Affine representations of abstract convex geometries

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Combinatorics
Geometry
Computation

- **Matroids** .... abstraction of dependence

  Application:
  - Finite geometry
  - Coding theory
  - Combinatorial optimization

- **Oriented Matroids** .... abstraction of dependence

  Application:
  - Convex polytopes
  - Computational geometry
  - Discrete geometry
  - Optimization

- **Convex geometries** .... abstraction of convexity

  Application:
  - Discrete geometry
  - Social choice theory
  - Mathematical psychology

- **Matroids** . . . . abstraction of dependence

  Every matroid can be represented
  as a homotopy-sphere arrangement.
  (Swartz, '02)

- **Oriented Matroids** . . . . abstraction of dependence

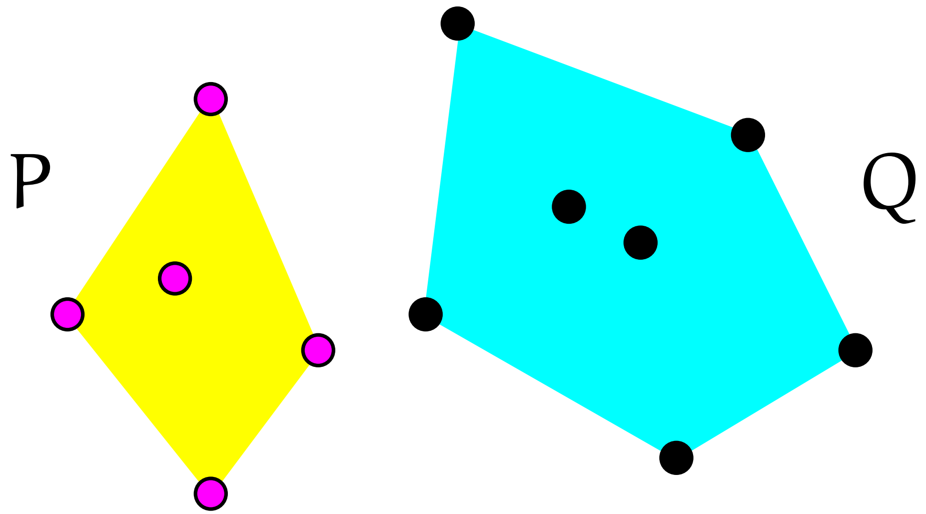
  Every oriented matroid can be represented
  as a pseudohyperplane arrangement.
  (Forkman–Lawrence, '78)

- **Convex geometries** . . . . abstraction of convexity

  ??????????

Our Theorem:

Any convex geometry is isomorphic to some generalized convex shelling,



determined by two point sets P and Q satisfying that $\mathrm{conv}(\mathrm{P}) \cap \mathrm{conv}(\mathrm{Q}) = \emptyset$.

This gives an affine representation of a convex geometry.

Our Theorem:

Any convex geometry is isomorphic to some generalized convex shelling.

In the rest of my talk

- Definition of a convex geometry
- Examples of a convex geometry
- Definition of a generalized convex shelling
- Our theorem
- Outline of the proof

(Edelman–Jamison '85)

E a nonempty finite set

$\mathcal{L}$ a nonempty family of subsets of E

$\boxed{\text{Def.}}$ $\mathcal{L} \subseteq 2^{E}$ is called a convex geometry on E
if $\mathcal{L}$ satisfies the following three conditions.

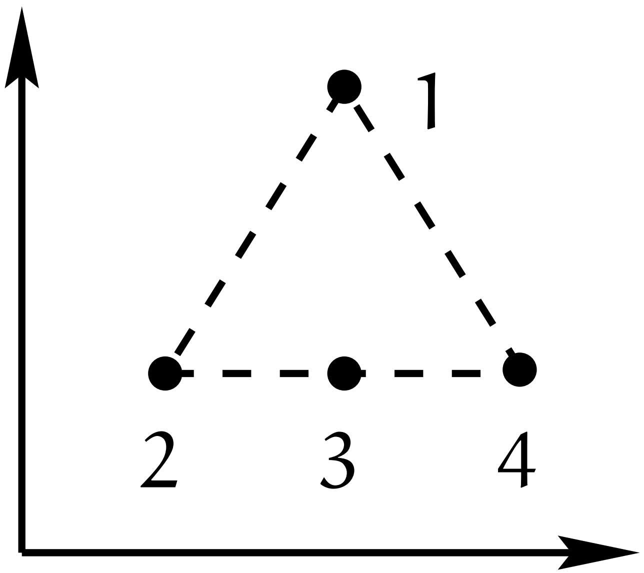
(1) $\emptyset \in \mathcal{L}$, $E \in \mathcal{L}$.

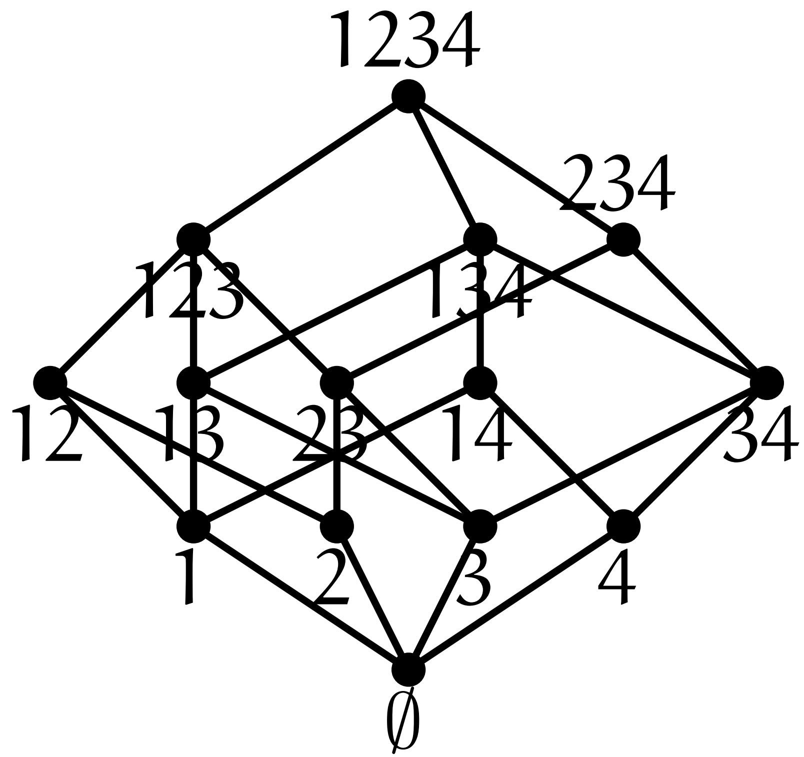
(2) $X, Y \in \mathcal{L} \Longrightarrow X \cap Y \in \mathcal{L}$.

(3) $X \in \mathcal{L} \setminus \{E\} \Longrightarrow \exists\, e \in E \setminus X$ s.t. $X \cup \{e\} \in \mathcal{L}$.

Q a finite point set in $\mathbb{R}^d$

Define: $\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X) \cap (Q \setminus X) = \emptyset\}$.





$\mathcal{L}$ is a convex geometry and called the convex shelling on Q.

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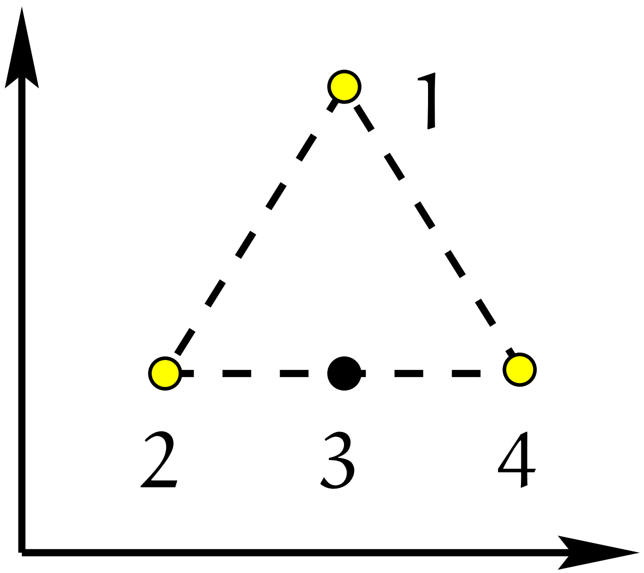

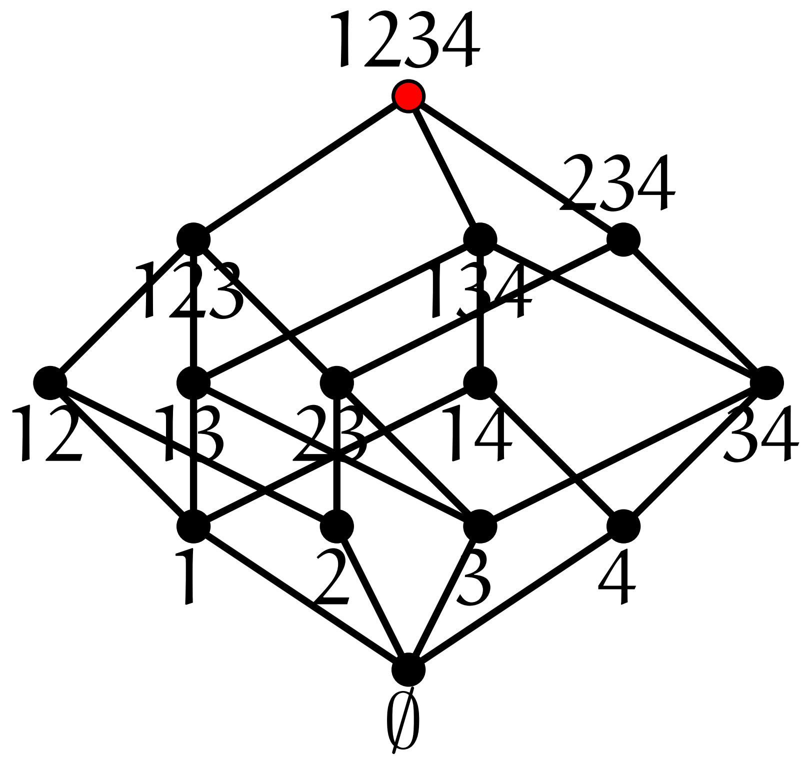

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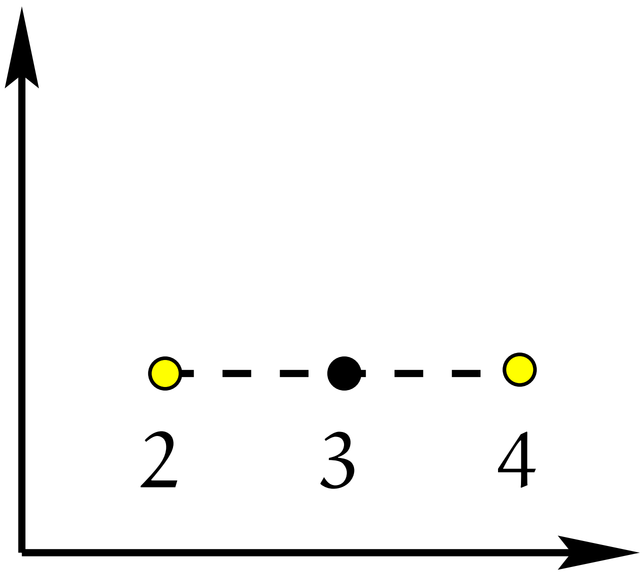

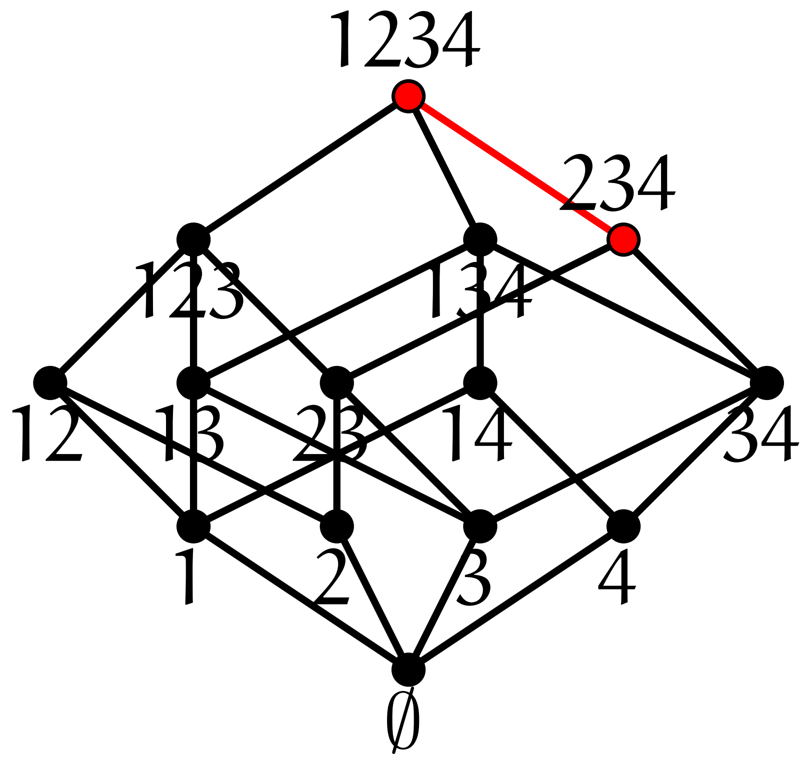

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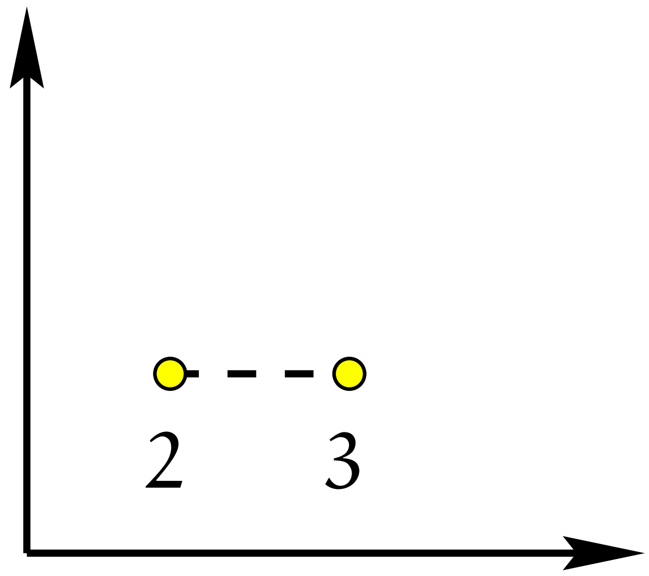

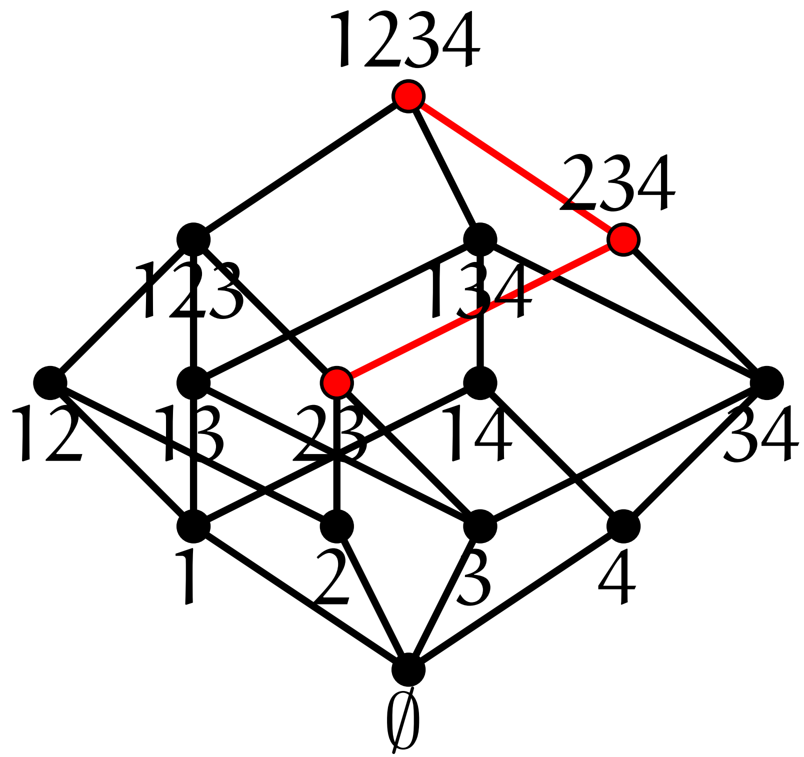

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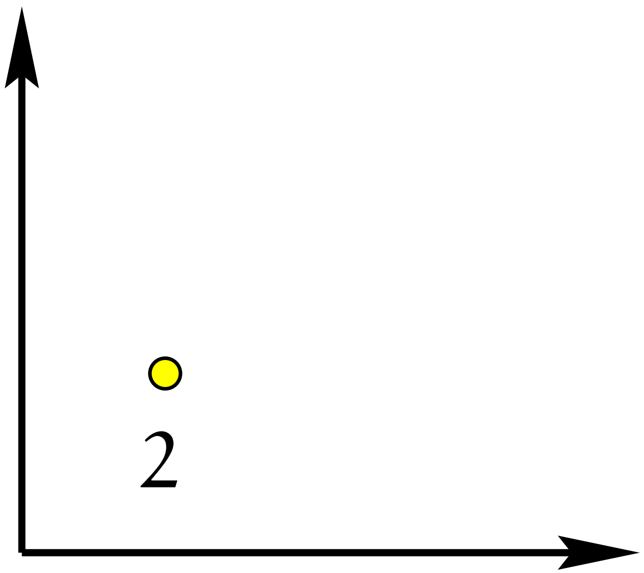

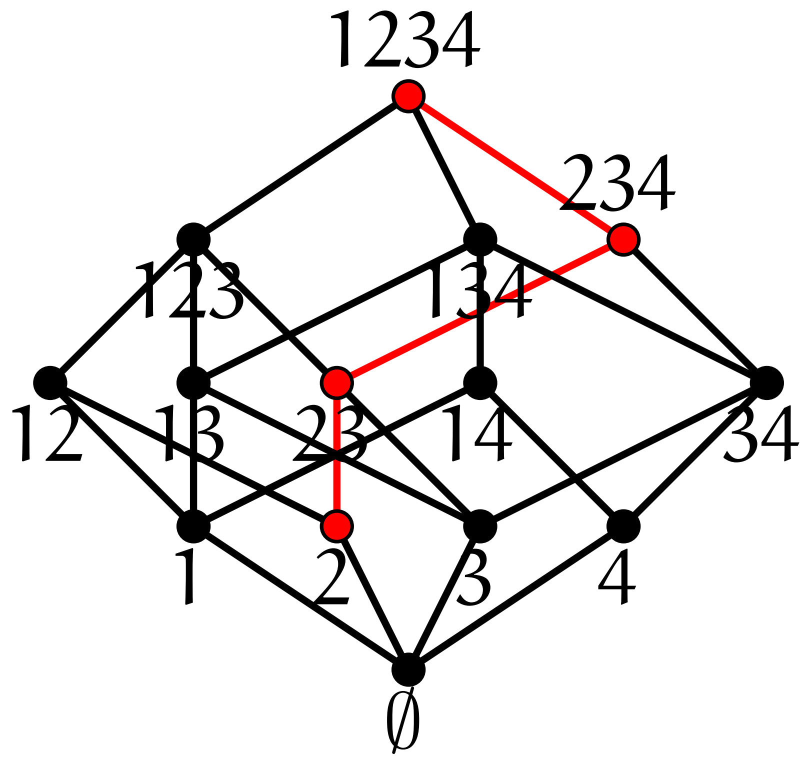

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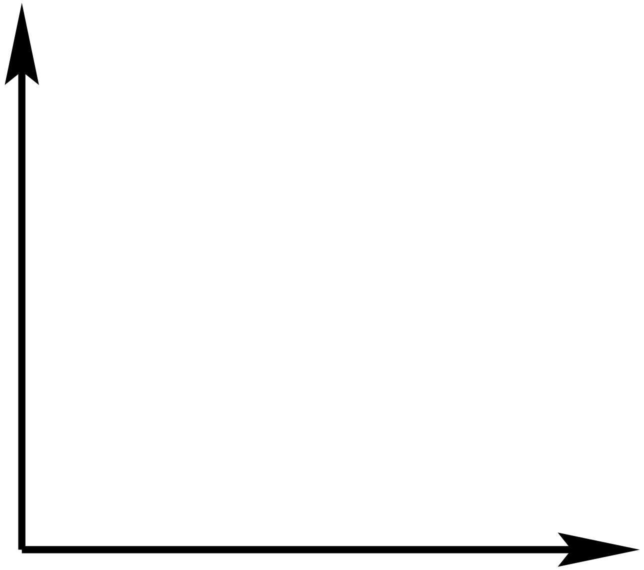

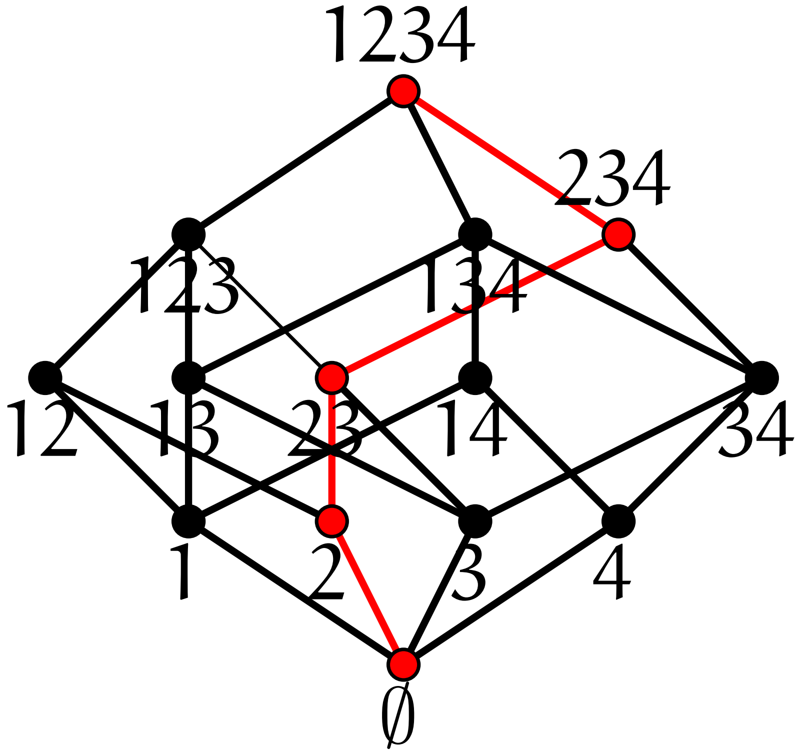

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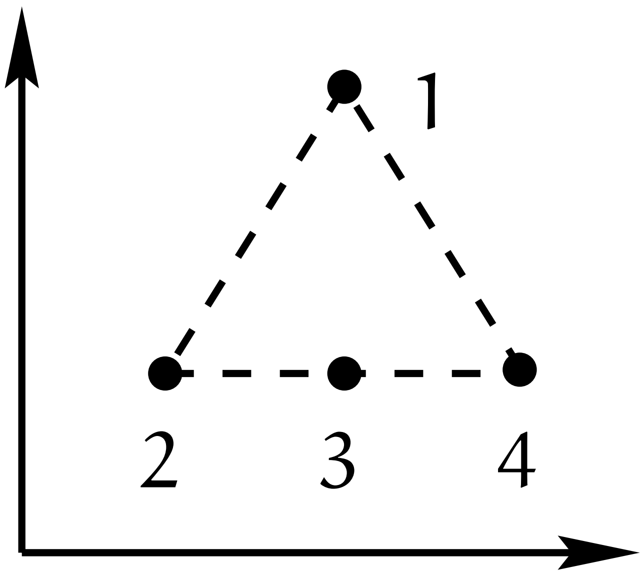
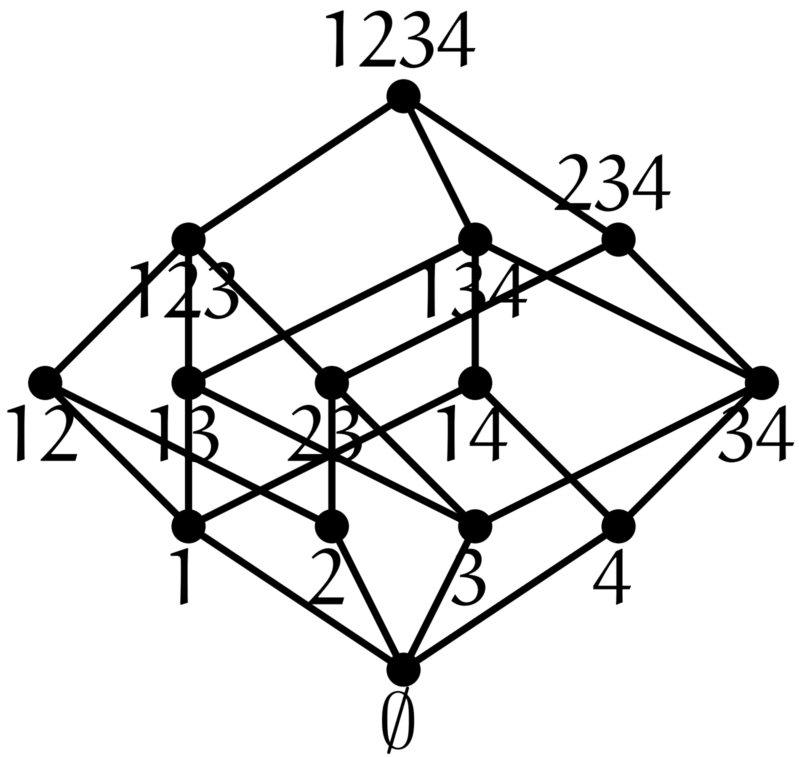
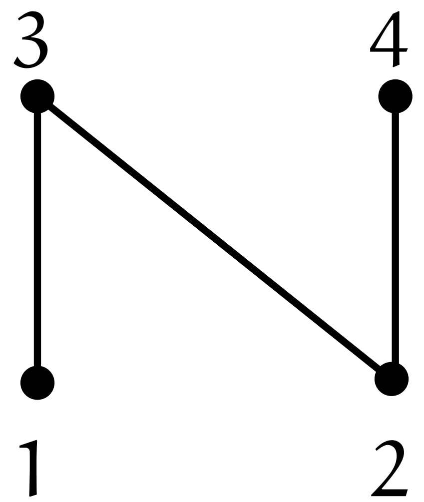


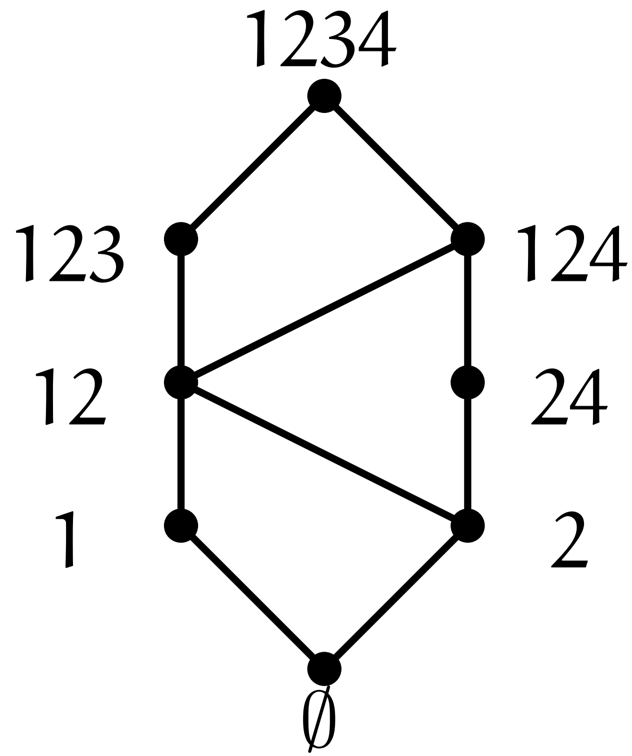


$\mathcal{L}$ is a convex geometry and called the convex shelling on Q.

$\mathcal{P} = (E, \leq)$ a partially ordered set

Define: $\mathcal{L} = \{X \subseteq E : e \in X, f \leq e \Rightarrow f \in X\}$.





$\mathcal{L}$ is a convex geometry on E and called the poset shelling of $\mathcal{P}$.

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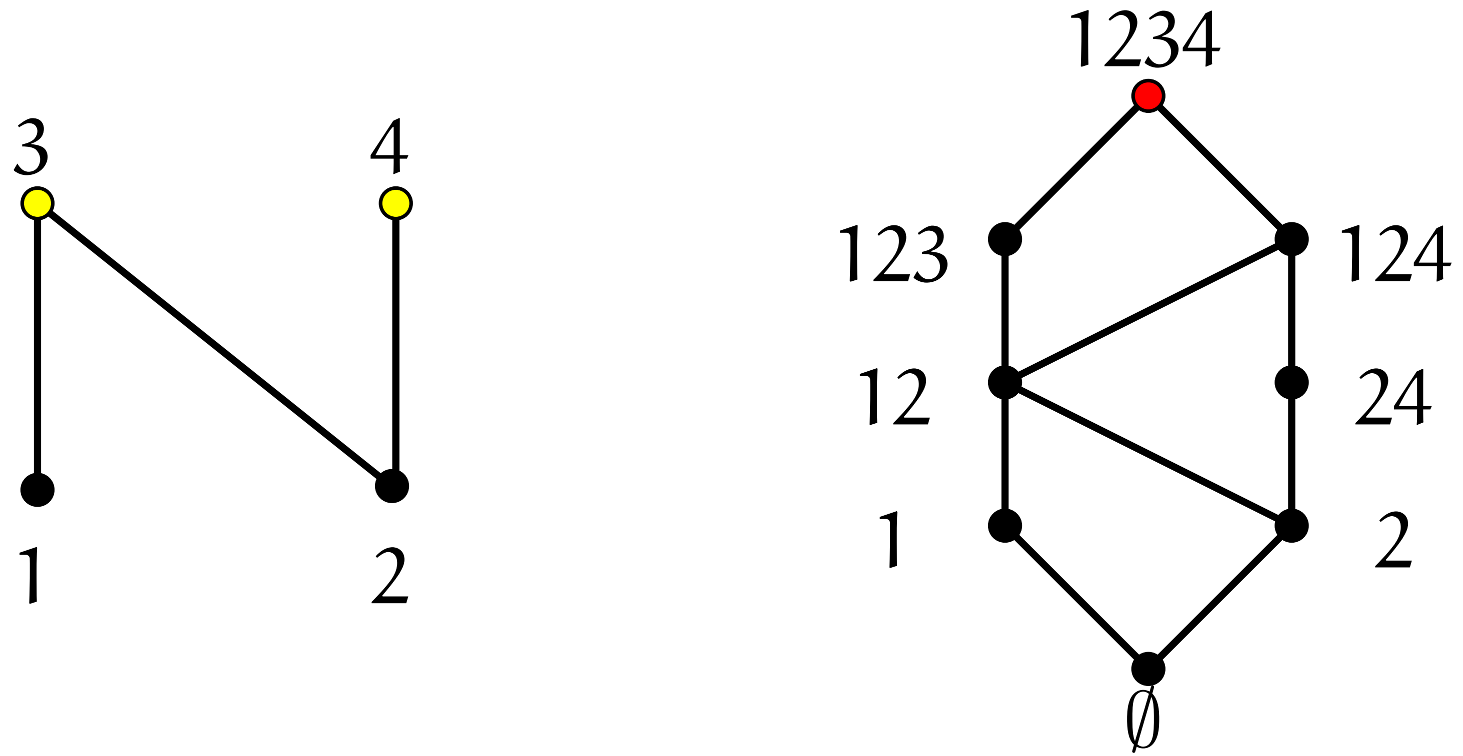

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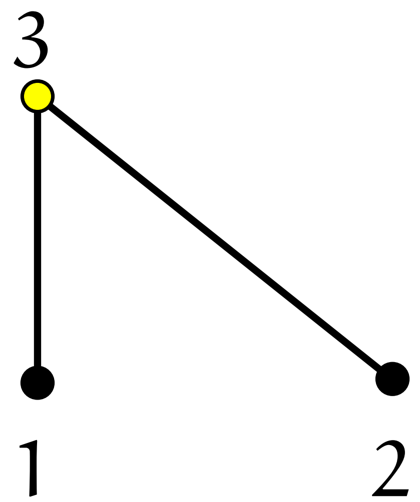

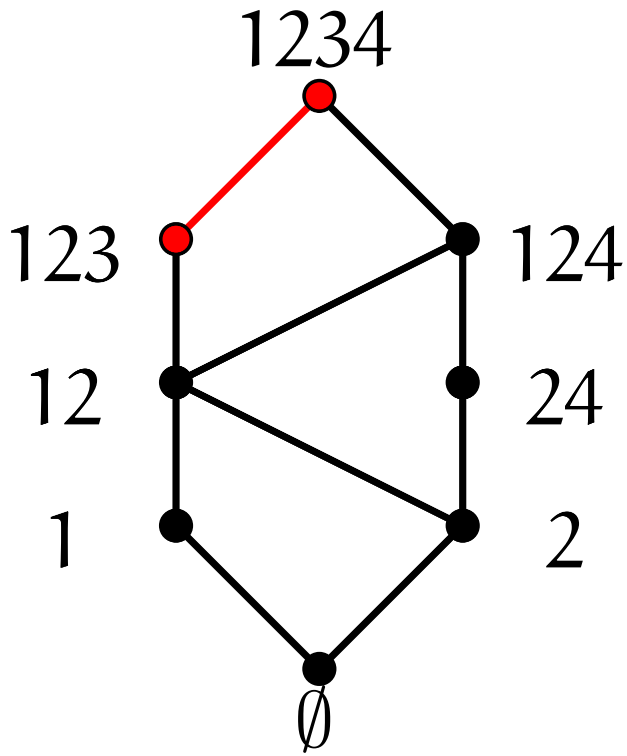

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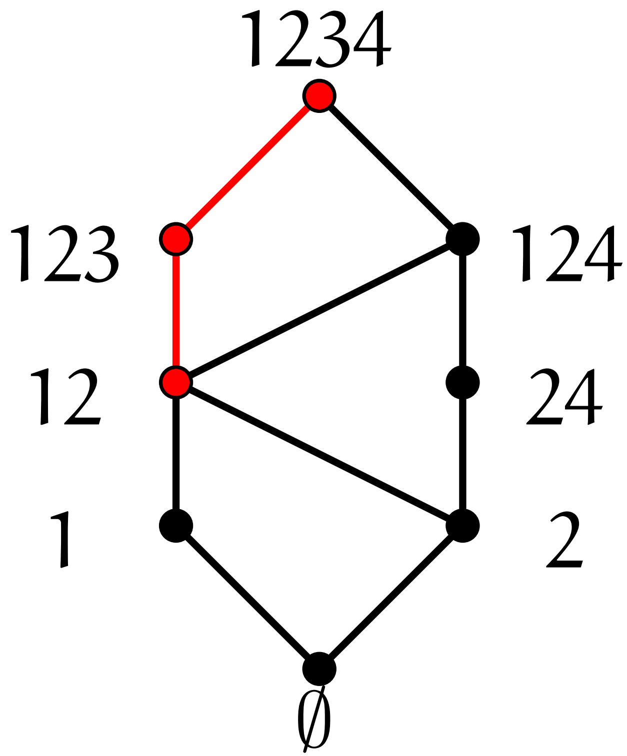

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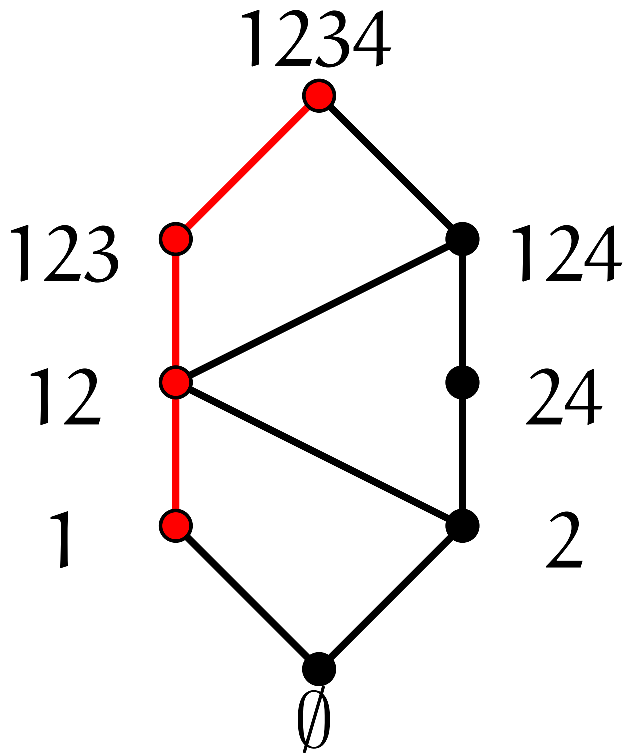

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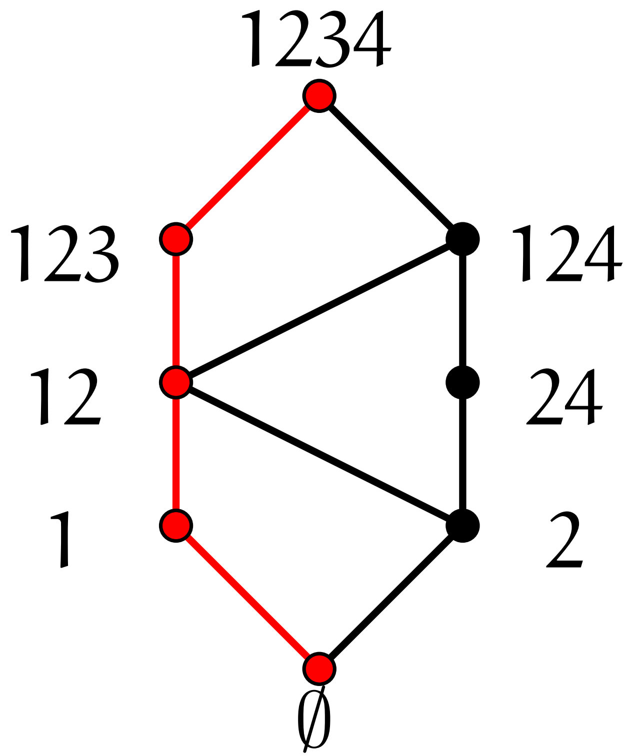

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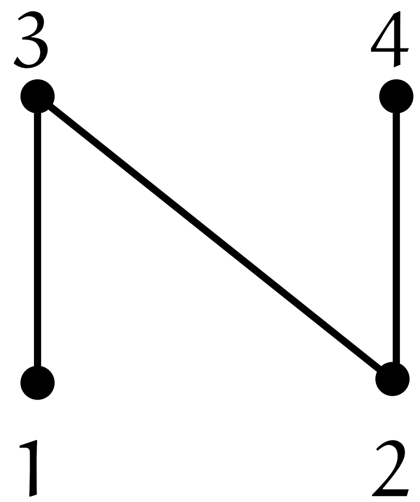
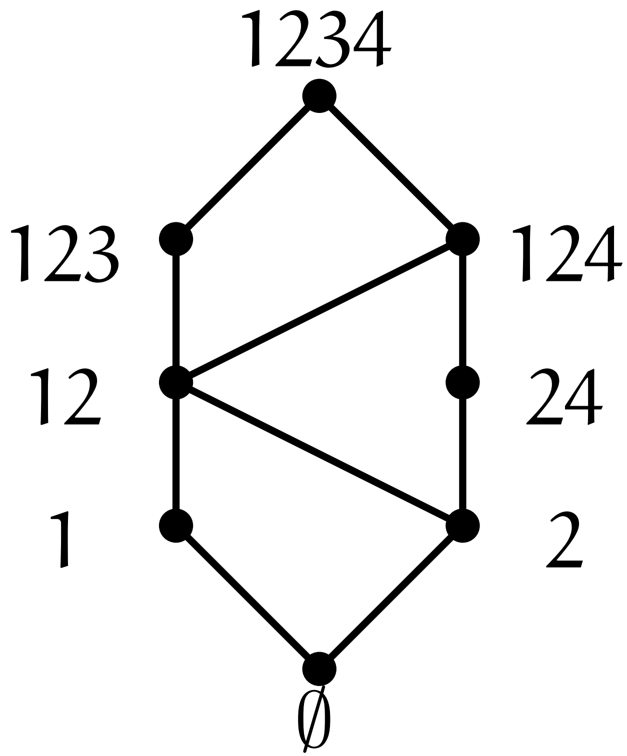


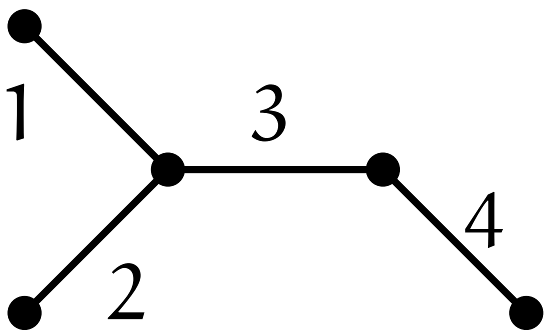


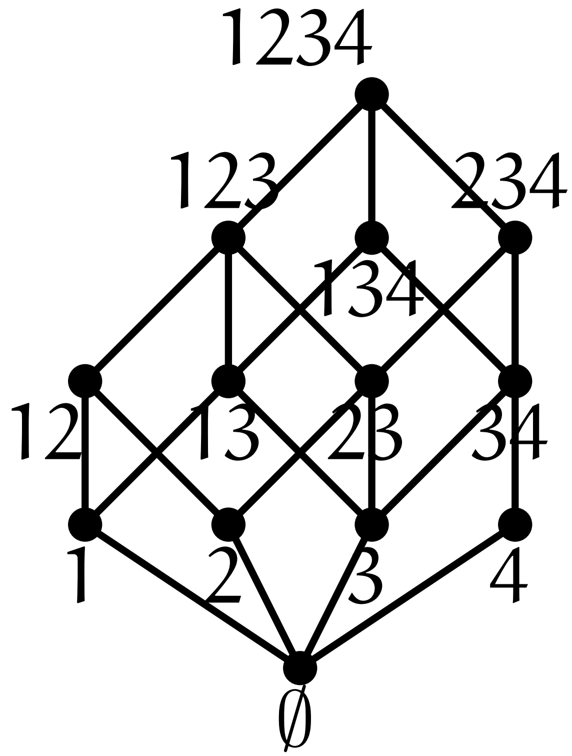
$\mathcal{L}$ is a convex geometry on E and called the poset shelling of $\mathcal{P}$.

$T = (V, E)$ a tree

Define:

$$\mathcal{L} = \{X \subseteq E : X \text{ forms a subtree of } T\}.$$

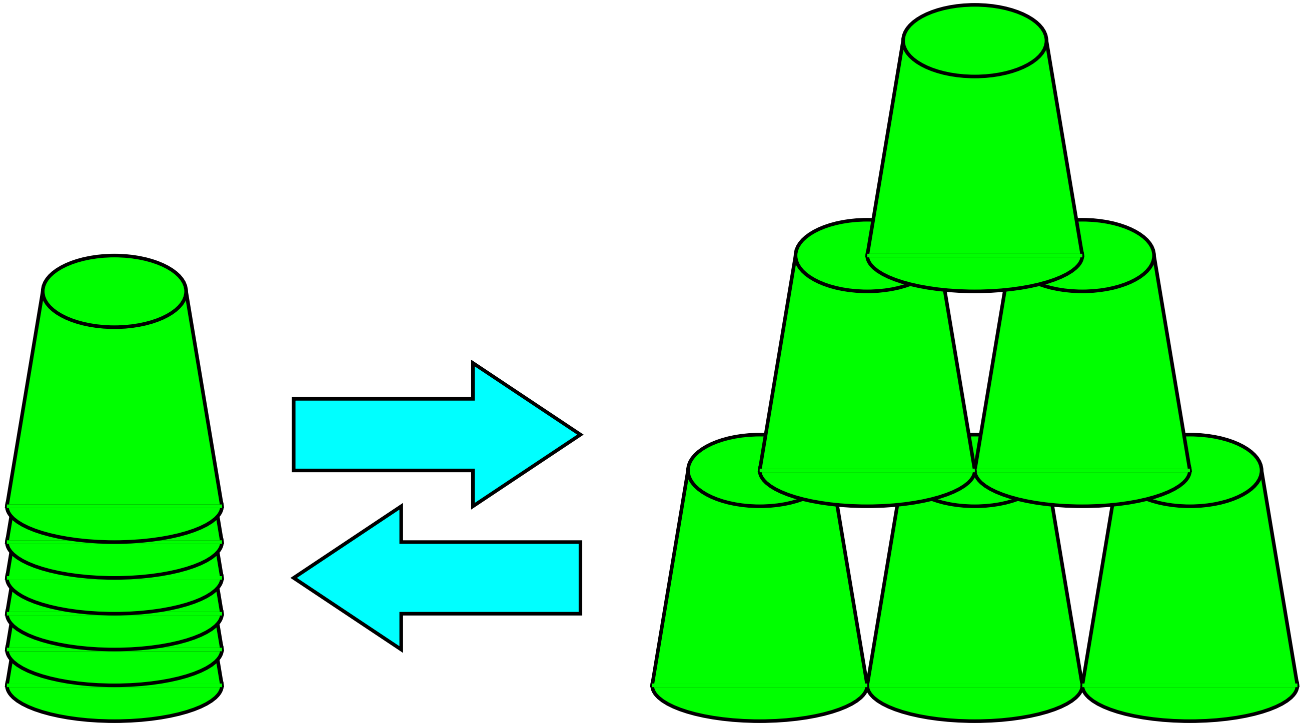




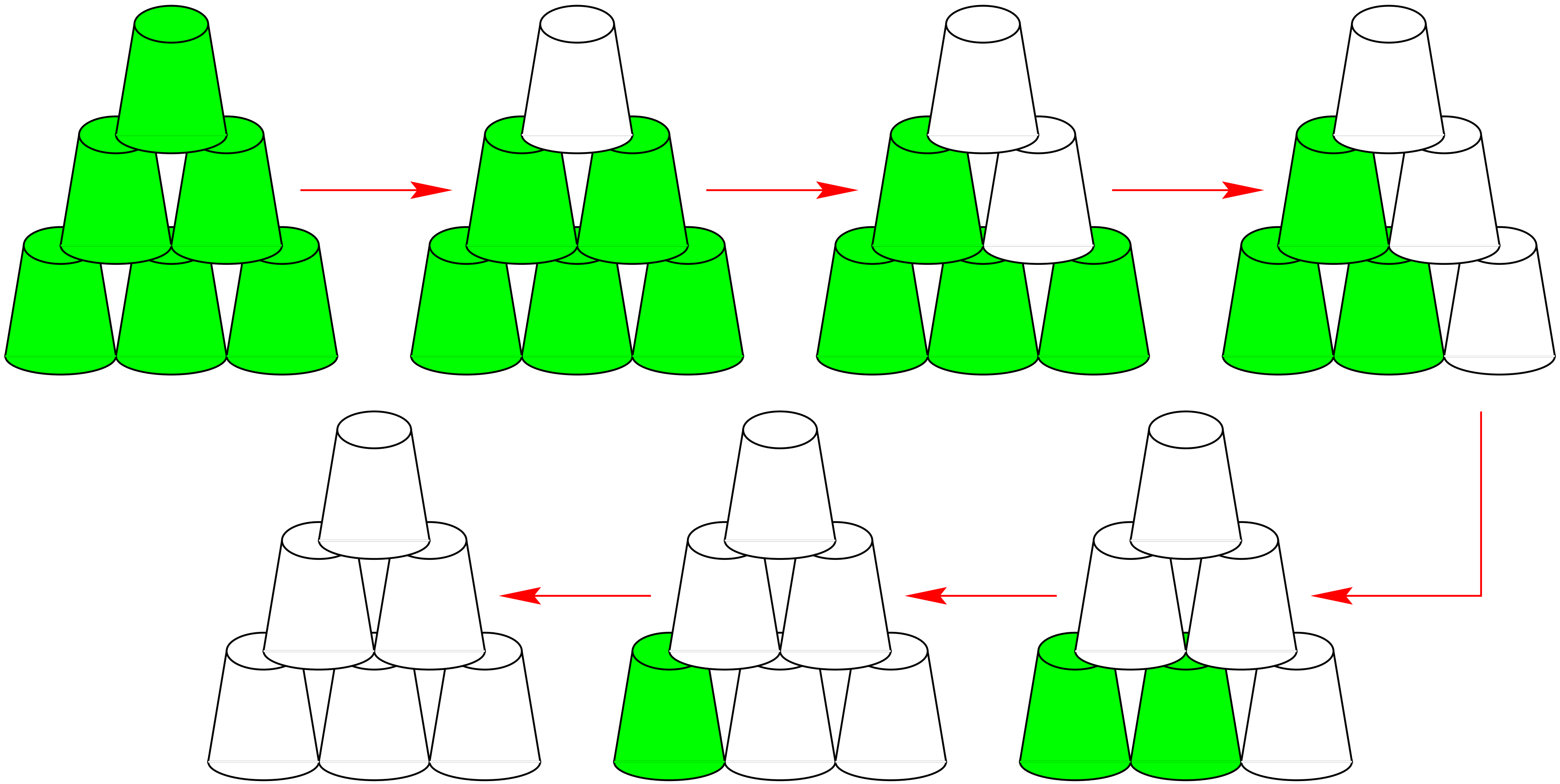
$\mathcal{L}$ is a convex geometry on E and called the tree shelling of T

What is "cupstacks"?

Construct the tower from the pile and get it back as quickly as possible.



A sequence in collapsing



Various objects yield convex geometries.

- From graphs
  - Tree shellings on vertices
  - Graph search
  - Simplicial elimination of chordal graphs
- From partially ordered sets
  - Poset double shellings
  - k-chains in partially ordered sets
- From finite point sets in $\mathbb{R}^d$
  - Lower convex shellings on point sets
- From oriented matroids
  - Convex shellings of acyclic oriented matroids
- ...

Our Theorem:

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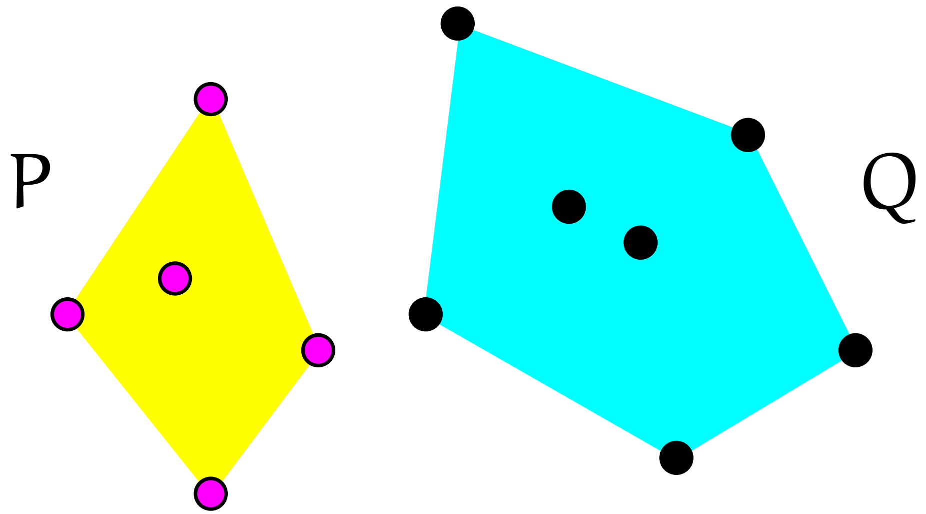
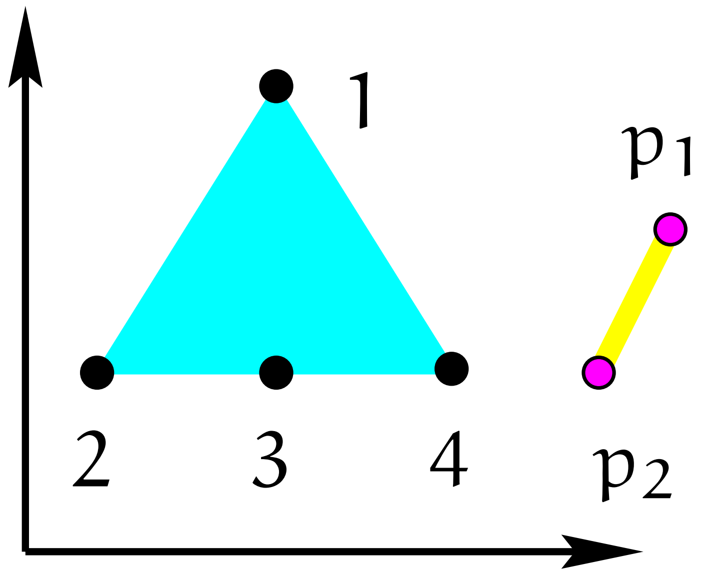


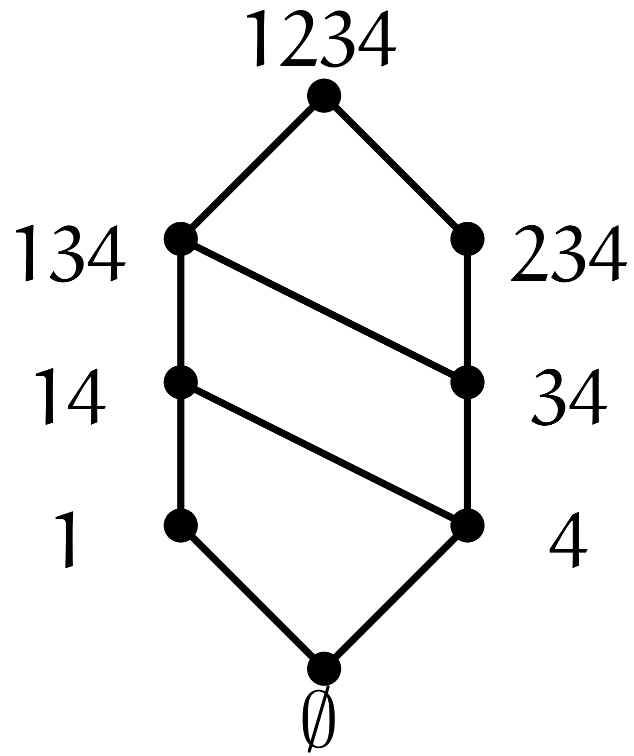
determined by two point sets P and Q
satisfying that $\mathrm{conv}(\mathrm{P}) \cap \mathrm{conv}(\mathrm{Q}) = \emptyset$.

This gives an affine representation of a convex geometry.

P,Q finite point sets in $\mathbb{R}^d$ satisfying $\operatorname{conv}(P) \cap Q = \emptyset$

Define: $\mathcal{L} = \{X \subseteq Q : \operatorname{conv}(X \cup P) \cap (Q \setminus X) = \emptyset\}$.





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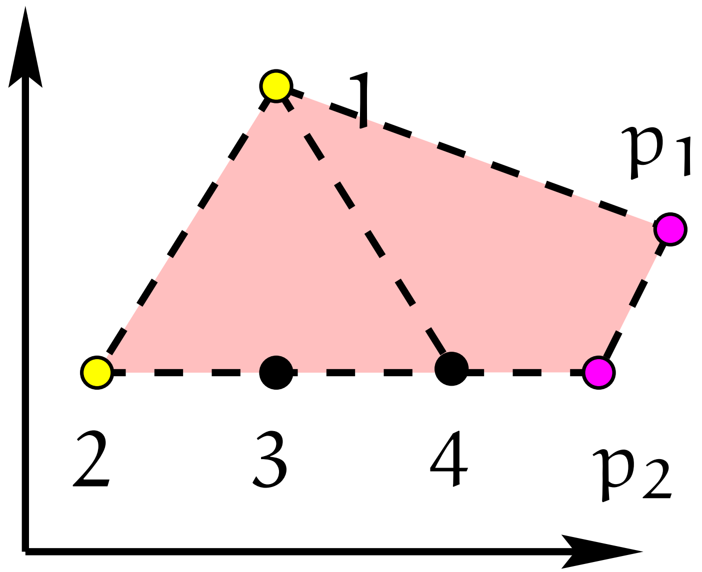

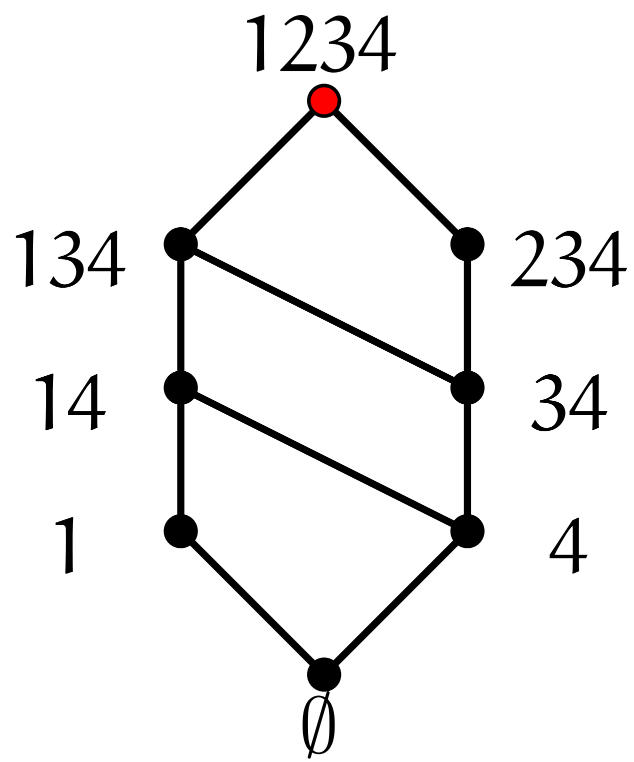

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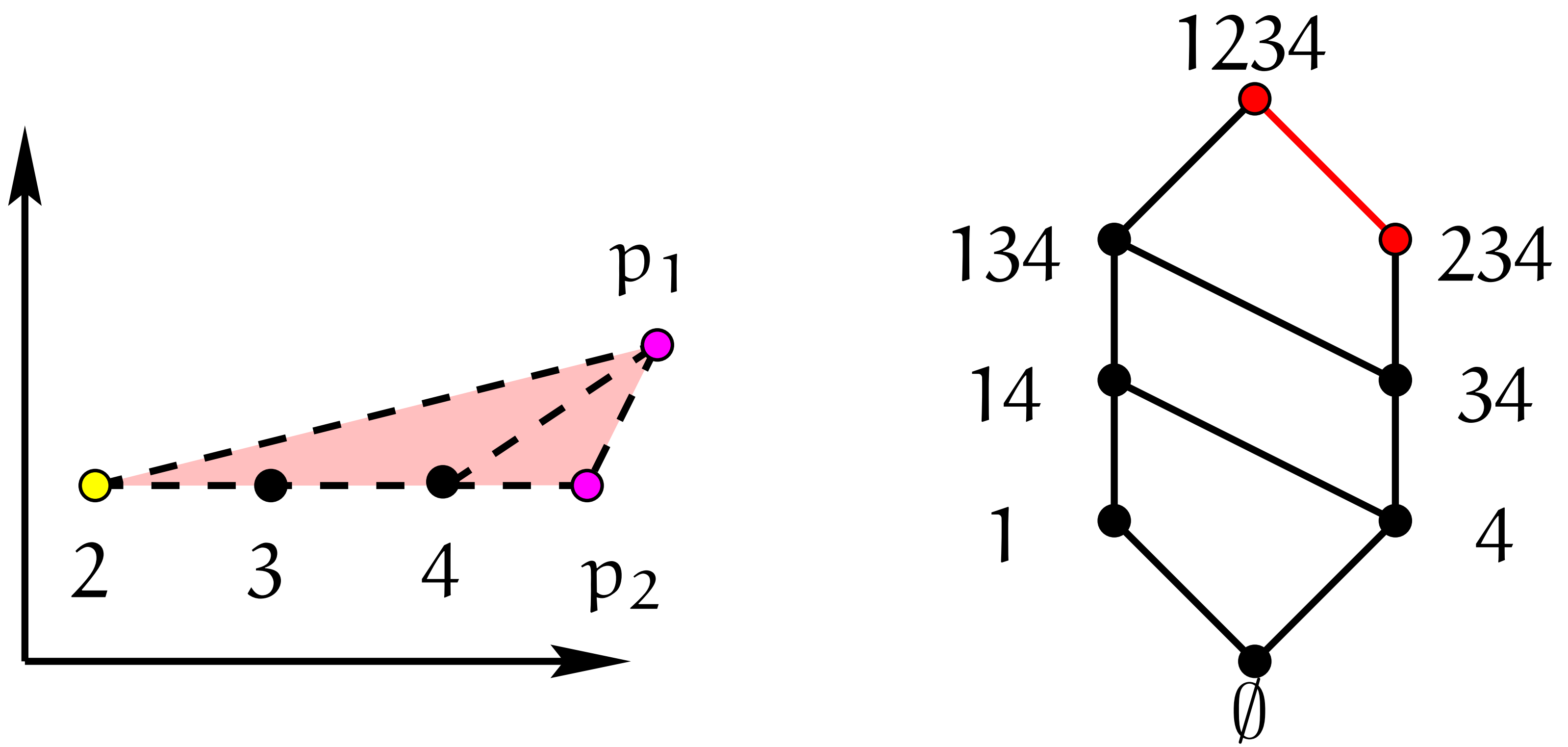

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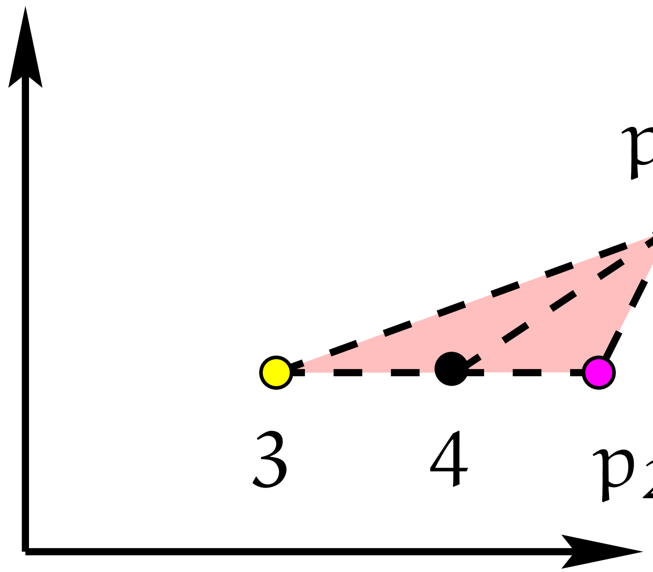

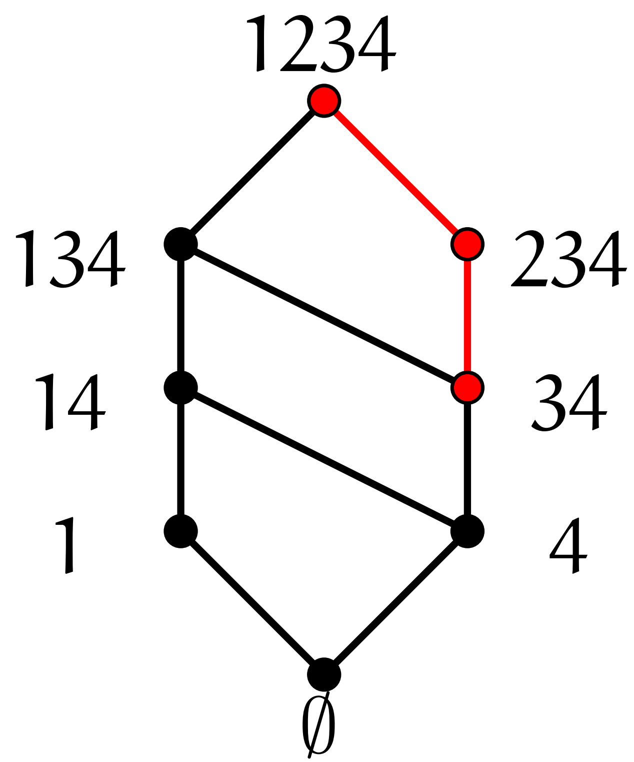

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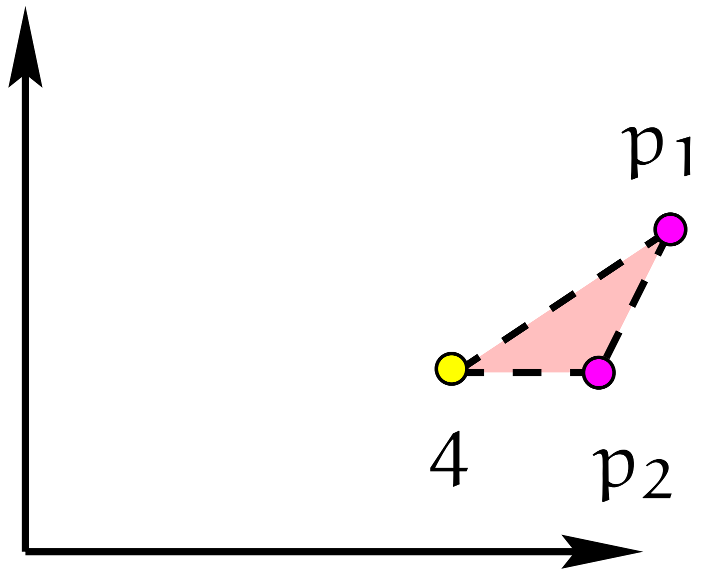

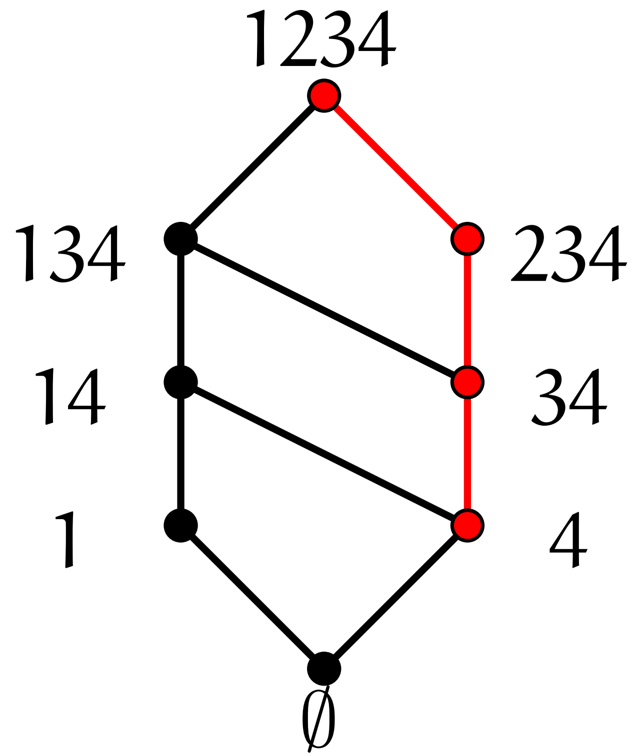

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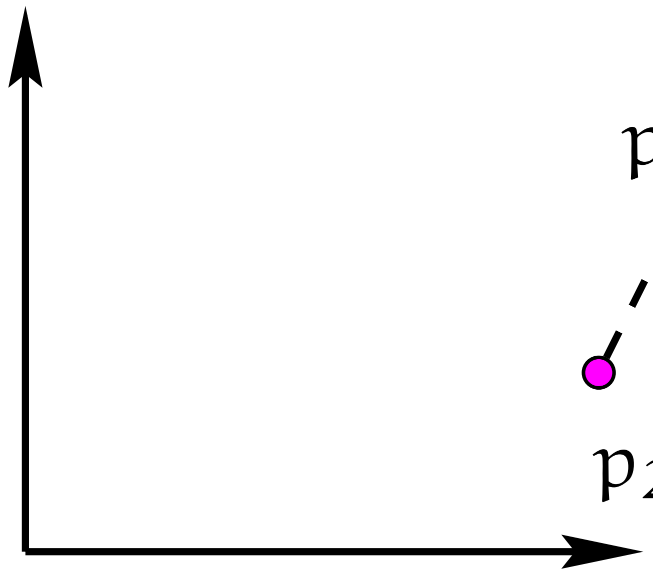

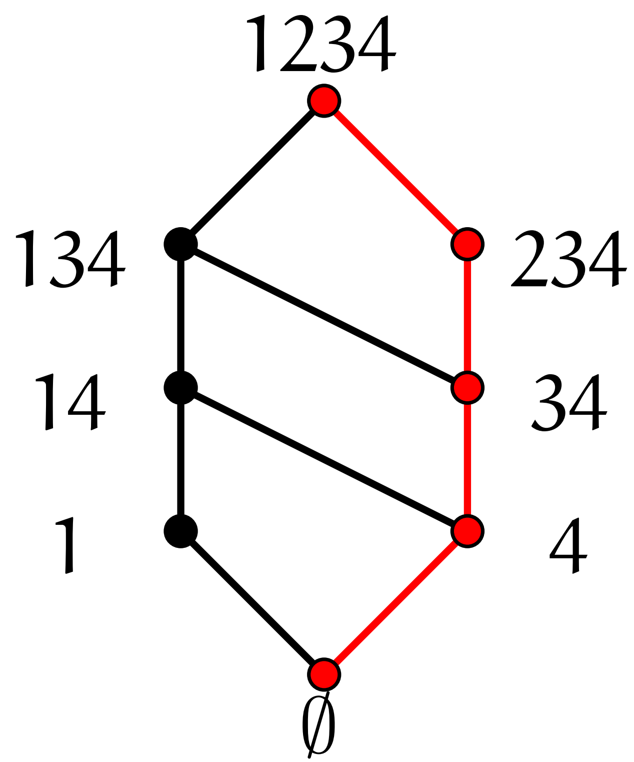

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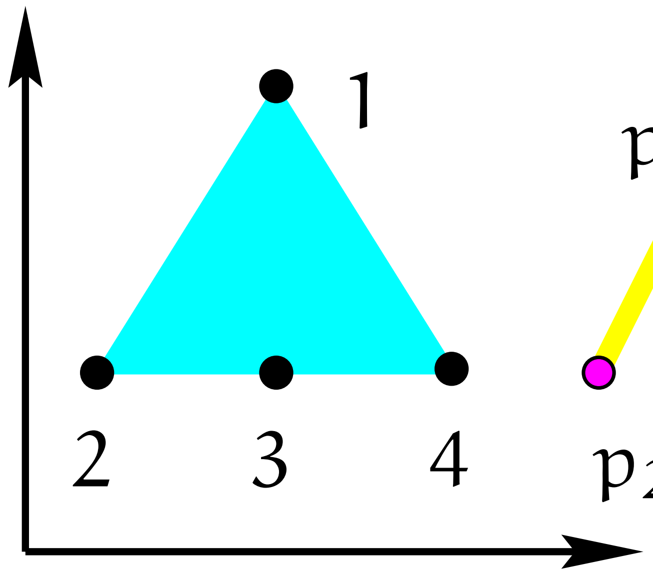

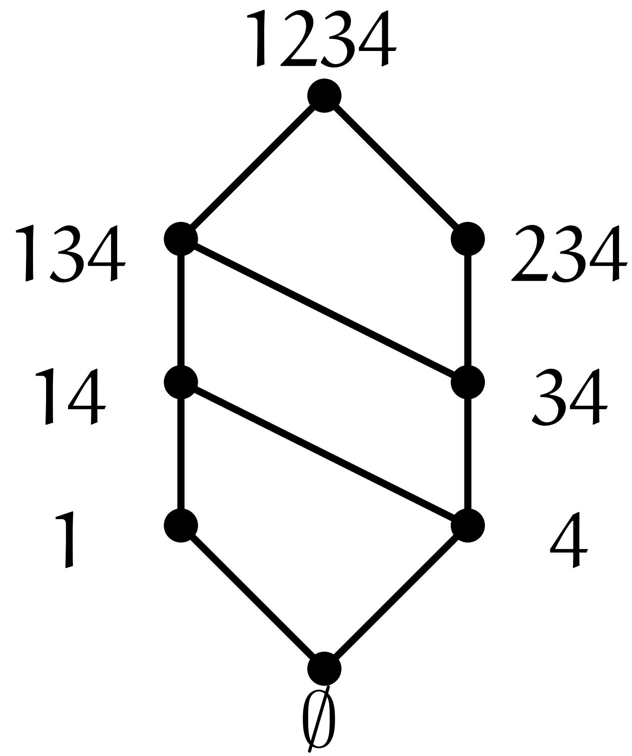

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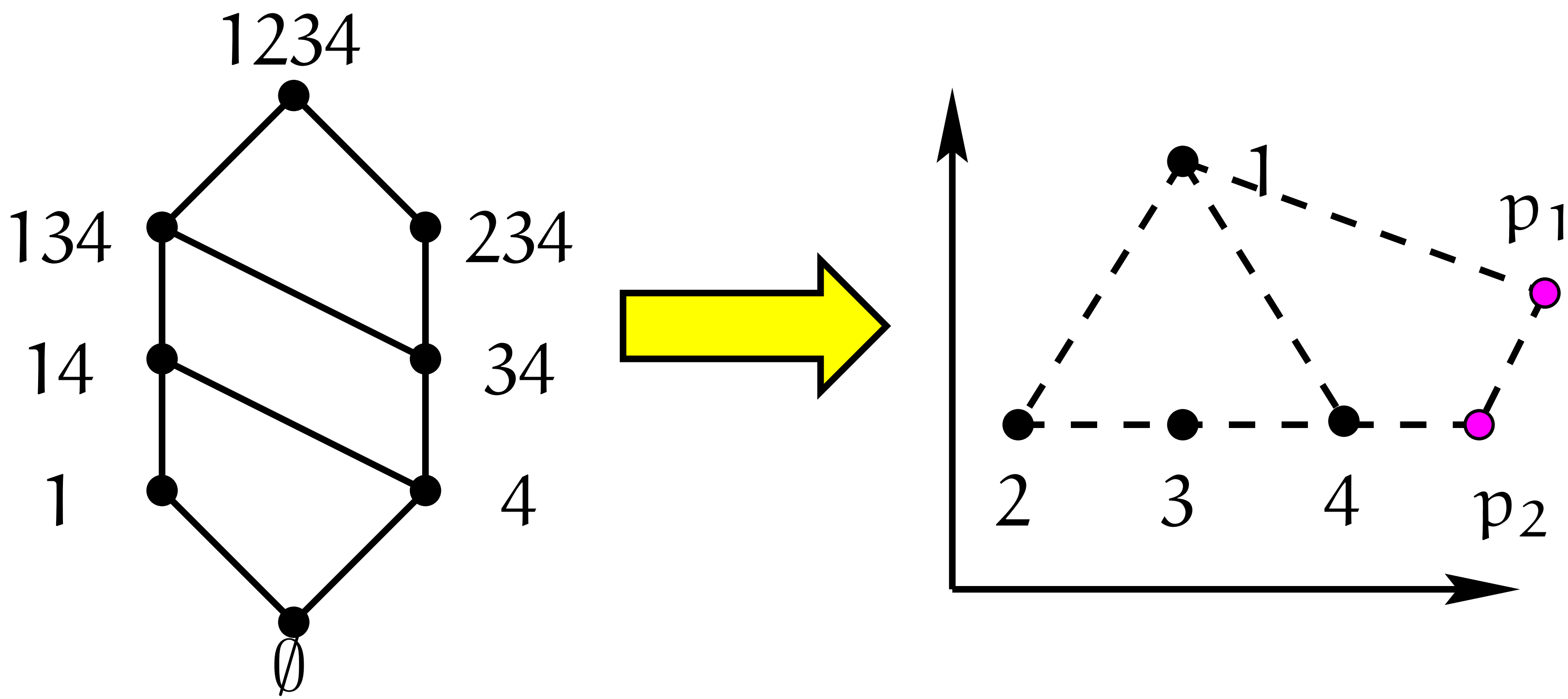
the generalized convex shelling on Q with respect to P.

Our Theorem

Any convex geometry is isomorphic to
some generalized convex shelling.

In other words,

For any convex geometry $\mathcal{L}$,
there exist finite point sets P and Q such that
$\mathcal{L}$ is isomorphic to
the generalized convex shelling on Q w.r.t. P.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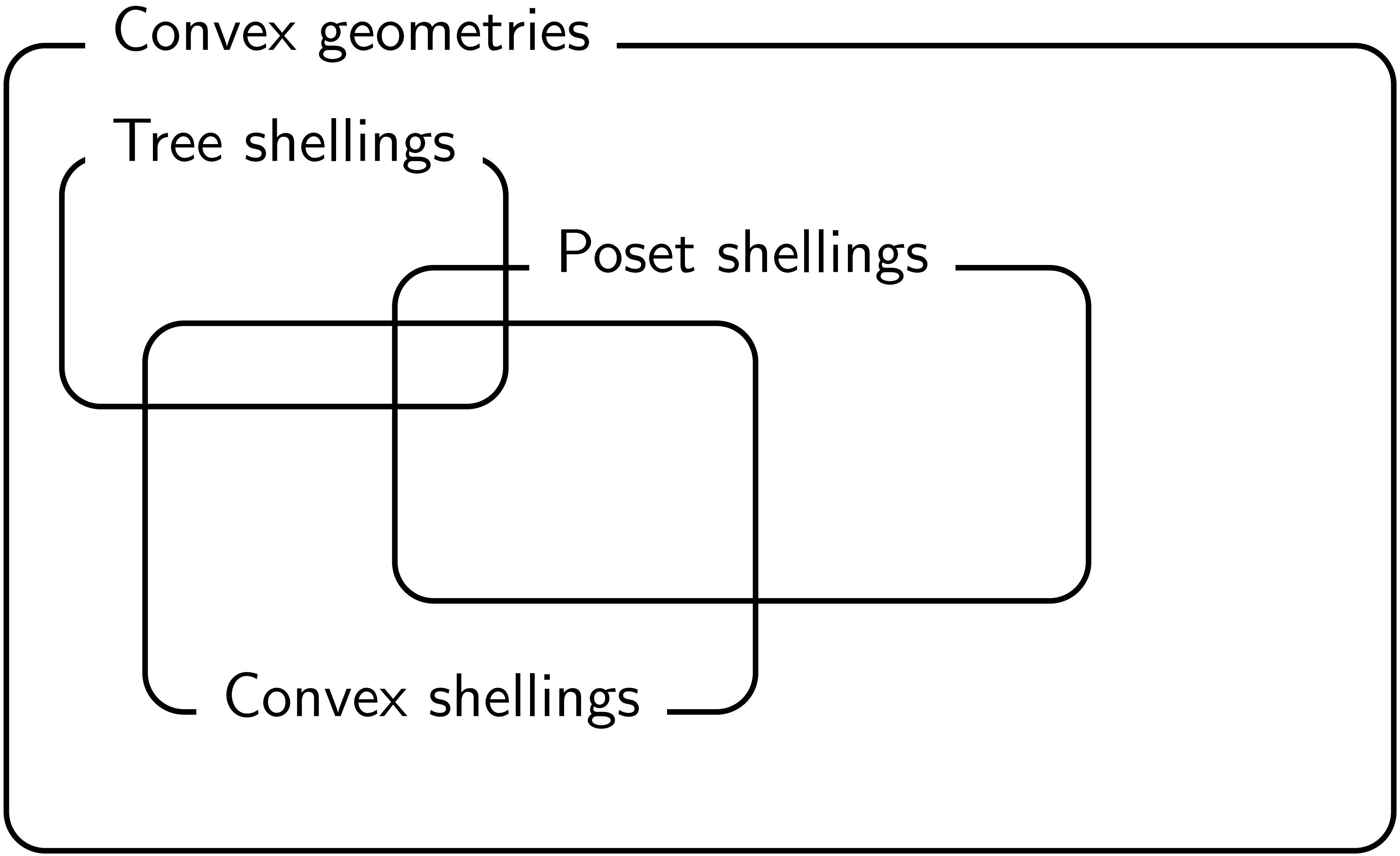
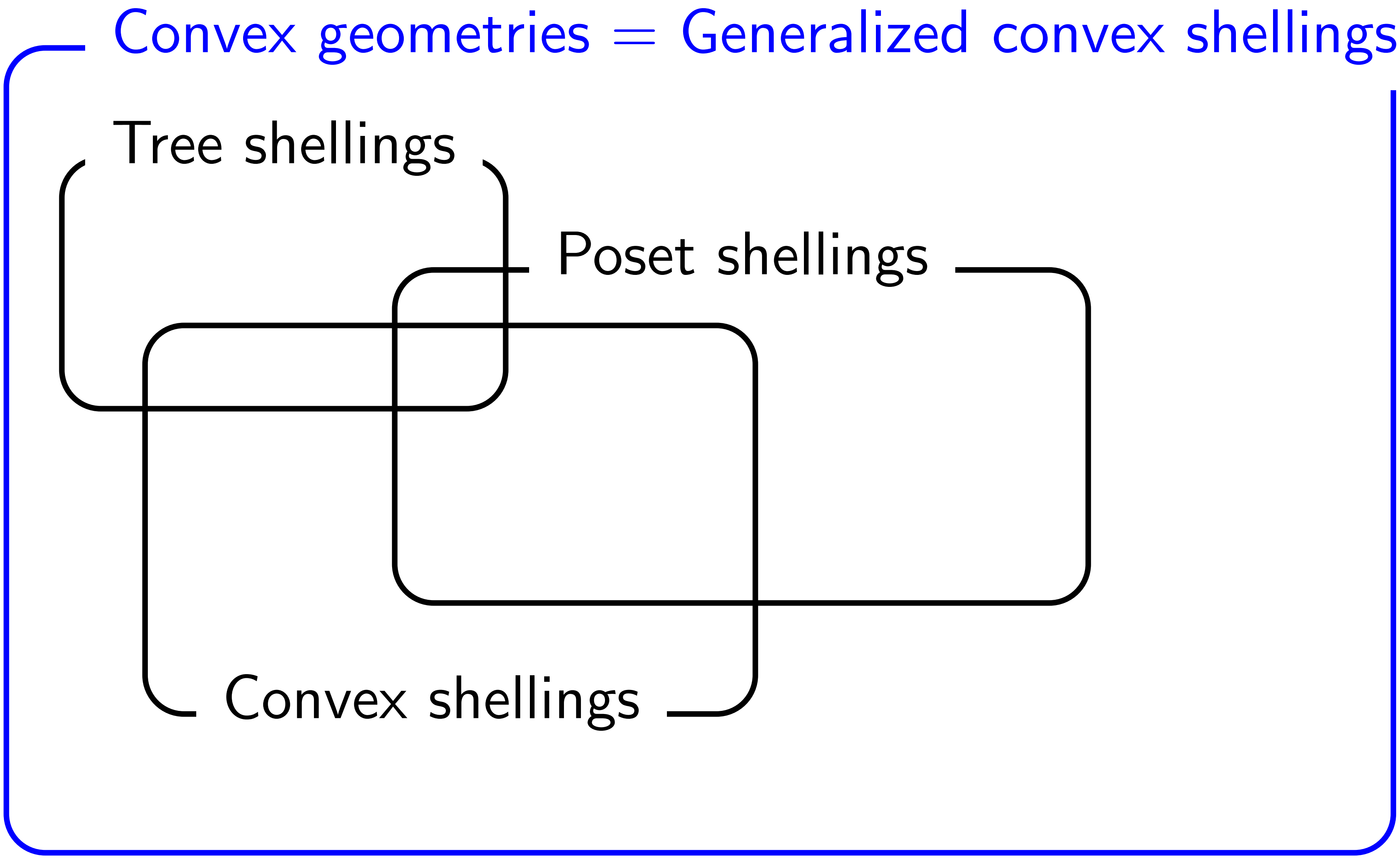
p2

What does the theorem mean? (1)




Convex geometries = Generalized convex shellings
Tree shellings
Poset shellings
Convex shellings

What does the theorem mean? (2)

- For oriented matroids and matroids, we have

Topological representation theorems.

- For convex geometries, we have

Affine representation theorem.

$\Longrightarrow$ An intrinsic simplicity of convex geometries

The proof goes along the following line.

We are given a convex geometry $\mathcal{L}$.

(1) Construct:
point sets P and Q from $\mathcal{L}$.

(2) Show:
$\mathcal{L} \ \widehat{\simeq}\ $ the generalized convex shelling on Q w.r.t. P.

To illustrate the proof, we will show a much weaker version.

What we will show

For any poset shelling $\mathcal{L}$
there exist point sets P and Q such that
$\mathcal{L}$ is isomorphic to
the generalized convex shelling on Q w.r.t. P.

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

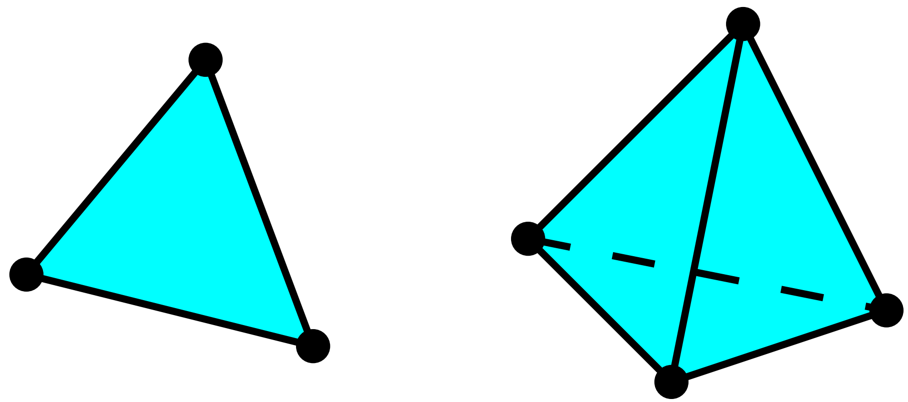
$\boxed{\text{Construction of } Q}$

We use the $(n-1)$-dimensional space $\mathbb{R}^{n-1}$.

For each $e \in E$, put a point $\mathbf{q}(e)$ such that

$\{\mathbf{q}(e) : e \in E\}$ is affinely independent,

$(\mathrm{conv}(\{\mathbf{q}(e) : e \in E\})$ is an $(n-1)$-simplex).



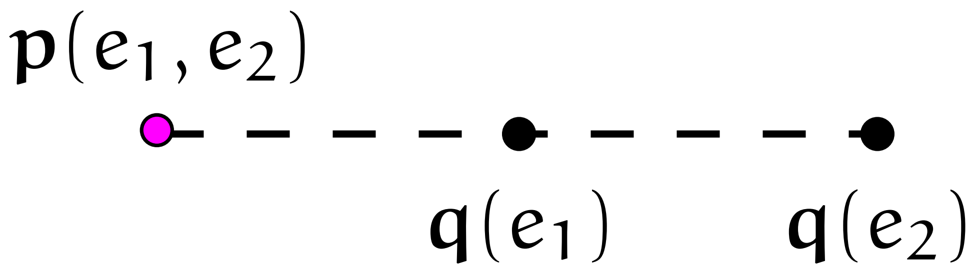
Let $Q = \{\mathbf{q}(e) : e \in E\}$.

Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

Construction of P

For each $e_1, e_2 \in E$ such that $e_1 < e_2$,

Put a point $\mathbf{p}(e_1, e_2)$ such that $\mathbf{q}(e_1) = \dfrac{\mathbf{p}(e_1, e_2) + \mathbf{q}(e_2)}{2}$.



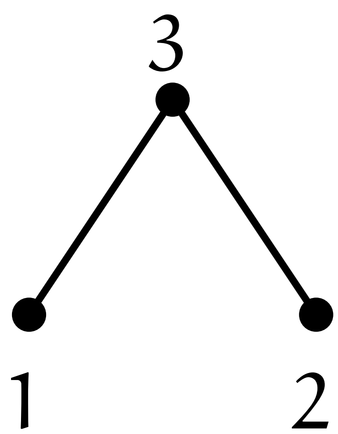
Let $P = \{\mathbf{p}(e_1, e_2) : e_1, e_2 \in E, e_1 < e_2\}$.

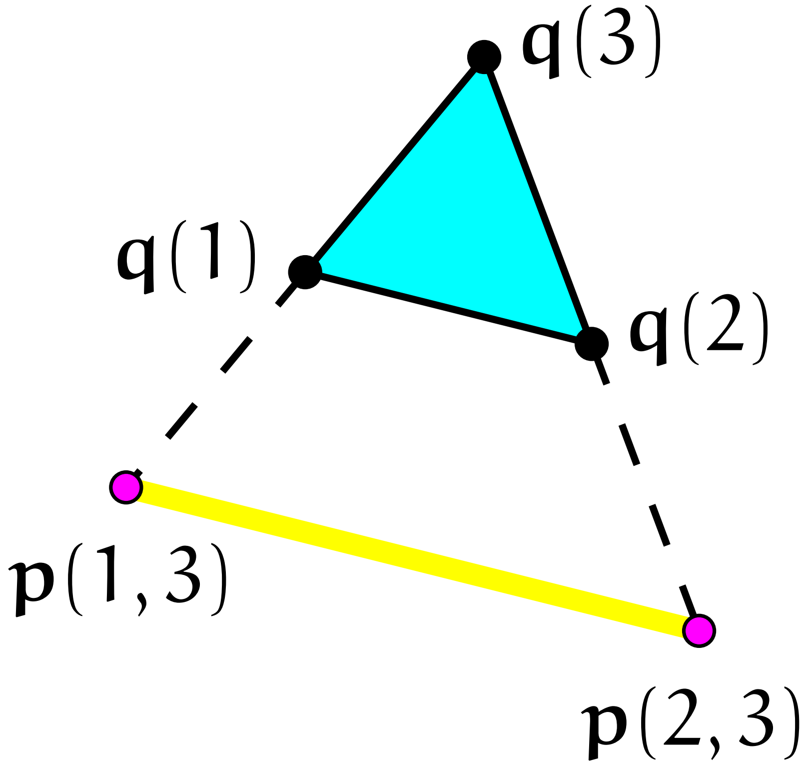
Given a partially ordered set $\mathcal{P} = (E, \leq)$. Let $n := |E|$.

$\boxed{\text{Construction of P}}$

For each $e_1, e_2 \in E$ such that $e_1 < e_2$,

Put a point $\mathbf{p}(e_1, e_2)$ such that $\mathbf{q}(e_1) = \dfrac{\mathbf{p}(e_1, e_2) + \mathbf{q}(e_2)}{2}$.





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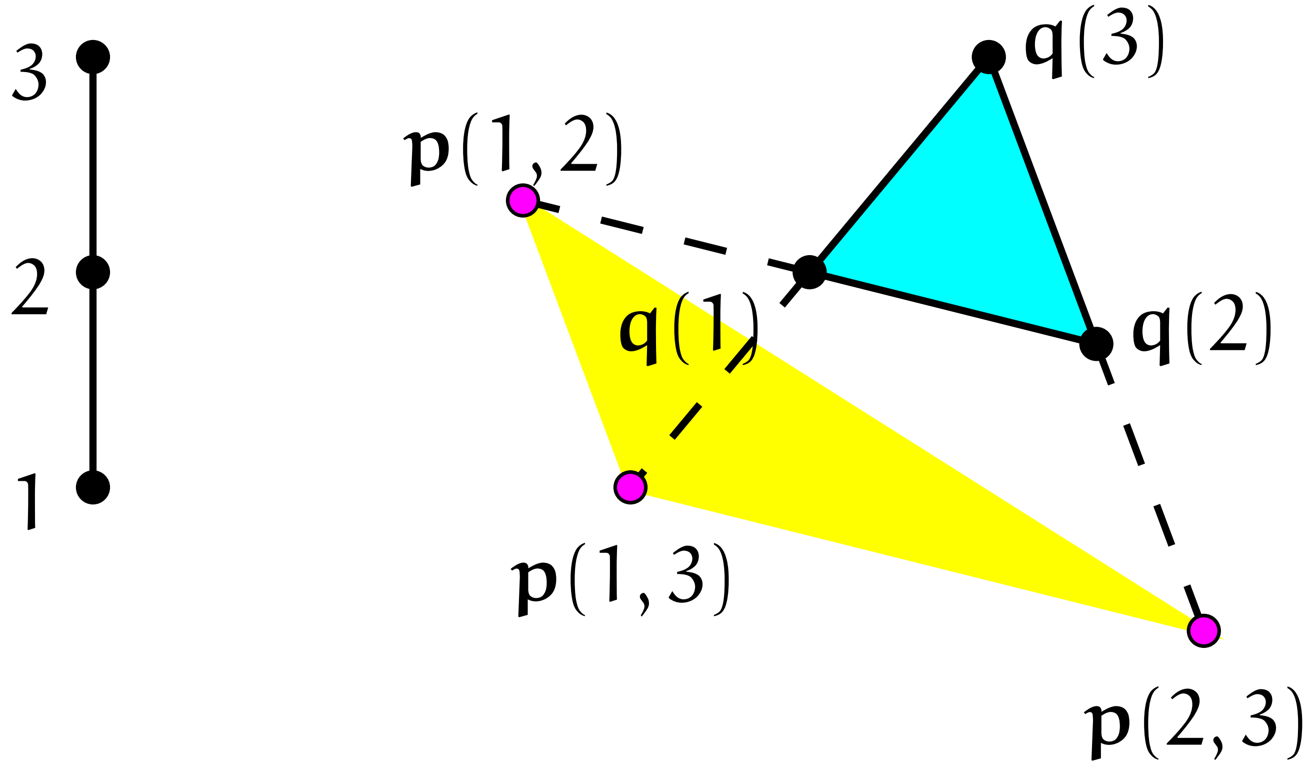

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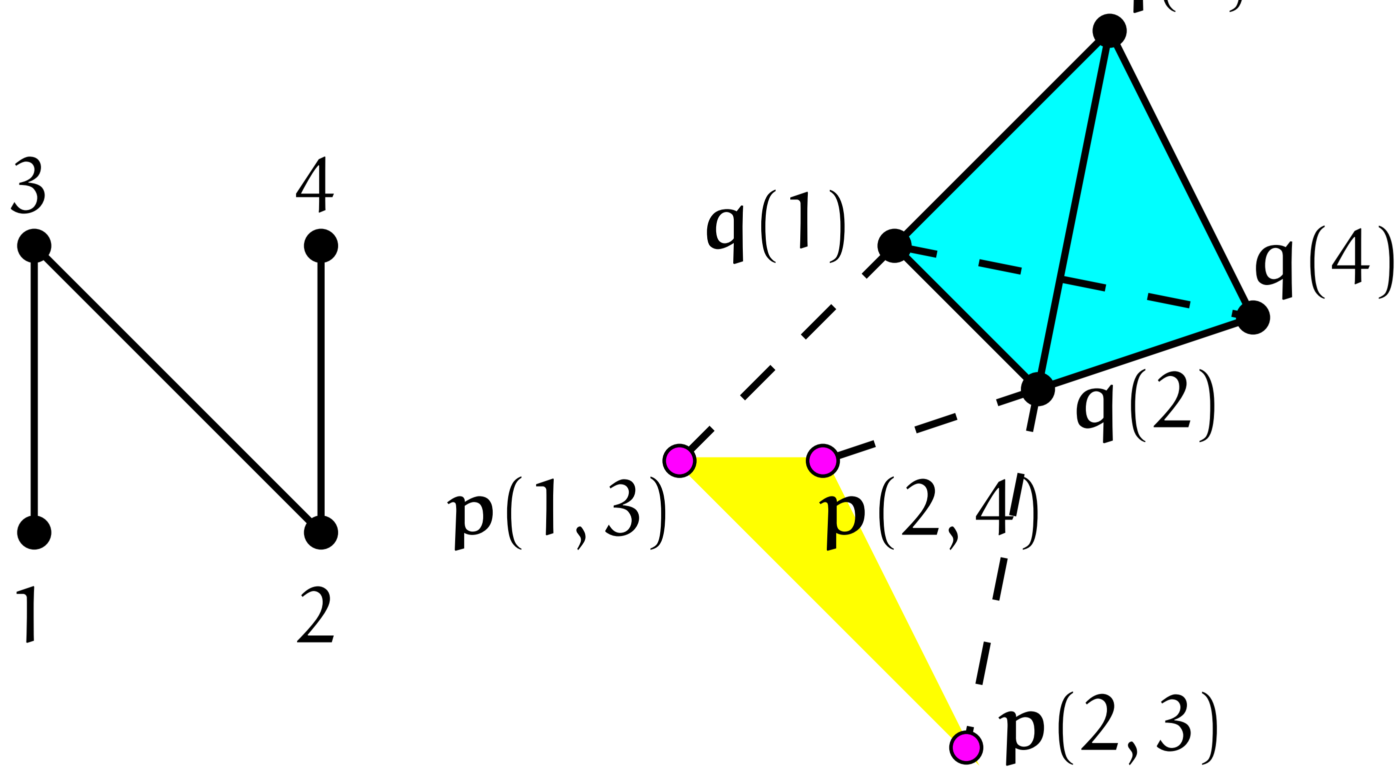


The proof goes along the following line.

We are given a convex geometry $\mathcal{L}$.

(1) Construct:
point sets P and Q from $\mathcal{L}$. . . . . DONE!

(2) Show:
$\mathcal{L} \cong$ the generalized convex shelling on Q w.r.t. P.

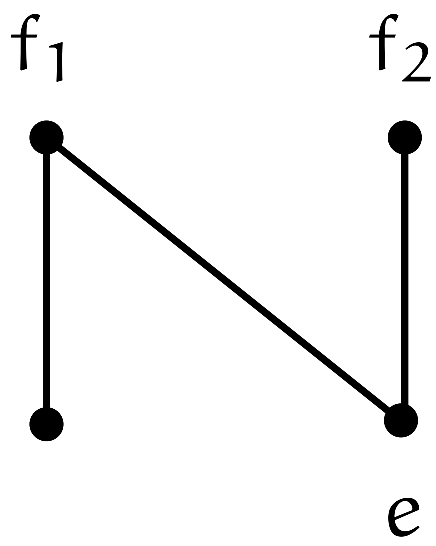
Claim the poset shelling of $\mathcal{P} = (E, \leq)$

$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in E$ allowed to be removed?

e is allowed to be removed $\Longleftrightarrow$ all f's such that $e < f$ have been already removed.



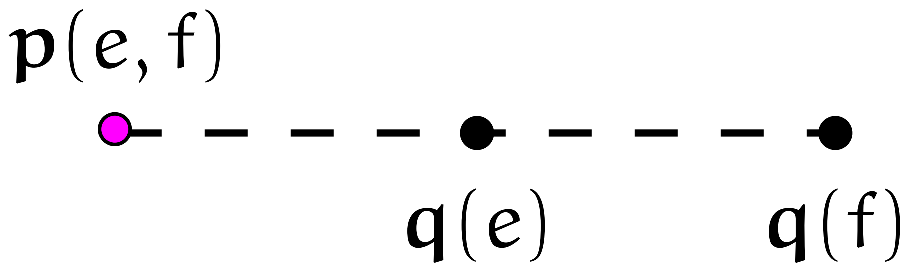
The final claim

Claim the poset shelling of $\mathcal{P} = (E, \leq)$
$\simeq$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(2) When is $\mathbf{q}(e) \in Q$ allowed to be removed?

$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(f)$'s s.t. $\mathbf{p}(e, f) \in P$ have been already removed.



"$\Longrightarrow$" is straightforward from the construction.
"$\Longleftarrow$" needs some investigations.

Claim the poset shelling of $\mathcal{P} = (E, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in E$ allowed to be removed?

e is allowed to be removed $\iff$ all f's such that $\boxed{e < f}$ have been already removed.

(2) When is $\mathbf{q}(e) \in Q$ allowed to be removed?

$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(f)$'s s.t. $\boxed{\mathbf{p}(e, f) \in P}$ have been already removed.

(3) $\boxed{e < f} \iff \boxed{\mathbf{p}(e, f) \in P}$.

Claim the poset shelling of $\mathcal{P} = (E, \leq)$
$\cong$ the generalized convex shelling on Q w.r.t. P.

Proof sketch.

(1) When is $e \in E$ allowed to be removed?

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$\mathbf{q}(e)$ is allowed to be removed $\iff$ all $\mathbf{q}(f)$'s s.t. $\boxed{\mathbf{p}(e, f) \in P}$ have been already removed.

Hence, the mapping "$e \longmapsto \mathbf{q}(e)$" is an isomorphism. [qed]

What was our theorem??

Our Theorem

Any convex geometry is isomorphic to some generalized convex shelling.

This theorem is expected to be useful for a lot of problems in convex geometries.

Further Work

How useful can it be?

[End of the talk]