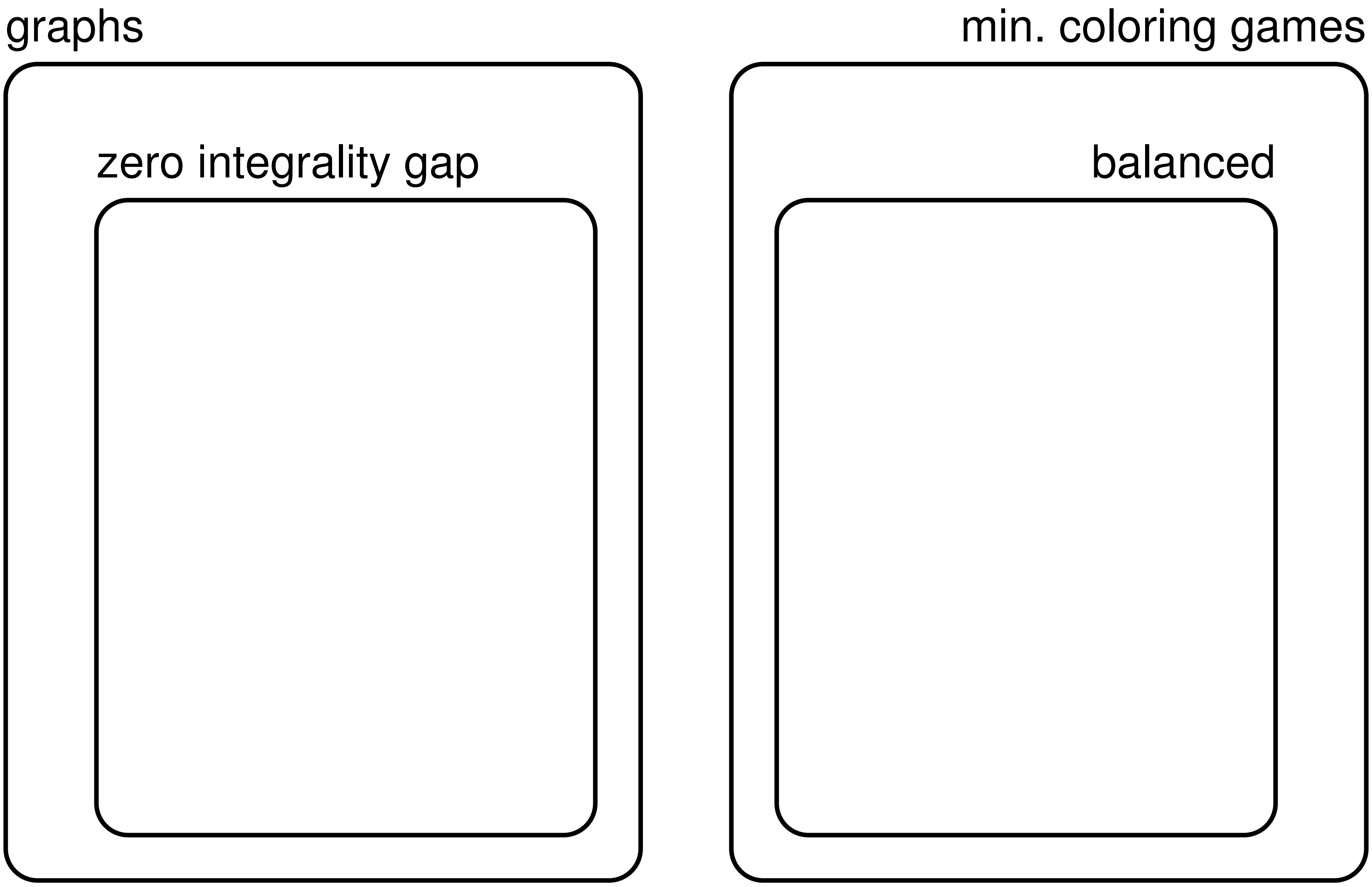
A min coloring game (V, χ_G) is balanced

$$\Updownarrow$$

The LP-relaxation of the above formulation of the min coloring problem has an integral opt. sol'n.

$\boxed{\text{Thm.}}$ (Deng, Ibaraki & Nagamochi '99)

It is NP-complete to decide the min coloring game of a given graph is balanced or not.


graphs
zero integrality gap
min. coloring games
balanced

Def.: A game (N, γ) is totally balanced if each of the subgames of (N, γ) is balanced.

Def.: Let $T \subseteq N$. A subgame of (N, γ) w.r.t. T is a game $(T, \gamma|_T)$ where $\gamma|_T(S) = \gamma(S)$ $(S \subseteq T)$.

Def.: A game (N,γ) is totally balanced
if each of the subgames of (N,γ) is balanced.

Def.: Let $T \subseteq N$. A subgame of (N,γ) w.r.t. T
is a game $(T,\gamma|_T)$ where $\gamma|_T(S) = \gamma(S)$ $(S \subseteq T)$.

Thm. (Deng, Ibaraki, Nagamochi & Zang '00)

A min coloring game (V,χ_G) is totally balanced

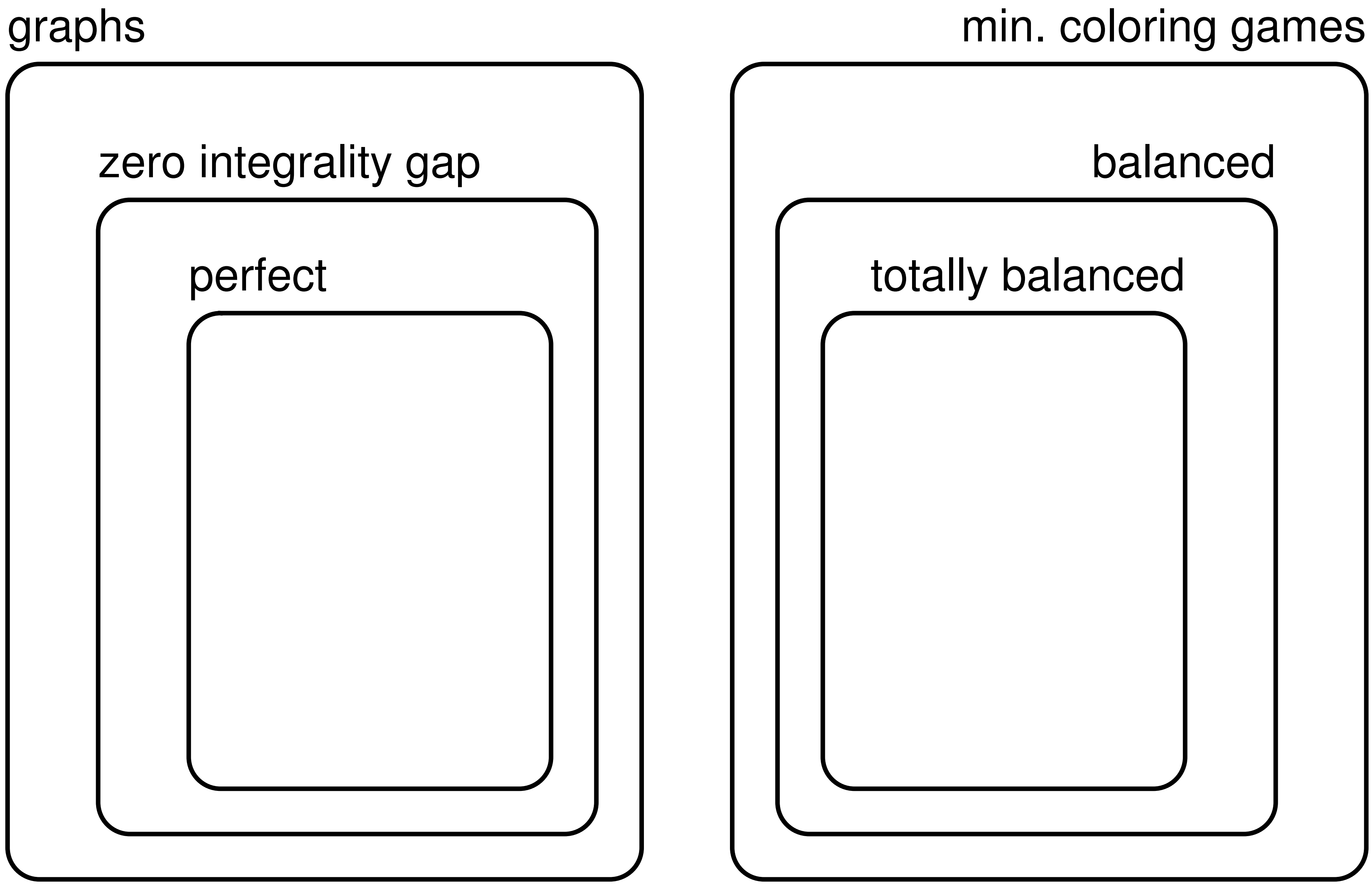
$\Updownarrow$

G is perfect.

Def.: G is perfect if for every induced subgraph $H \subseteq G$
$\chi(H) =$ max size of the cliques in H.

(A clique is a vertex subset which induces a complete graph.)

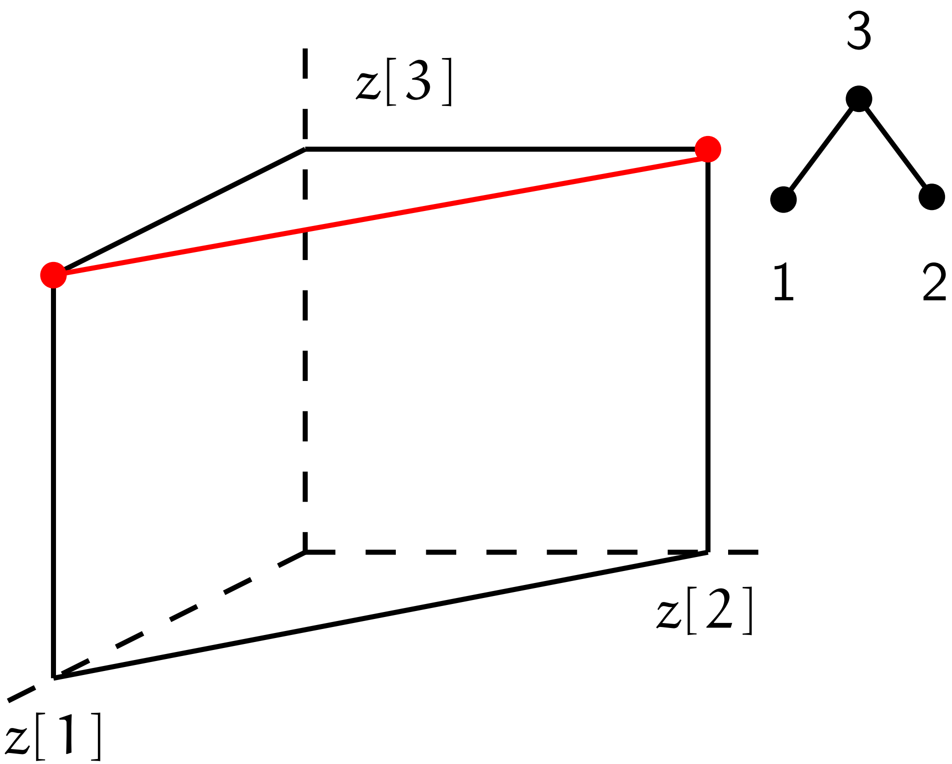
Correspondence



Thm.

G a perfect graph

core $=$ conv(the char. vectors of the maximum cliques of G).



$$\text{Core} = \text{conv}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

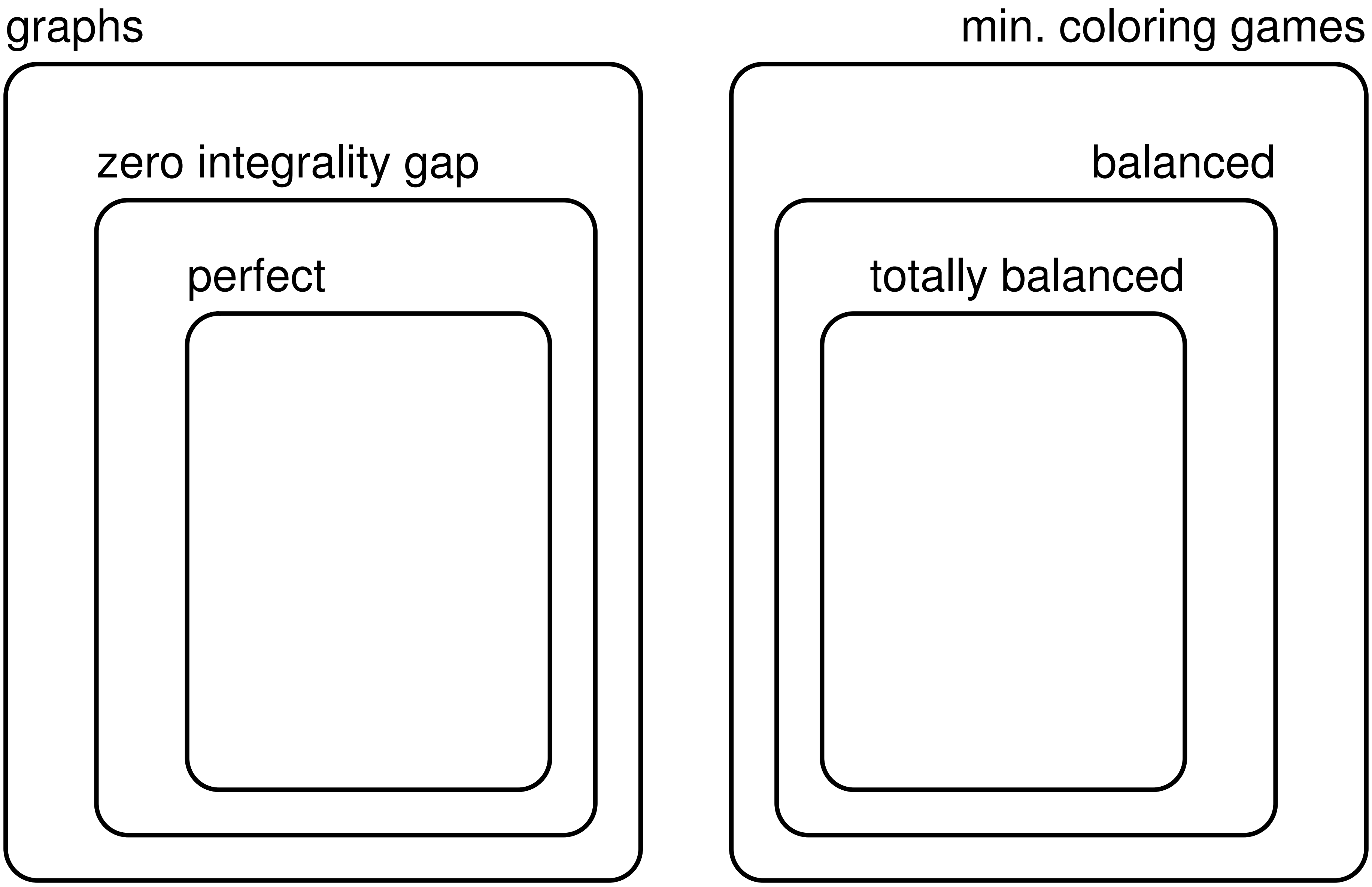
$\boxed{\text{Cor.}}$ We can do the following in poly. time.

(1) Find a core allocation for a perfect graph.
(≈ Find a maximum clique in poly. time.)

(2) Decide whether a given vector belongs to the core or not for a pefect graph.
(≈ The membership problem for the clique polytope.)

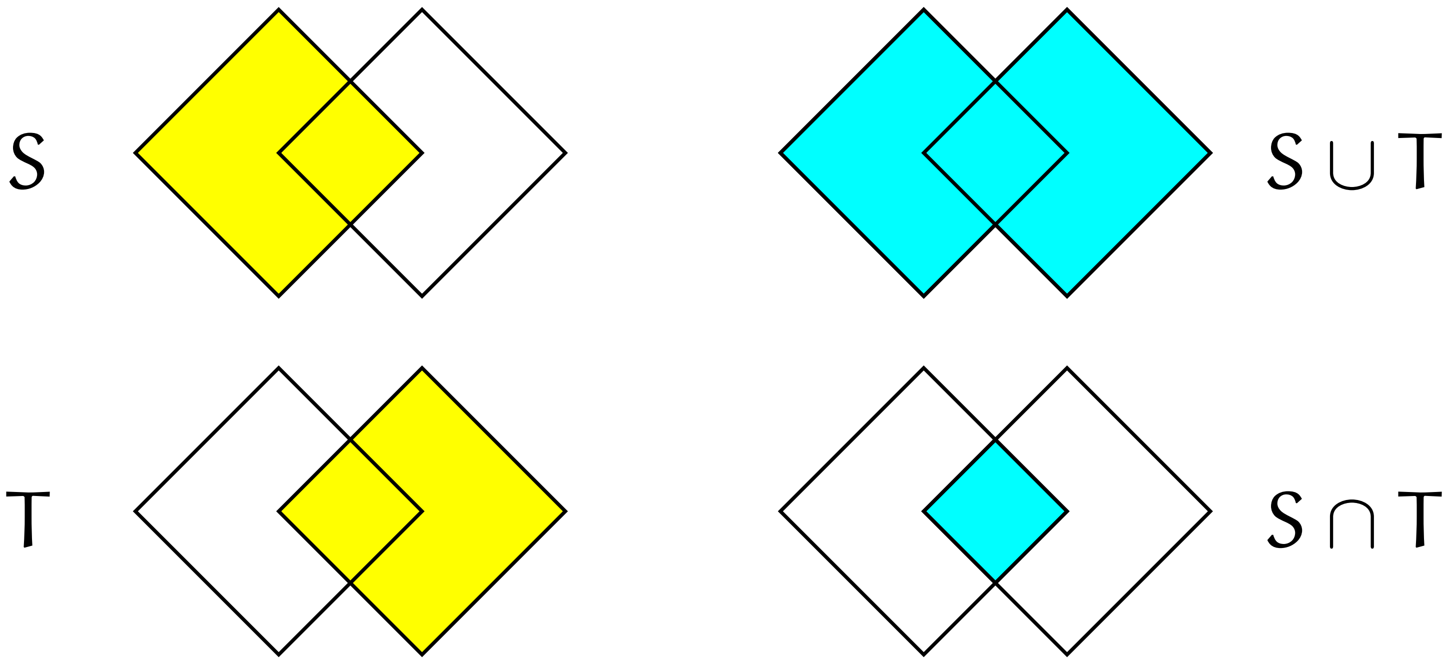
They are the consequences of the previous theorem and a result by Grötschel, Lovász & Schrijver ('83).

Correspondence



Def. A game (N, γ) is submodular if

$$\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T) \quad \forall\, S, T \subseteq N.$$



Submodular games are extensively studied and known to have a lot of nice properties. Among them, ...

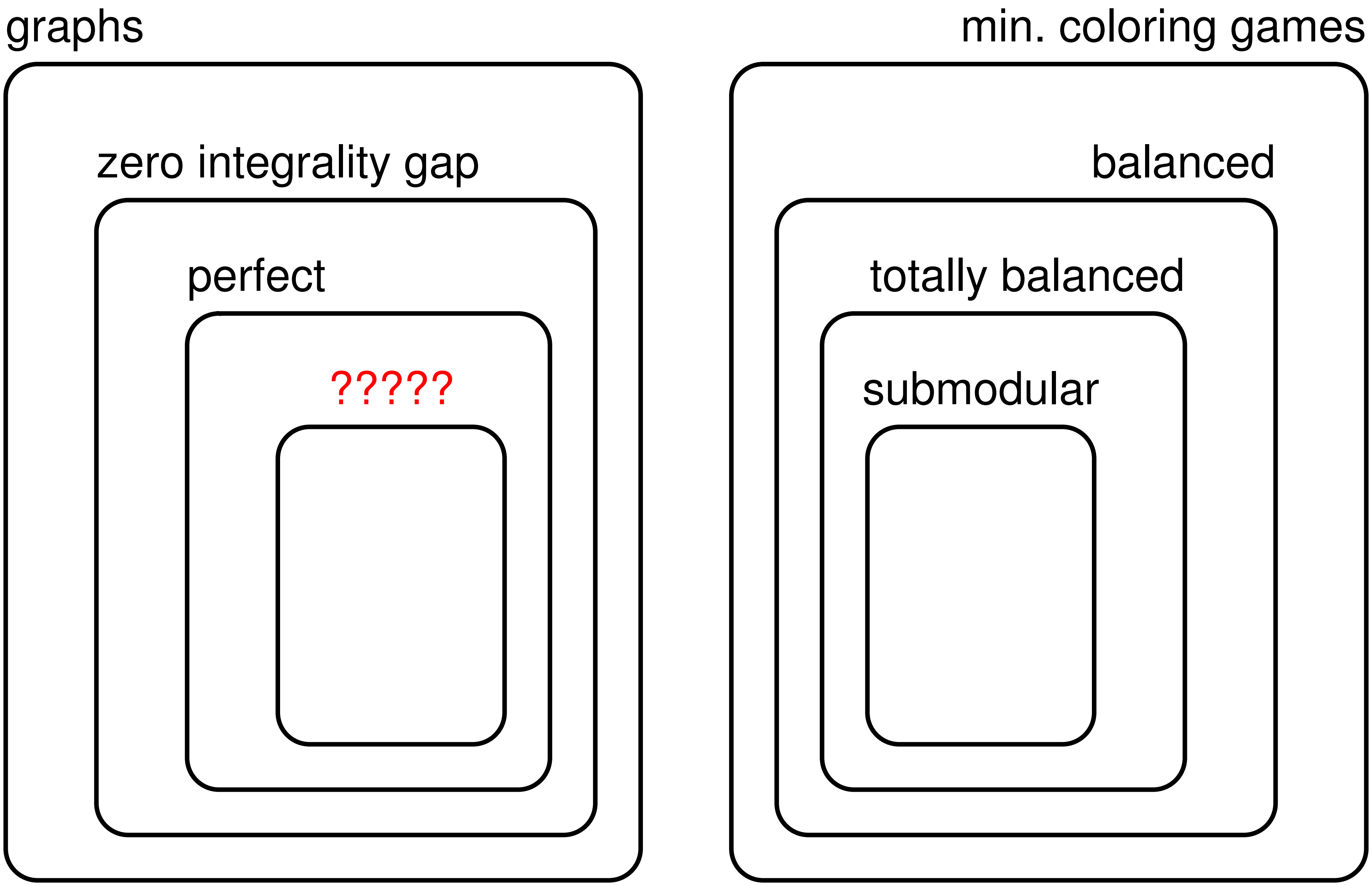
Thm. (Shapley '71, Edmonds '70)

(N, γ) is submodular $\Longrightarrow$ (N, γ) is totally balanced.

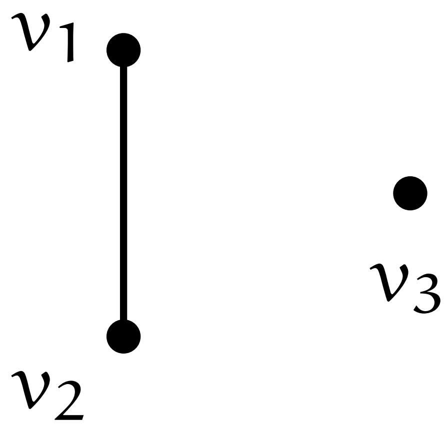
Remark

Edmonds '70 showed the above theorem in the context of submodular-type optimization, which is a generalization of matroid optimization and plays a significant role in combinatorial optimization.

cooperative game theory	$\leftrightarrow$	combinatorial optimization
games	$\leftrightarrow$	set functions
core	$\leftrightarrow$	base polyhedron

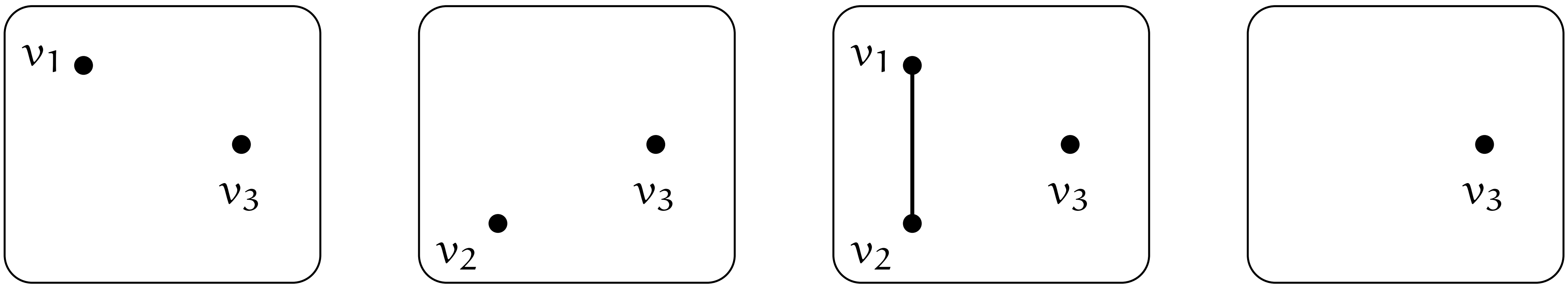

graphs
zero integrality gap
perfect
?????
min. coloring games
balanced
totally balanced
submodular

For $G = K^1 \cup K^2$,
(V, χ_G) is not submodular.



Let $S = \{v_1, v_3\}, T = \{v_2, v_3\}$

$$\underbrace{\chi_G(S)}_{1} + \underbrace{\chi_G(T)}_{1} < \underbrace{\chi_G(S \cup T)}_{2} + \underbrace{\chi_G(S \cap T)}_{1}.$$



$\boxed{\text{Thm.}}$ (Okamoto '03)

The following statements are equivalent.

(1) A min coloring game (V, χ_G) is submodular.

(2) G contains no $K^1 \cup K^2$ as its induced subgraph.

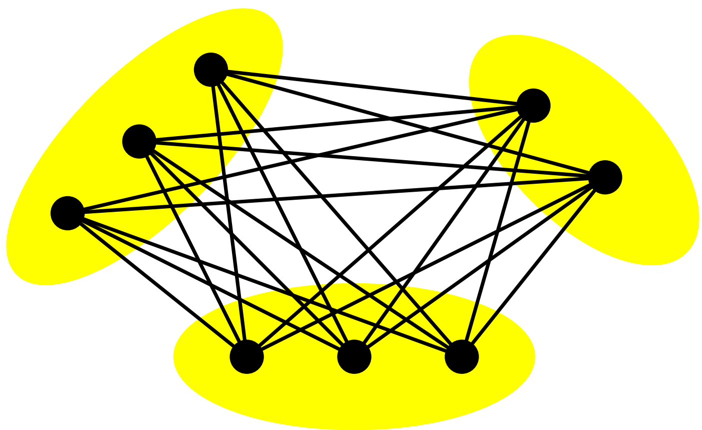
$\boxed{\text{Thm.}}$ (Okamoto '03)

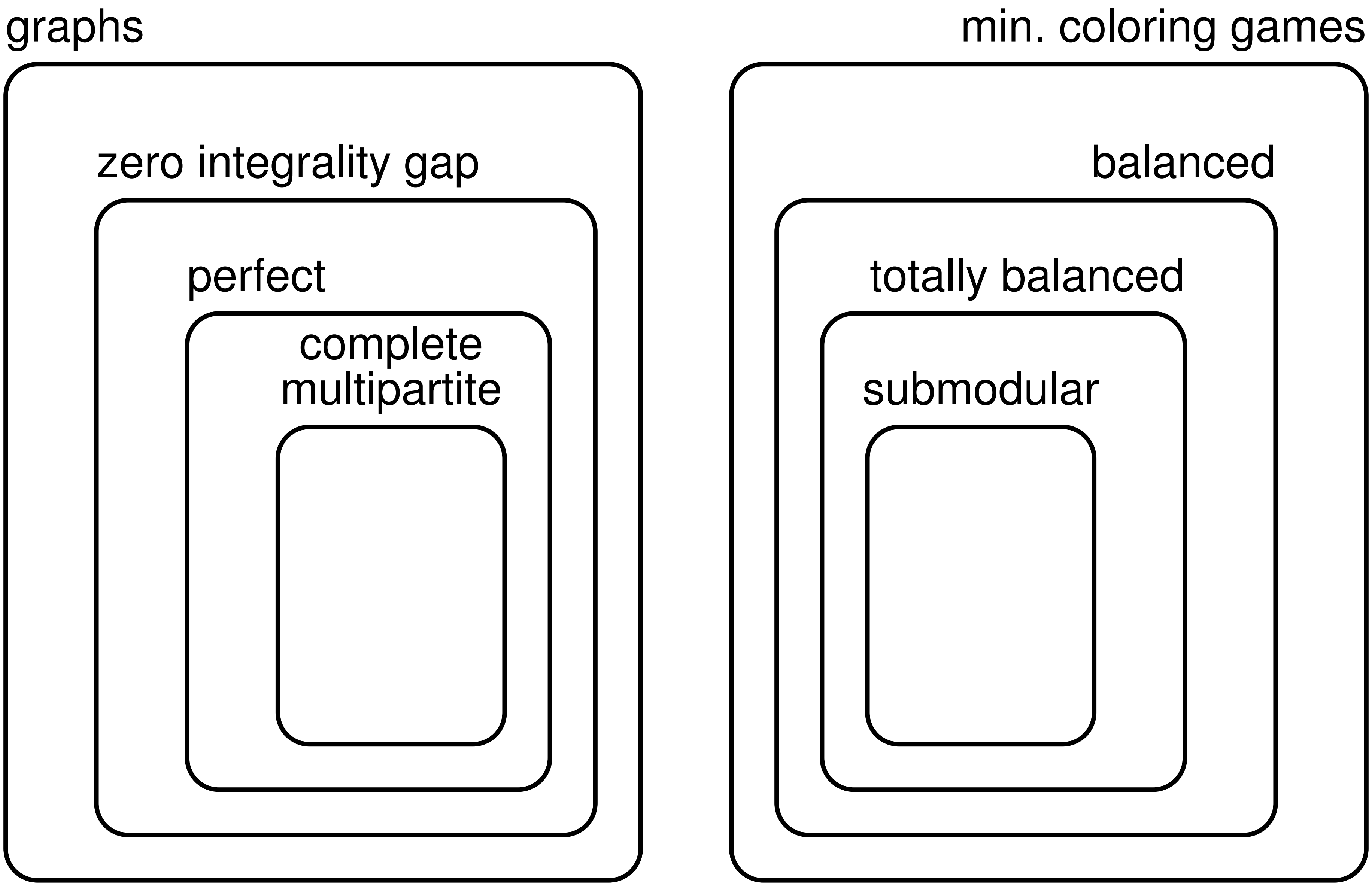
The following statements are equivalent.

(1) A min coloring game (V, χ_G) is submodular.

(2) G contains no $K^1 \cup K^2$ as its induced subgraph.

(3) G is a complete multipartite graph.




graphs
zero integrality gap
perfect
complete multipartite
min. coloring games
balanced
totally balanced
submodular

Focus on cores and nucleoli of minimum coloring games

- Def.: minimum coloring game
- Def.: core
- Properties:
  Balancedness, Total balancedness, Submodularity
- Characterization: the core for a perfect graph

. . . .

- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

What is good/bad for the core??

Good :)

- Easy to investigate.
- Much is known.

Bad :(

- Might be empty.
- Even if not empty,
  there might be many allocations in the core.
  (We need another criterion to choose one of them.)

The nucleolus is another fairness concept,
which uniquely exists for every min coloring game.

Let (N, γ) a game
$z \in \mathbb{R}^N$ a cost allocation
$S \subseteq N$ (often called a coalition)

Def. An excess $e(S, z)$ is defined as

$$e(S, z) \ := \ \sum_{i \in S} z[i] - \gamma(S).$$

Interpretation: The smaller $e(S, z)$, the happier S with z.

Fact $z \in \mathbb{R}^N$ belongs to the core of (N, γ)
$\iff e(N, z) = 0$ and $e(S, z) \leq 0$ $(\forall\ S \subseteq N)$.

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.
- Arrange these excesses in non-increasing order to obtain $\theta_z \in \mathbb{R}^{2^{|N|}-2}$. $(\theta_z[i] \geq \theta_z[j]$ if $i \leq j$.)

Example

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

$$\theta_z = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -1\right)^\top$$

S	$\gamma(S)$	$e(S, z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
$\{1, 2\}$	1	$1/2$
$\{1, 3\}$	2	$-1/2$
$\{2, 3\}$	2	-1
$\{1, 2, 3\}$	2	(0)

$\boxed{\text{Def.}}$ The nucleolus of (N, γ) is defined as

$$\boldsymbol{\nu}(\mathrm{N}, \gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^{\mathrm{N}} : \begin{array}{l} \boldsymbol{z} \text{ lex-mins } \boldsymbol{\theta}_{\boldsymbol{z}} \text{ over all cost alloc's } \boldsymbol{y} \\ \text{s.t. } \boldsymbol{y}[\mathrm{i}] \le \gamma(\{\mathrm{i}\}) \quad \forall \mathrm{i} \in \mathrm{N} \end{array} \right\}.$$

$\boxed{\text{Interpretation:}}$ The smaller $e(S, \boldsymbol{z})$, the happier S with $\boldsymbol{z}$.

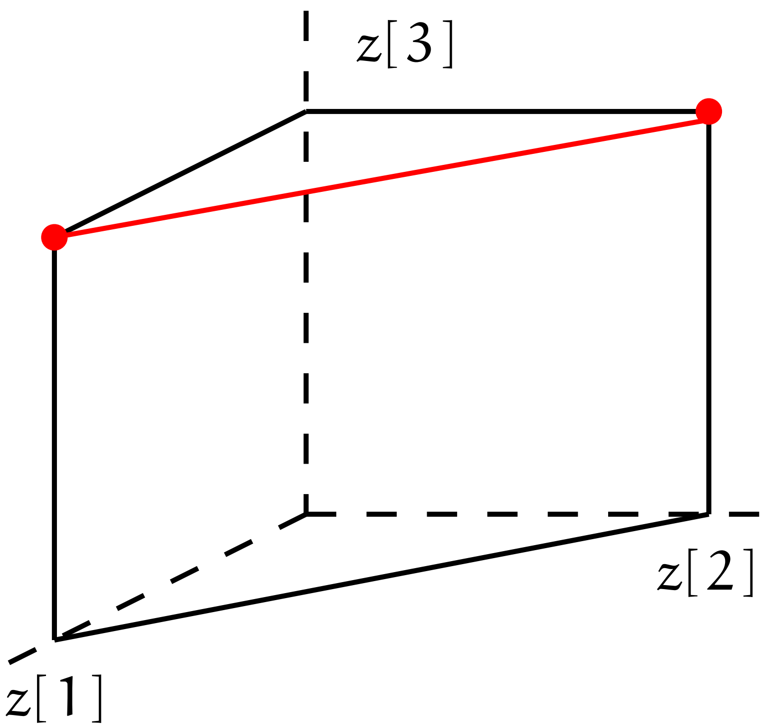
$\Rightarrow$ Want an allocation which minimizes max excess.

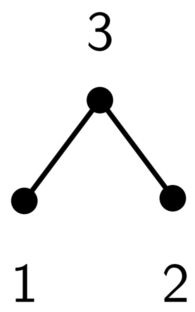
$\boxed{\text{Thm.}}$ (Schmeidler '69)

The nucleolus consists of a single vector.

So we usually say $\boldsymbol{\nu}(\mathrm{N}, \gamma) = \boldsymbol{z}$ instead of $\boldsymbol{\nu}(\mathrm{N}, \gamma) = \{\boldsymbol{z}\}$.

Example: nucleolus



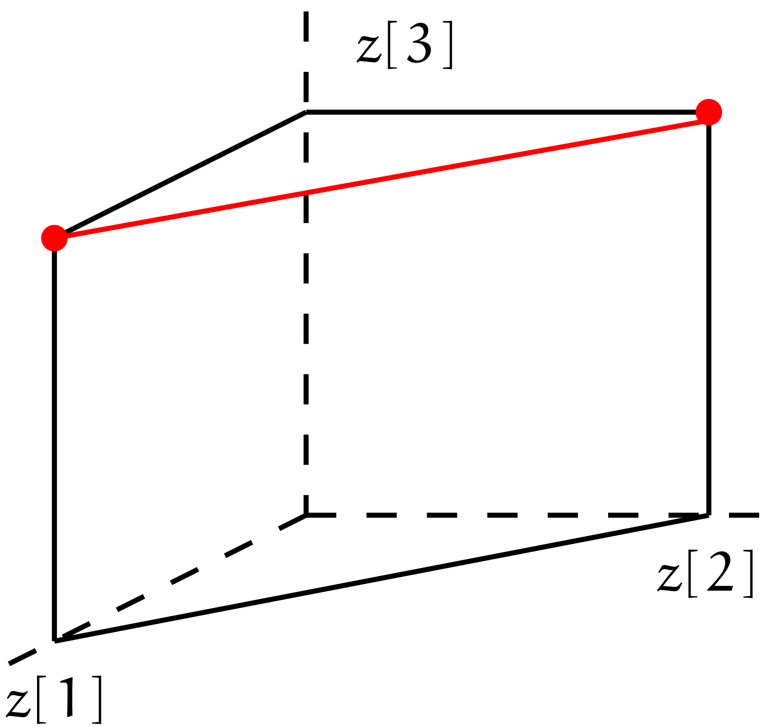


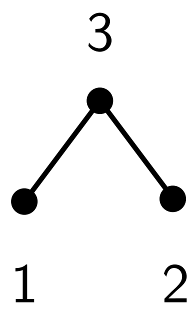
$$\nu = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top \text{ is the nucleolus.}$$

$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\nu)$	$-1/2$	$-1/2$	0	0	$-1/2$	$-1/2$

$$e(S,\nu) := \sum_{i\in S} \nu[i] - \gamma(S).$$





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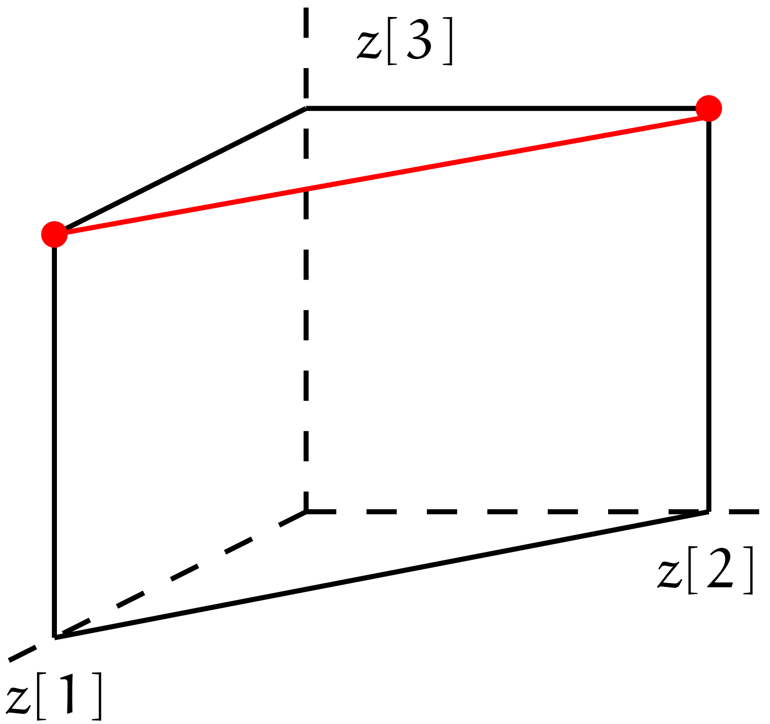
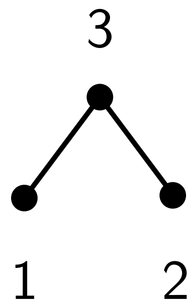
$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

$\boxed{\text{Fact}}$ the core is nonempty $\Rightarrow$ the nucleolus $\in$ the core.

Example: nucleolus





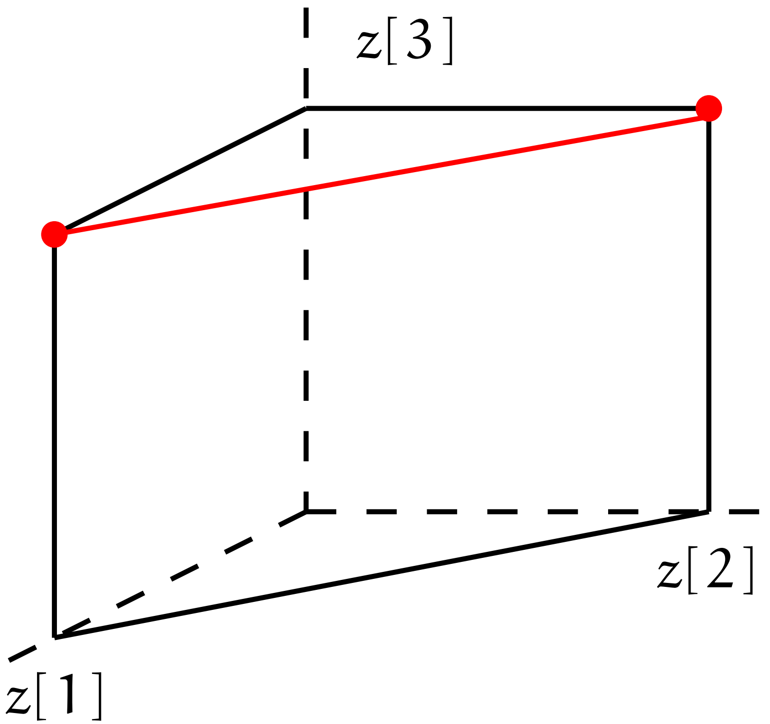
$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$ is the nucleolus.

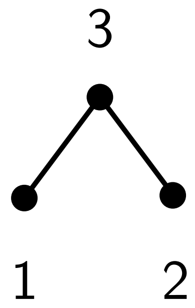
$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

Let $\boldsymbol{z} = \lambda(1,0,1)^\top + (1-\lambda)(0,1,1)^\top \qquad (0 \le \lambda \le 1)$
$= (\lambda, 1-\lambda, 1)^\top.$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,\boldsymbol{z})$	$\lambda-1$	$-\lambda$	0	0	$\lambda-1$	$-\lambda$

Example: nucleolus





$$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$$ is the nucleolus.

$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

$$\theta_z = \begin{cases} (0, 0, -\lambda, -\lambda, \lambda - 1, \lambda - 1)^\top & \text{if } 0 \leq \lambda \leq 1/2 \\ (0, 0, \lambda - 1, \lambda - 1, -\lambda, -\lambda)^\top & \text{if } 1/2 \leq \lambda \leq 1 \end{cases}$$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\gamma(S)$	1	1	1	1	2	2
$e(S,z)$	$\lambda - 1$	$-\lambda$	0	0	$\lambda - 1$	$-\lambda$

Consider the computation of the nucleolus.

Thm. (Faigle, Kern & Kuipers '98)

It is NP-hard for totally balanced games.

Thm. (Kuipers '96, Faigle, Kern & Kuipers '01)

It can be done in poly. time for submodular games.

On the computation of the nucleolus

	General	$\supseteq$	Min col. game
games	NP-hard		NP-hard
$\cup\mid$	$\Uparrow$		
balanced	NP-hard		???
$\cup\mid$	$\Uparrow$		
totally balanced	NP-hard		???
$\cup\mid$			
submodular	poly	$\Rightarrow$	poly

Obs. The computation of the nucleolus of a min coloring game is NP-hard.

Proof Suppose we get the nucleolus $\boldsymbol{\nu}$ in poly time.

$\Rightarrow$ Compute $\sum_{i \in V} \boldsymbol{\nu}[i] = \chi(G)$.

$\Rightarrow$ We have obtained $\chi(G)$ in poly time.

$\Rightarrow$ $P = NP$. [qed]

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\vert$		
zero duality gap		???
$\cup\vert$		
perfect		???
$\cup\vert$		
complete multipartite		Poly

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\!\mid$		
zero duality gap		???
$\cup\!\mid$		
perfect		???
$\cup\!\mid$		
O-good		characterization
$\cup\!\mid$		
complete multipartite		Poly

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the core.

Namely,

$$\boldsymbol{\nu}[i] = \frac{\#\text{ of maximum cliques containing } i}{\#\text{ of maximum cliques}}.$$

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the core.

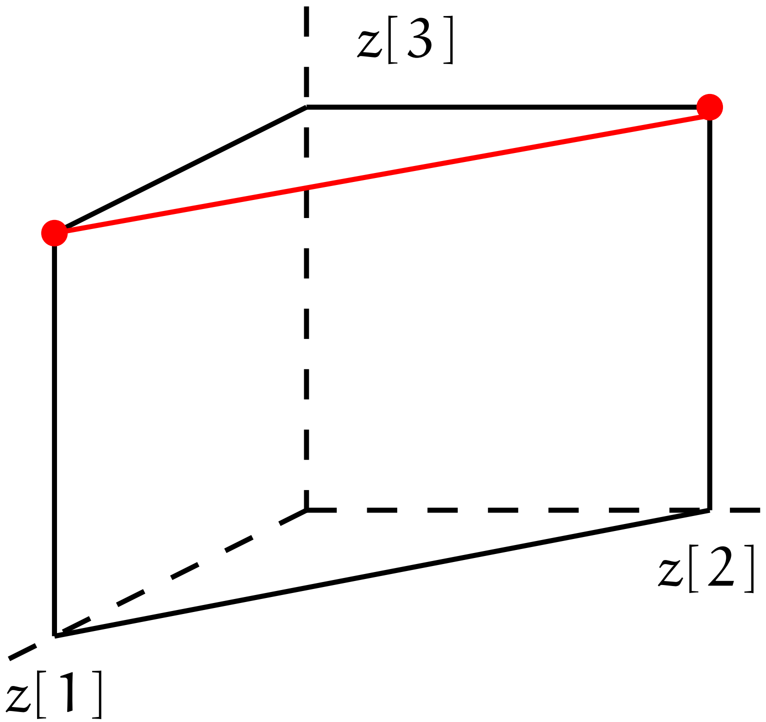
Remark

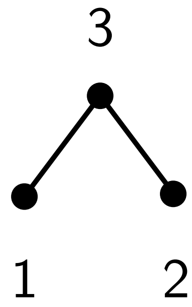
(1) We omit the def. of O-good perfect graphs.

(2) The class of O-good perfect graphs contains

- the graphs with unique maximum cliques
- the complete multipartite graphs
- the chordal graphs (especially the forests).

A graph is chordal if every induced cycle is of length 3.



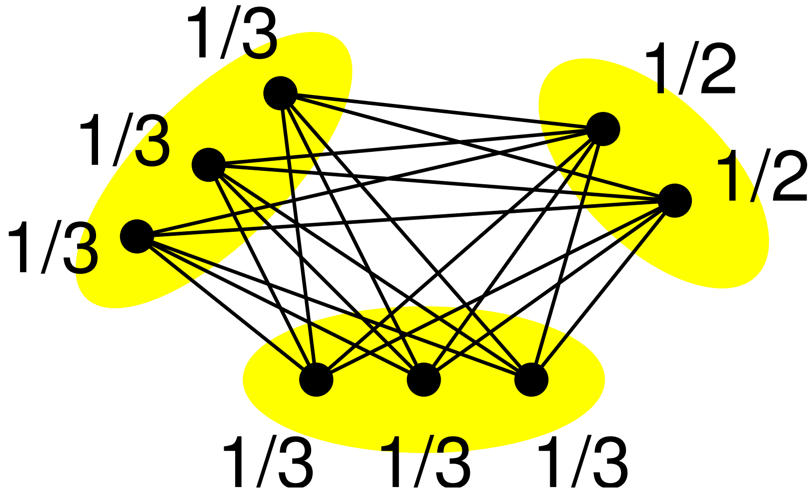


$$\boldsymbol{\nu} = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top \text{ is the nucleolus.}$$

$$\theta_{\boldsymbol{\nu}} = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

Indeed, this graph is $\begin{cases} \text{a forest} \\ \text{complete multipartite.} \end{cases}$

Consider a complete multipartite graph.

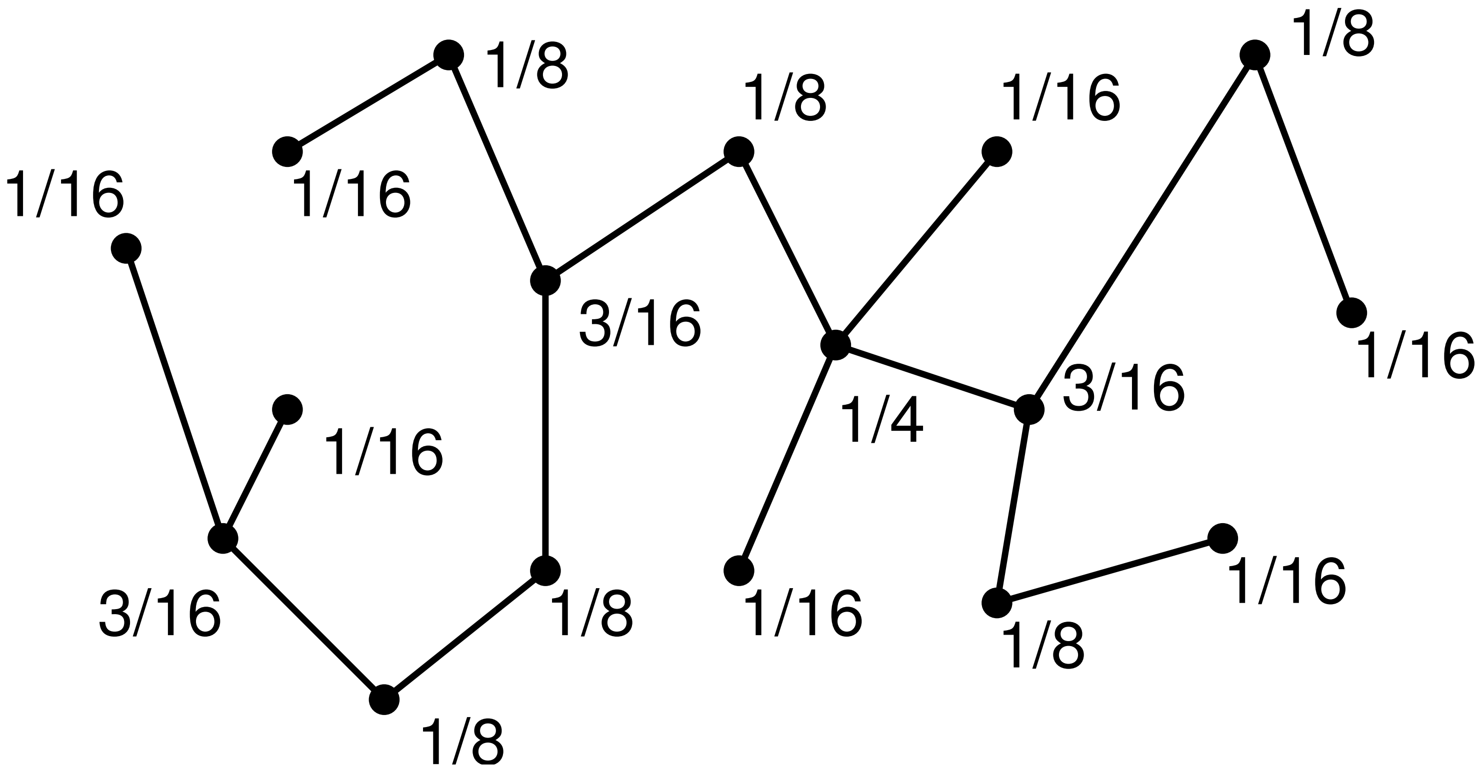


We have

$$\nu[i] = \frac{1}{n_i},$$

where n_i is $\#$ of vertices of the class to which i belongs.

Consider a forest.



We have

$$\boldsymbol{\nu}[i] = \frac{\deg(i)}{|E|},$$

where $\deg(i)$ is $\#$ of edges incident to i.

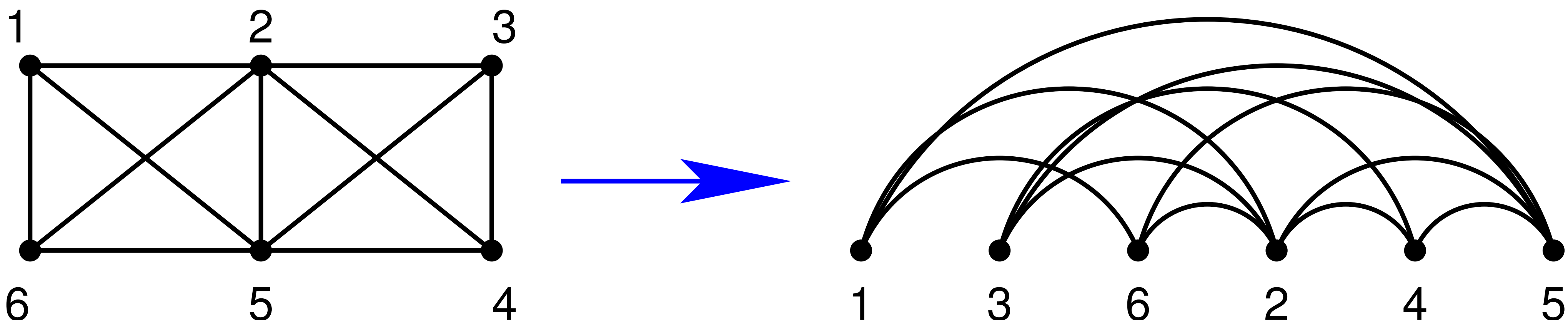
For chordal graphs, we can use the following theorem to compute the nucleoli.

$\boxed{\text{Thm.}}$ (Fulkerson & Gross '65)

In a chordal graph, # of maximal cliques $\leq$ # of vertices.

Moreover, we can enumerate them in poly time.

$\boxed{\text{Proof Sketch}}$



(1) Use the LP formulation for the nucleolus computation.

(Peleg)

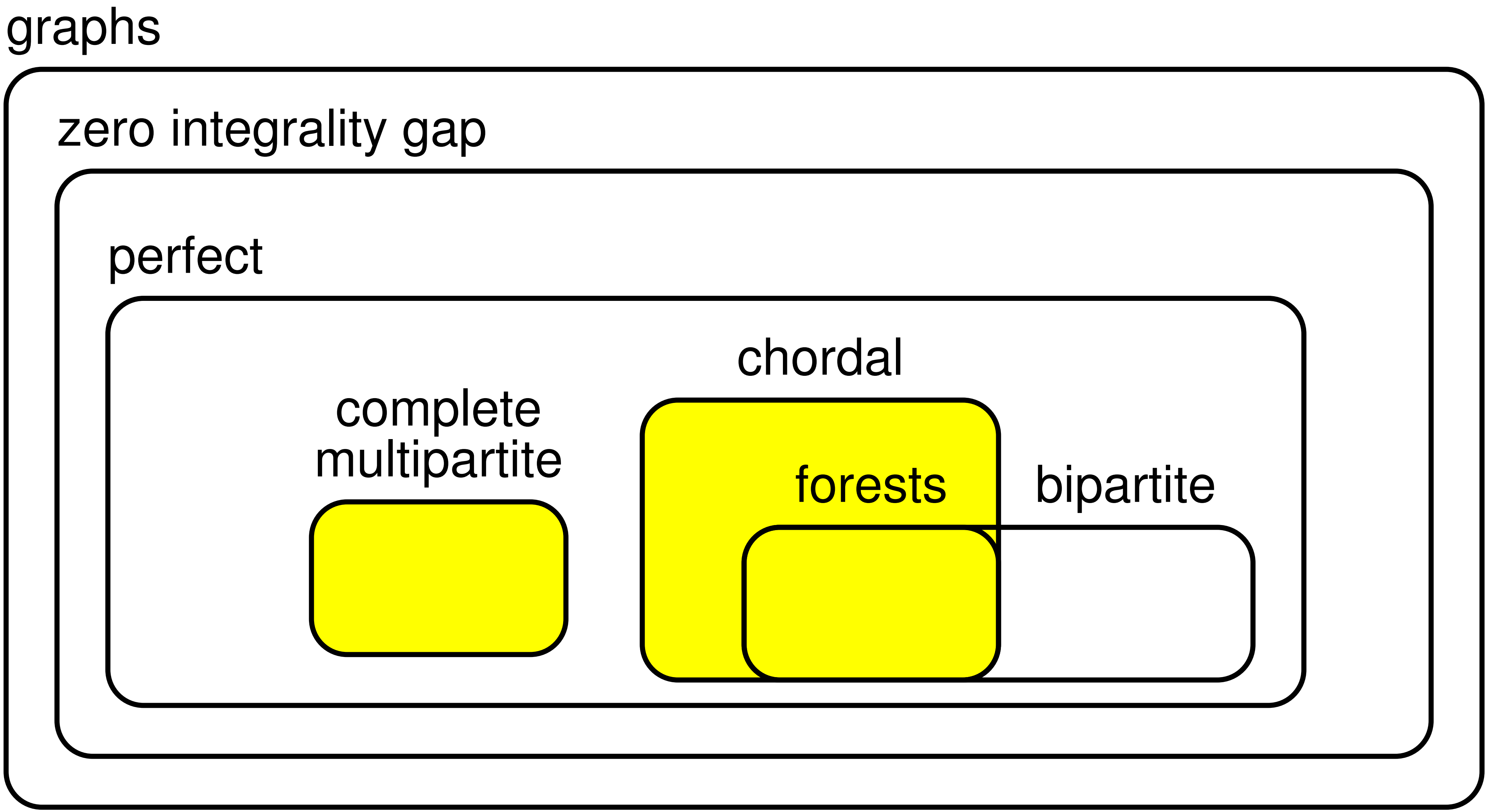
By solving a sequence of LP problems, we can obtain the nucleolus (not in poly time).

(2) Identify the essential coalitions. (Huberman '80)

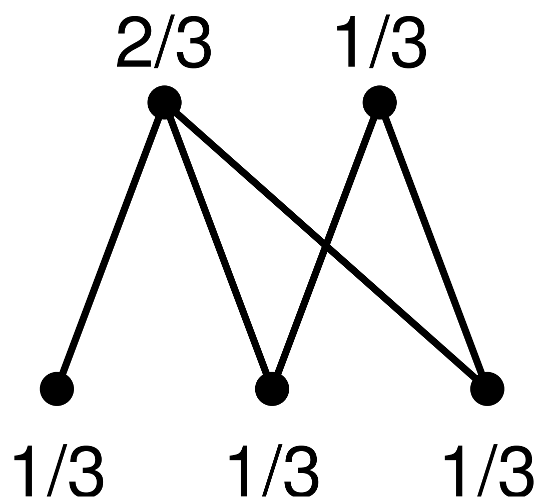
The essential coalitions reduce the work load.
$S \subseteq V$ is essential $\Leftrightarrow$ S is an independent set.

(3) Analyze the LP.

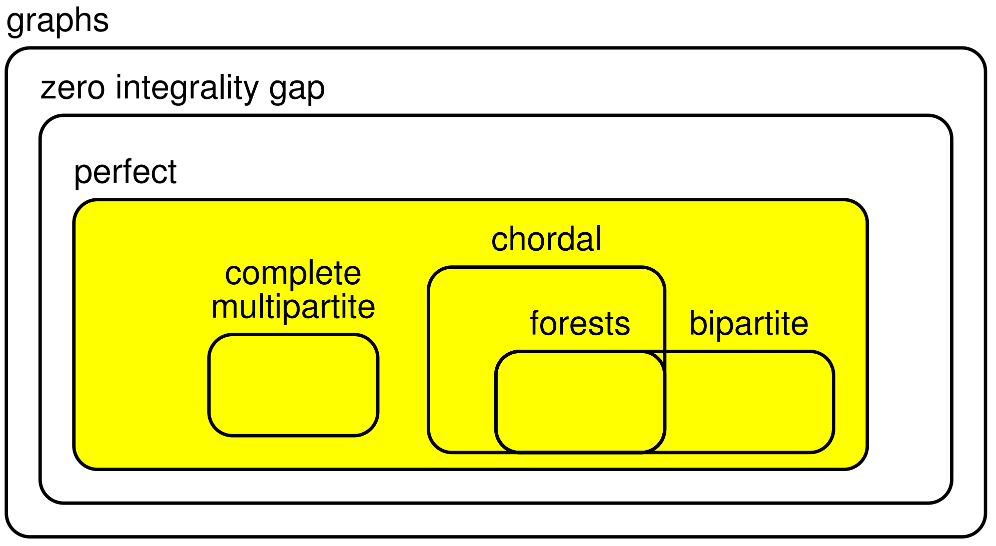
For O-good perfect graphs, we can precisely tell what are the optimal solutions in the LP problems with help of the characterization of the extreme points of the core.


graphs
zero integrality gap
perfect
chordal
complete
multipartite
forests
bipartite

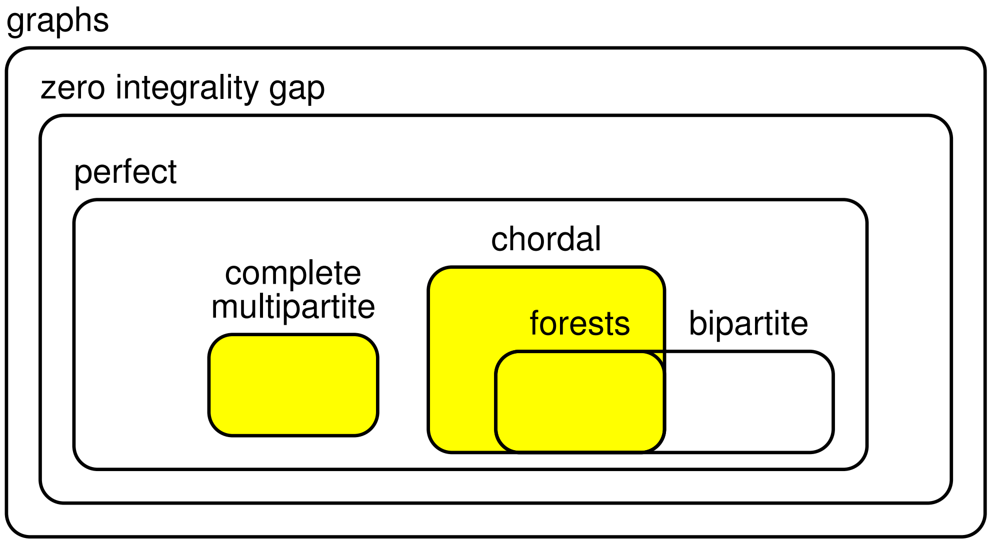
There is a bipartite graph for which the nucleolus is not the barycenter of the char. vectors of the maximum cliques.



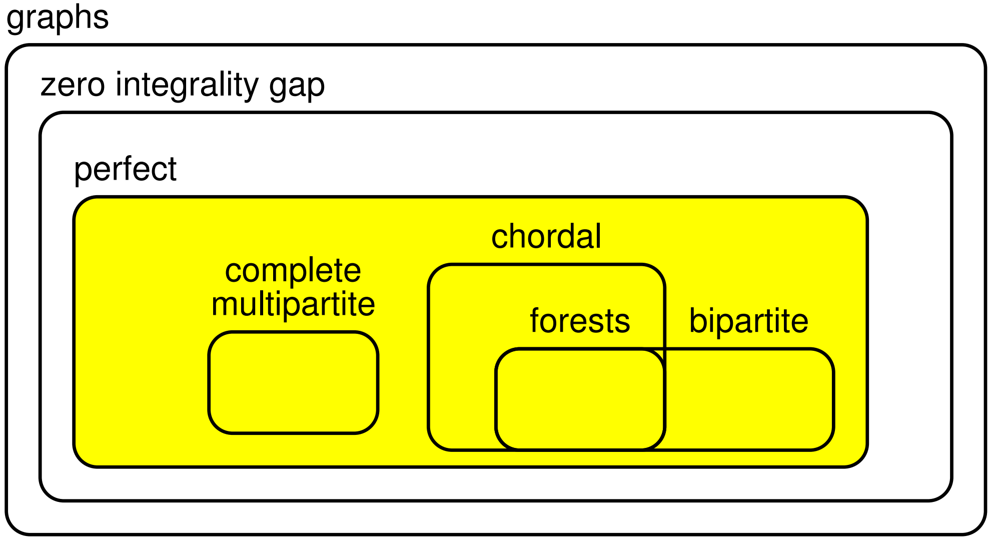
Core



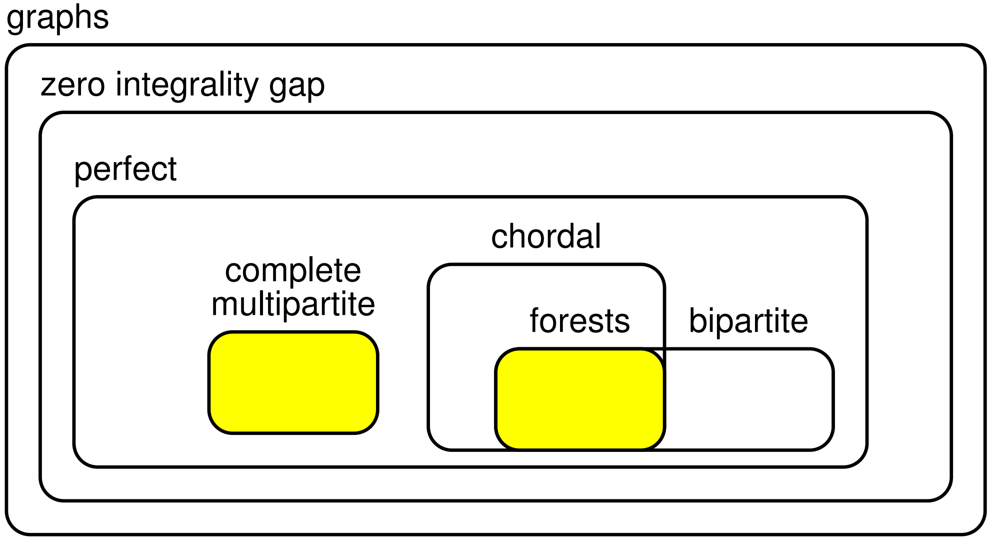
Nucleolus



τ-value



Shapley value



Open problems: minimum coloring games

- Nucleolus for perfect graphs ??
- Nucleolus for bipartite graphs ??
- Shapley value for perfect graphs ??
- Shapley value for bipartite graphs ??
- Cost allocations for other kinds of graphs ??

The list of what we discussed

- Def.: minimum coloring game
- Def.: core
- Properties:
  Balancedness, Total balancedness, Submodularity
- Characterization: the core for a perfect graph
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  (optimization theory)
- They want to allocate the benefit in a fair way.
  (cooperative game theory)

Status of algorithmic problems on cooperative games

- As many cooperative games as optimization problems!!
- Many algorithmic problems remain unsolved!!

$\Longrightarrow$ Why not work on them??

[End of the talk]