Matroid Representation of Clique Complexes

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Combinatorics
Geometry
Computation

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Background

- An independence system can be represented as the intersection of finitely many matroids.
- The clique complex of a graph is an independence system.

Question

- How many matroids do we need for clique complexes?

Motivation

Given later.

V a nonempty finite set

Def.: an independence system on V is a nonempty family $\mathcal{I}$ of subsets of V s.t.

$$X \in \mathcal{I} \quad \Longrightarrow \quad Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1, 2, 3, 4\}$;

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}\}$$

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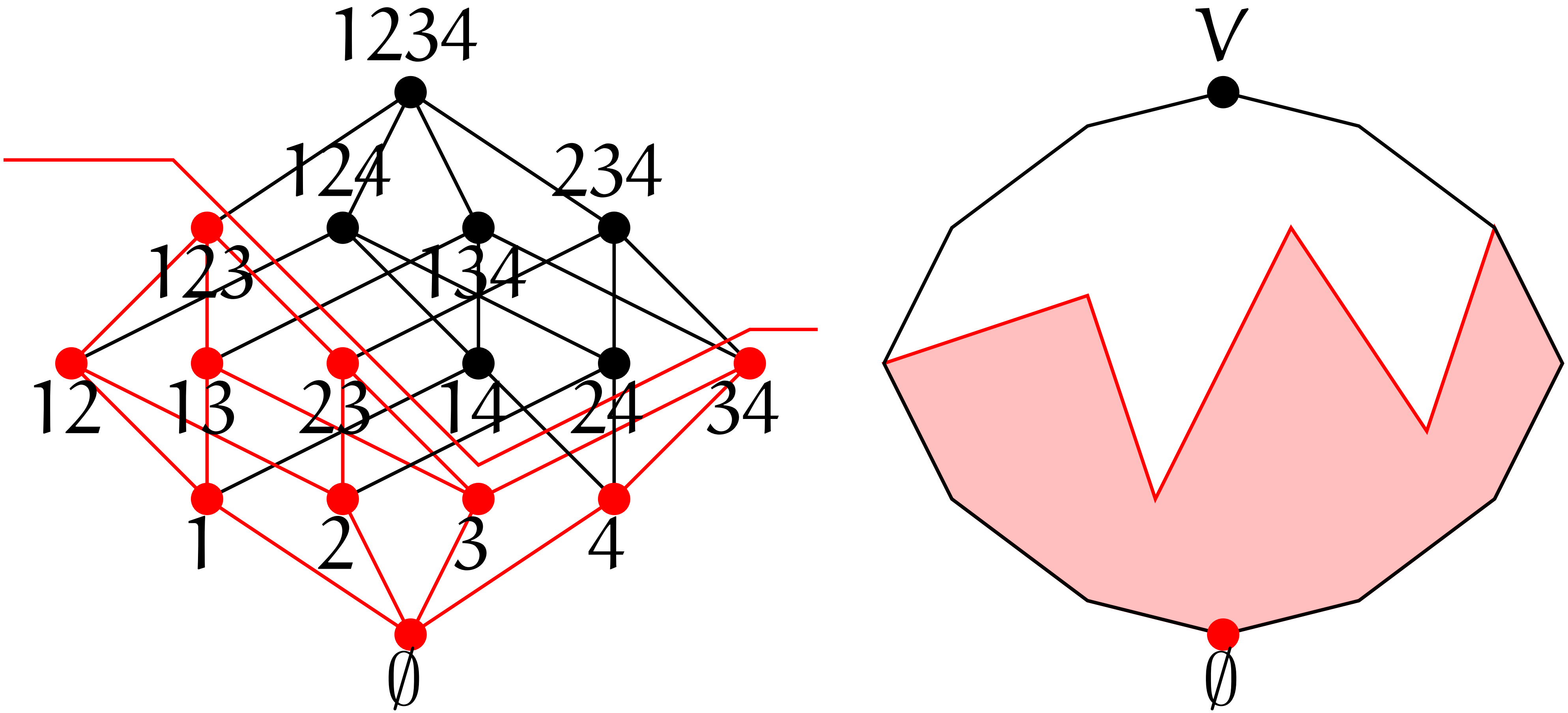
$\boxed{\text{Remark:}}$

also called an abstract simplicial complex.

Typical independence systems from ...

- From graphs
  - the family of the cliques
  - the family of the forests
  - the family of the matchings
- From partially ordered sets
  - the family of all chains
- From polytopes
  - the family of all faces of a simplicial polytope
- From game theory
  - the family of losing coalitions of a weighted majority game

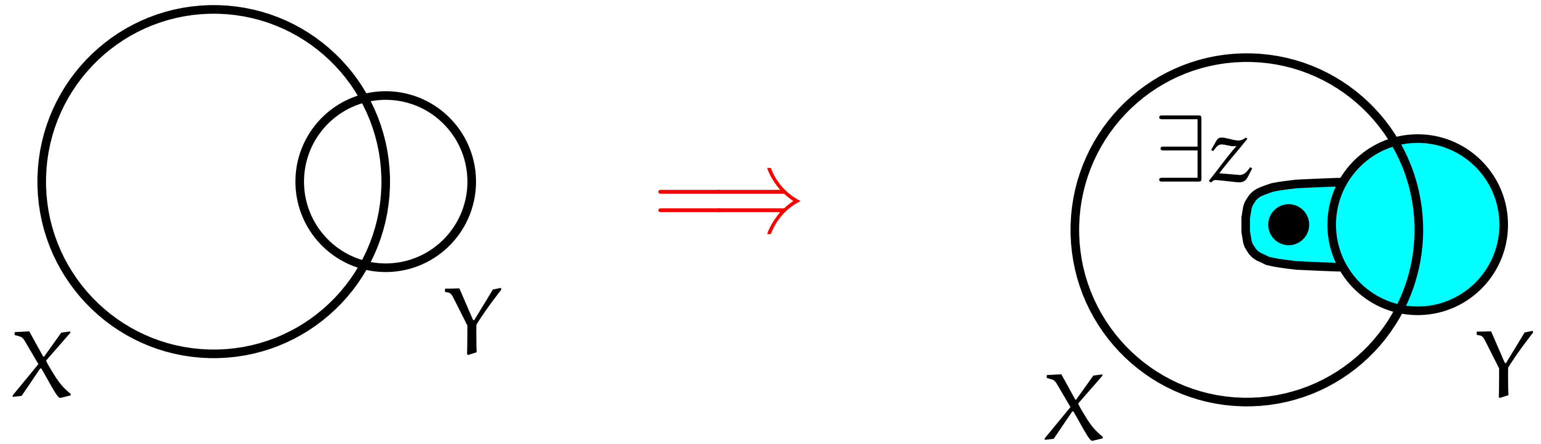
$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 34, 123\}$.



(known as the face poset of $\mathcal{I}$,
when we see $\mathcal{I}$ as an abstract simplicial complex)

Def.: An indep. system $\mathcal{I}$ is called a matroid if $\mathcal{I}$ satisfies the augmentation axiom:

$$X, Y \in \mathcal{I}, |X| > |Y| \Longrightarrow \exists z \in X \setminus Y \text{ s.t. } \underline{\underline{Y \cup \{z\}}} \in \mathcal{I}.$$



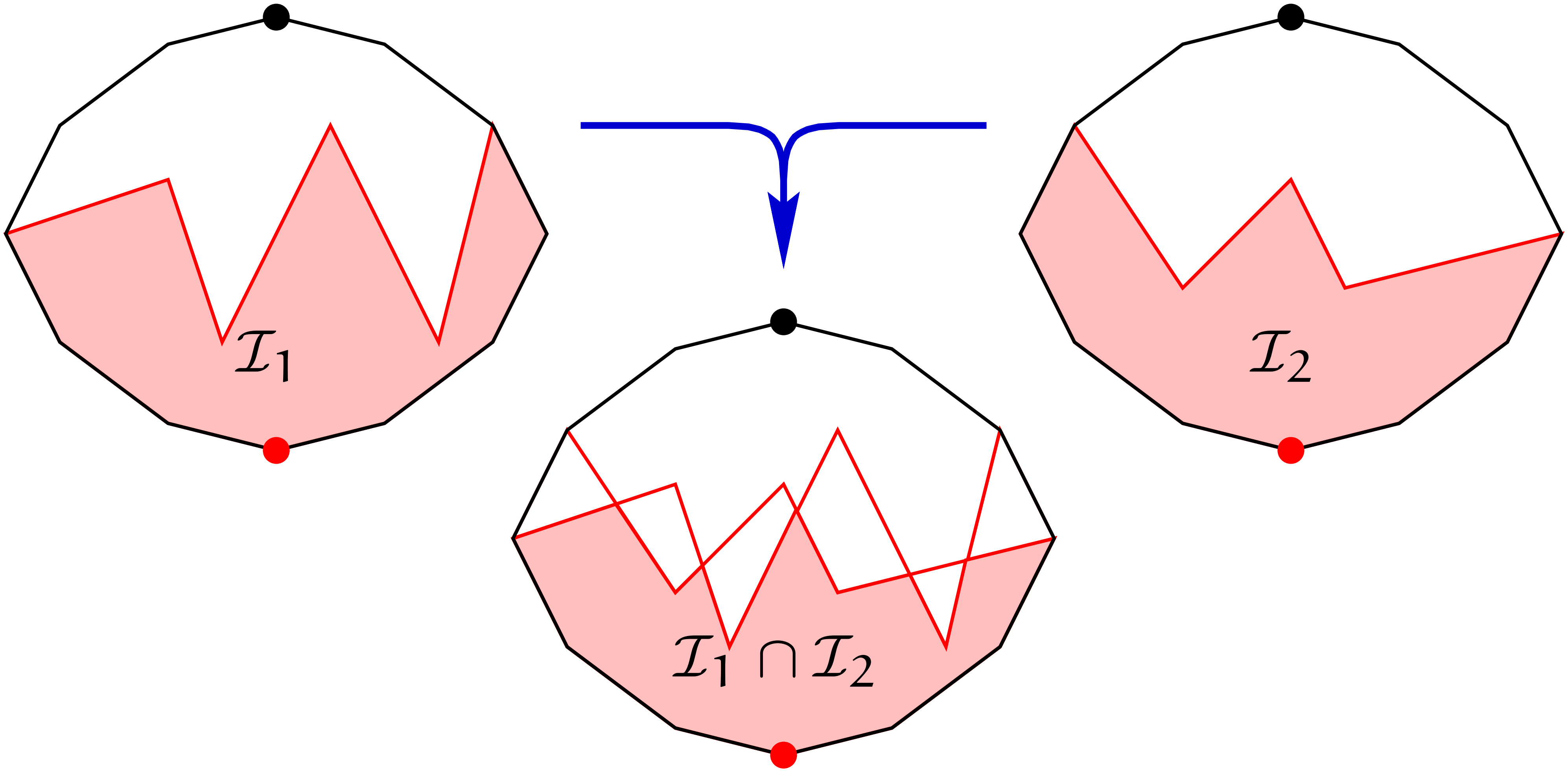
Appearance of matroids:

Combinatorial optimization, Combinatorial geometry, Combinatorial topology, Coding theory, Design theory, etc.

Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$



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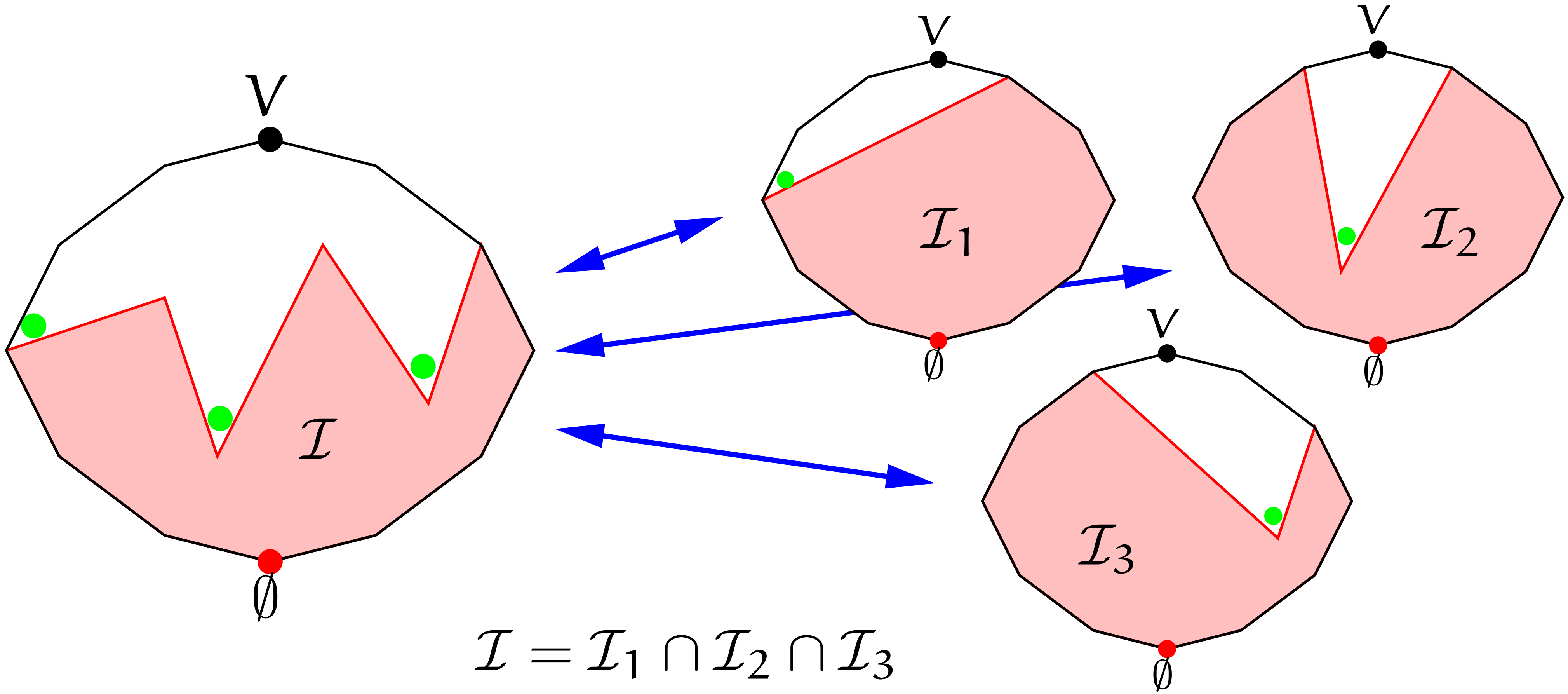
- The intersection of more indep. systems is defined analogously.
- The intersection of indep. systems is again an indep. system.
- The intersection of matroids is
  an indep. system
  but not necessarily a matroid.
  However ...

Fact: Every indep. system is the intersection of finitely many matroids.

Proof. Min. sets not in $\mathcal{I}$ are shown by ●.



$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$$

Question:

How many matroids do we need to represent a given indep. system as their intersection??

. . . .

The proof in the prev. page says that:

this number $\leq$ the number of $\bullet$ for $\mathcal{I}$.

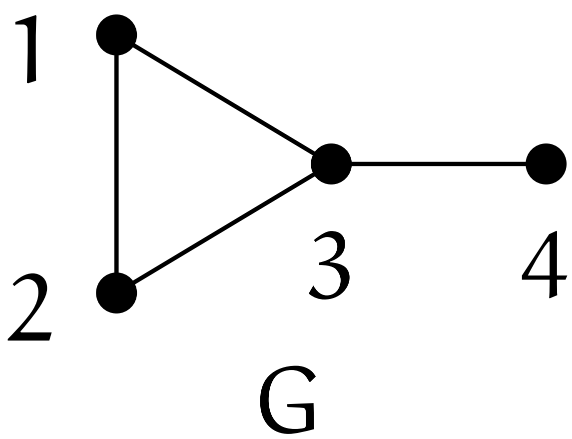
But, we might do better.

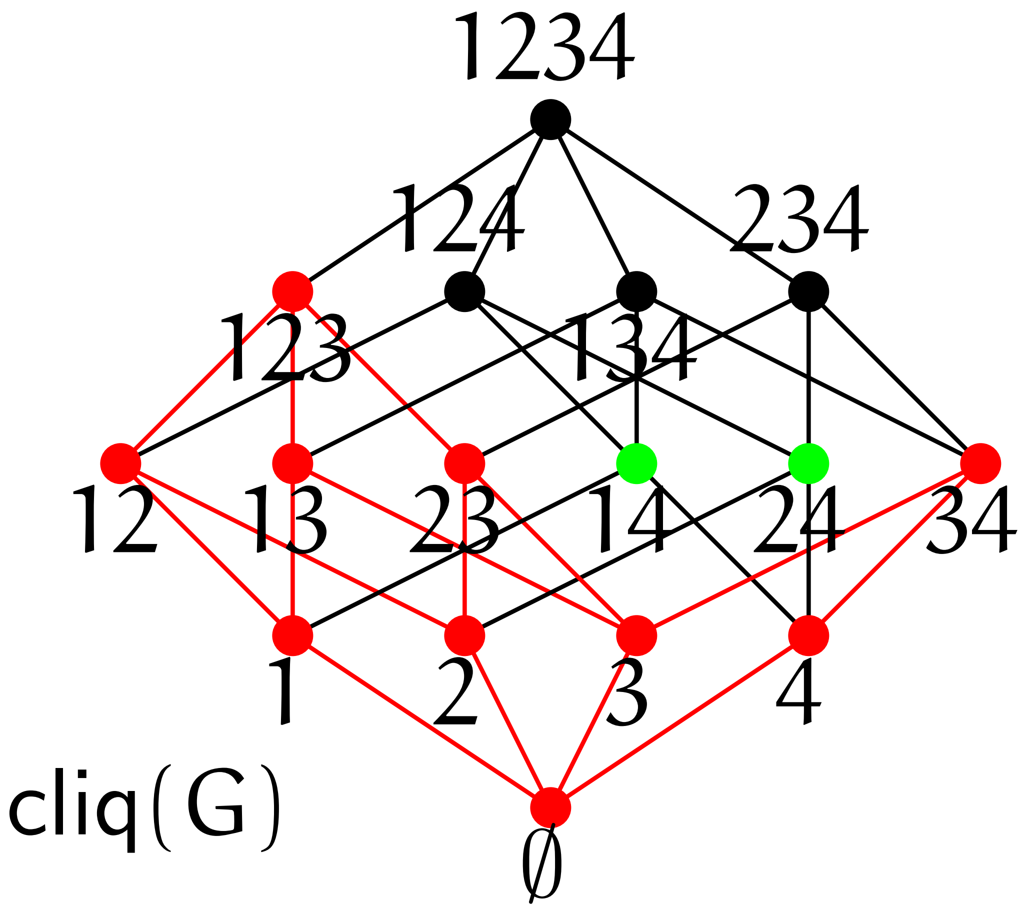
★ We study this number for clique complexes.

Def.: The clique complex of a graph $G = (V, E)$ is the family of all cliques of G.

Example:

$V = \{1, 2, 3, 4\}$





Question:

How many matroids do we need to represent a clique complex as their intersection??

Result:

We investigate and characterize these numbers.

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represent a clique complex as their intersection??

$\boxed{\text{Result:}}$

We investigate and characterize these numbers.

However, you may ask...

(1) Why do we consider such a representation??

(2) Why do we consider clique complexes??

Q. Why do we consider such a representation?

Ans. in the following proposition.

$\boxed{\text{Q.}}$ Why do we consider such a representation?

$\boxed{\text{Ans.}}$ in the following proposition.

$\boxed{\text{Prop.:}}$ (Jenkyns, '76; Korte–Hausmann, '78)

(Translation of their prop. to clique complexes)

cliq(G) is the intersection of k matroids
$\Longrightarrow$ Greedy Alg. is a k-approx. alg. for
Max weighted clique problem in G.

★ $k \approx$ how complex G is w.r.t. clique problem.

Motivation (2)

Q. Why do we consider clique complexes?

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Q. Why do we consider clique complexes?

Ans. The class of clique complexes includes some important classes like the classes of

- the family of the matchings of a graph G
  $=$ cliq(the complement of the line graph of G)
  studied by Fekete–Firla–Spille '03
  in the same framework as ours.
- the family of the chains of a poset P
  $=$ cliq(the comparability graph of P)
- $\ldots$

In the rest of my talk

(1) clique complex
which is a matroid

- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids

- Key concept: stable-set partition

(3) other results

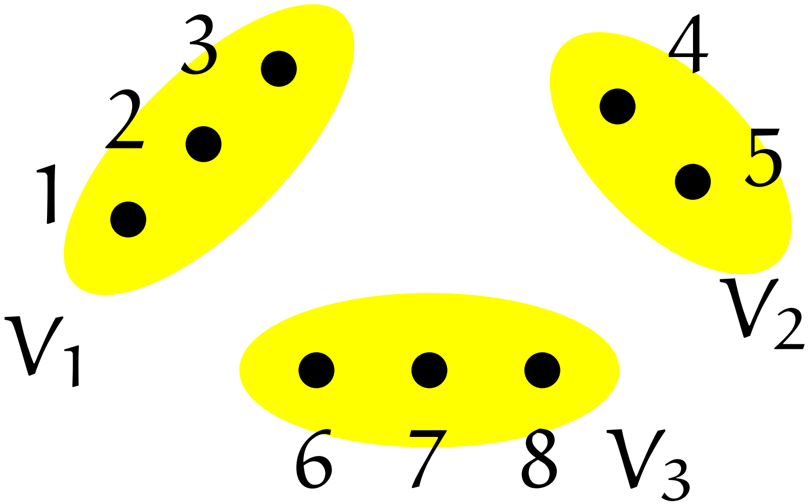
Def.: Partition matroid

V a nonempty finite set
$\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ a partition of V

Def.: A partition matroid $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is

$$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \ldots, r\}.$$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$



$\mathcal{I}(\mathcal{P}) =$
$\{146, 147, 148, 156, 157, 158,$
$246, 247, 248, 256, 257, 258,$
$346, 347, 348, 356, 357, 358,$
and their subsets$\}$

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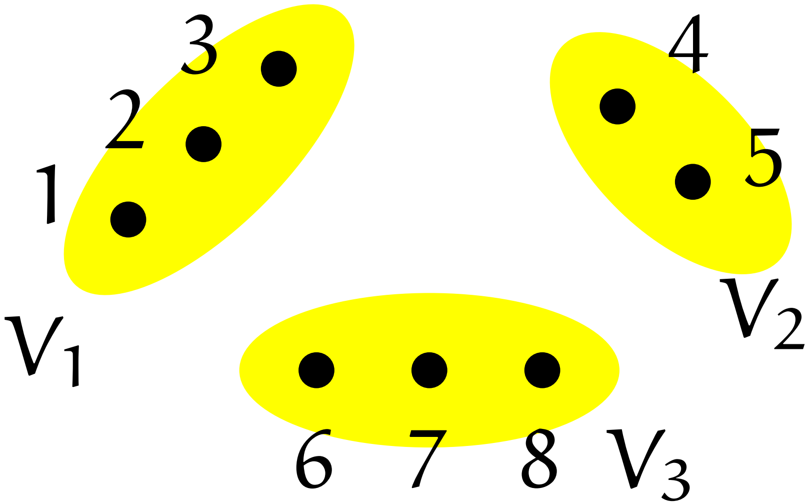
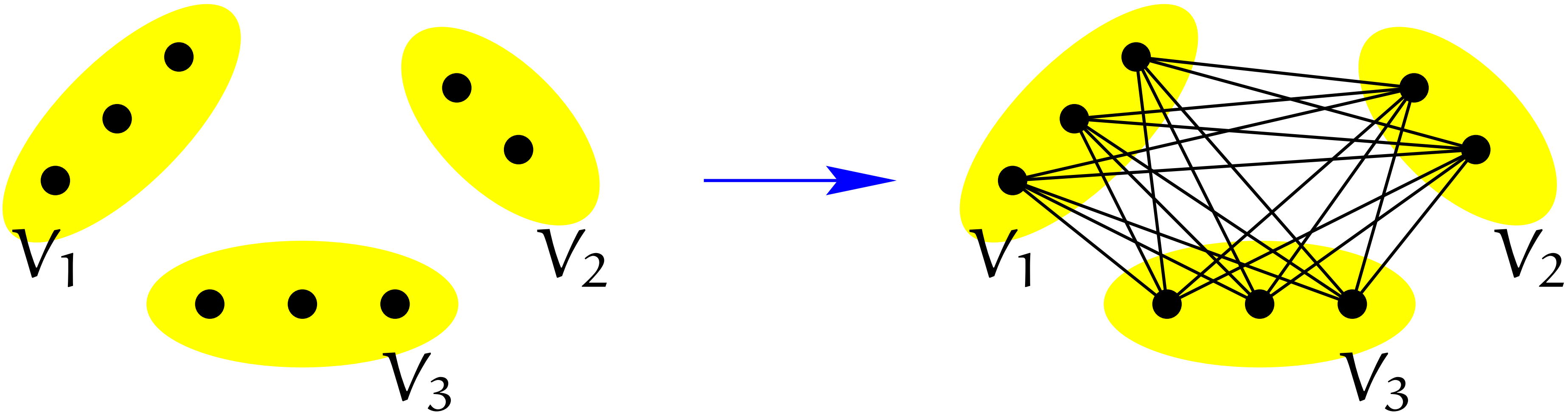


Notice: In fact, $\mathcal{I}(\mathcal{P})$ is a matroid.

Partition matroids are clique complexes

Lem.: Partition matroids are clique complexes.

Proof. Given a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V.



Construct a complete r-partite graph $G_{\mathcal{P}}$ s.t. each part of $G_{\mathcal{P}}$ corresponds to V_i.

Then, $\mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$. [qed]

Remark: G a graph

$$\text{G is complete multipartite} \\ \Downarrow \\ \text{cliq(G) is a matroid.}$$

Thm 1: G a graph

$$\text{G is complete multipartite}$$
$$\Updownarrow$$
$$\text{cliq}(G) \text{ is a matroid.}$$

Proof. Omitted.

In the rest of my talk

(1) clique complex
which is a matroid

- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids

- Key concept: stable-set partition

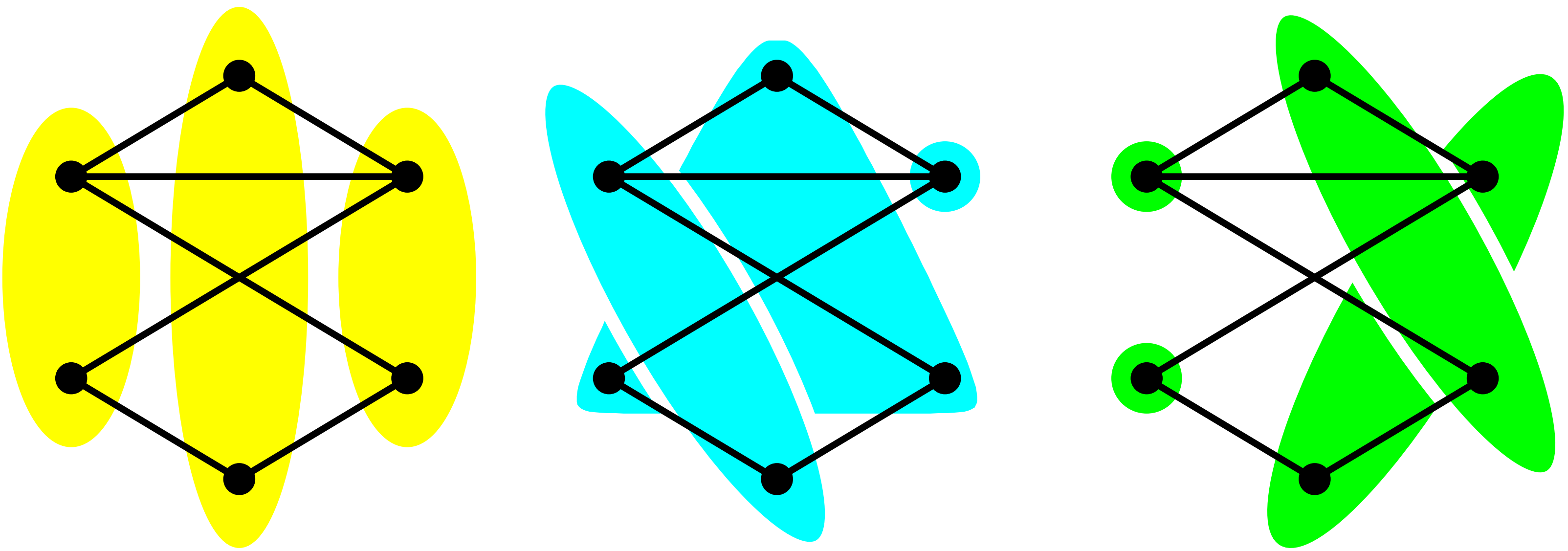
(3) other results

Def.: Stable-set partitions

$G = (V, E)$ a graph

Def.: A stable-set partition of G is a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of V s.t. V_i is a stable set of G $(i = 1, \ldots, r)$.

Example:



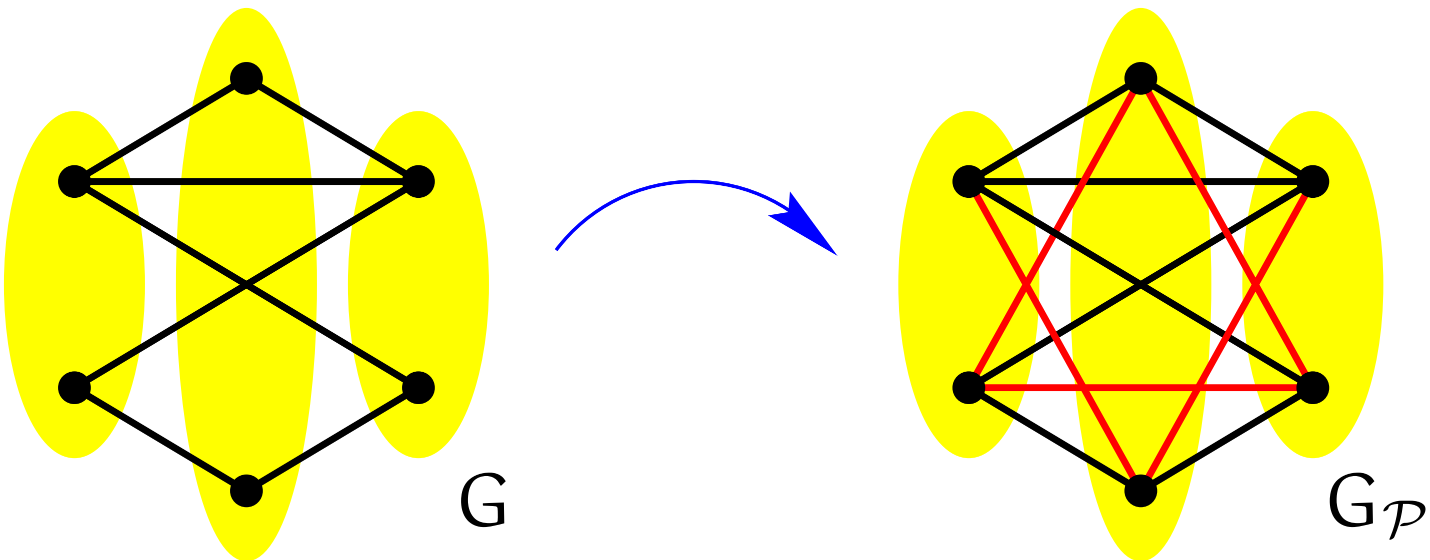
Construct a graph from a stable-set partition

G a graph
$\mathcal{P}$ a stable-set partition of G

Construct a complete r-partite graph $G_{\mathcal{P}}$ from $\mathcal{P}$.

$\boxed{\text{Observation:}}$ An edge of G is an edge of $G_{\mathcal{P}}$.

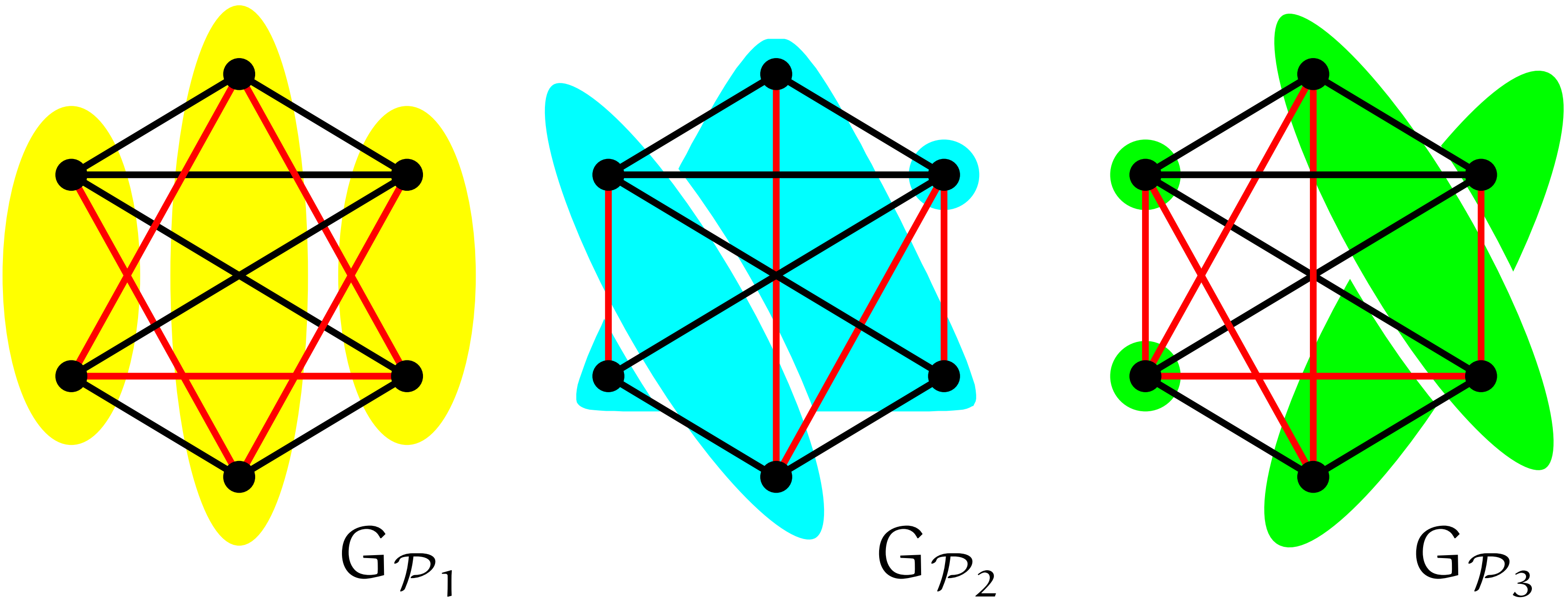
This means: $\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.



Three stable-set partitions

Generally,
$\mathrm{cliq}(G) \subseteq \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$.

However, in this example,



$$\mathrm{cliq}(G) = \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$$

That is because

Three stable-set partitions

Generally,

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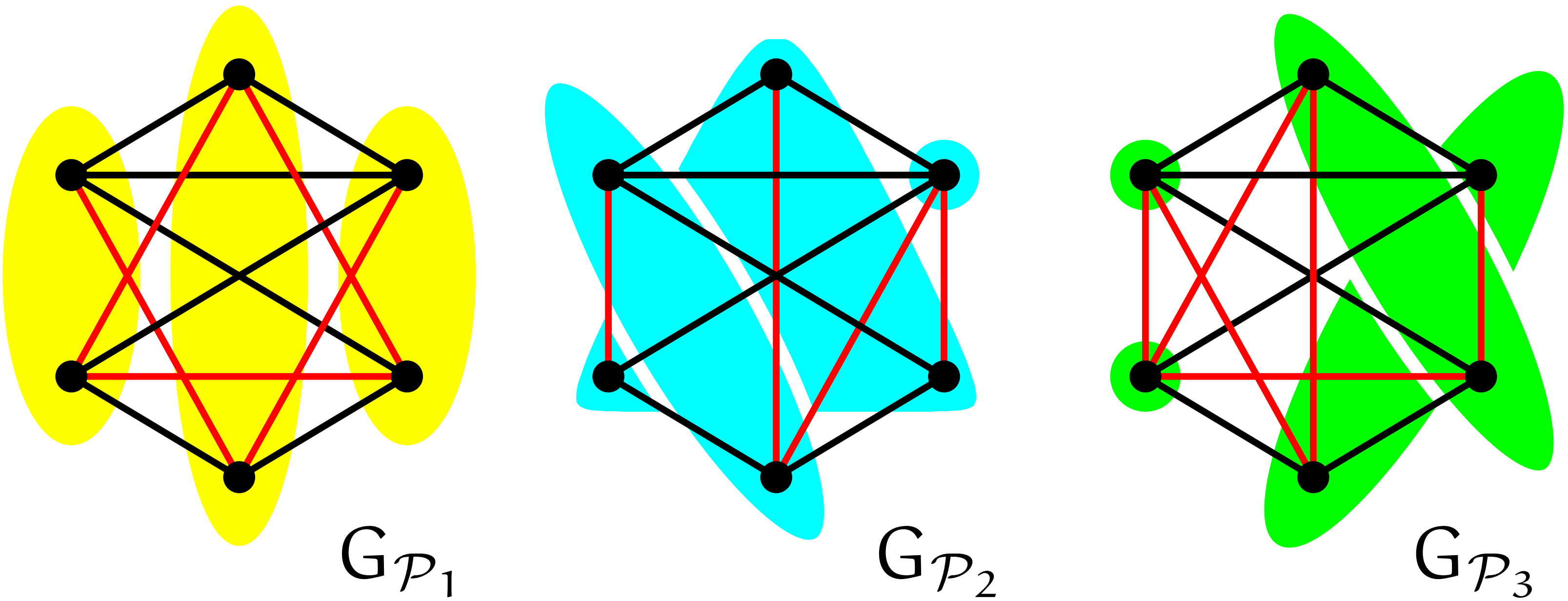


$$\mathrm{cliq}(G) = \mathrm{cliq}(G_{\mathcal{P}_1}) \cap \mathrm{cliq}(G_{\mathcal{P}_2}) \cap \mathrm{cliq}(G_{\mathcal{P}_3})$$

No red edge is contained in all of these graphs.

$G = (V, E)$ a graph

$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ stable-set partitions of G

$$\mathrm{cliq}(G) = \bigcap \{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$$

$$\Updownarrow$$

No red edge is contained in all $G_{\mathcal{P}_i}$'s.

In other words...

Remark: $G = (V, E)$ a graph

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. $\mathrm{cliq}(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

$$\Updownarrow$$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t. no red edge is contained in all $G_{\mathcal{P}_i}$'s.

Thm 2: $G = (V, E)$ a graph

cliq(G) is the intersection of k matroids

$$\Updownarrow$$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t.
cliq$(G) = \bigcap\{\mathcal{I}(\mathcal{P}_i) : i = 1, \ldots, k\}$

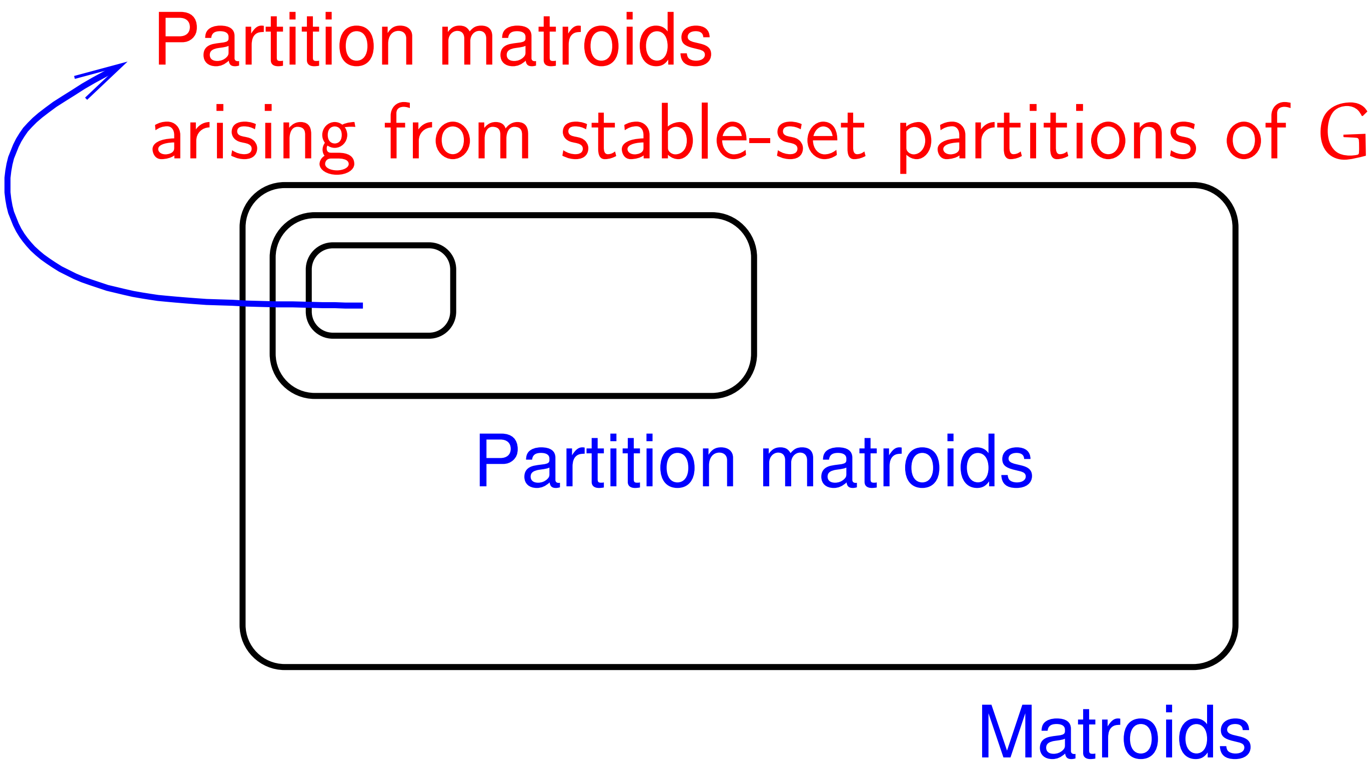
$$\Updownarrow$$

$\exists$ k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of G s.t.
no red edge is contained in all $G_{\mathcal{P}_i}$'s.

Proof. Omitted.

What does this theorem mean? (1)

Given a graph G.



When we want a representation of $\mathrm{cliq}(G)$ by k matroids, just search k partition matroids from stable-set partitions of G.

What does this theorem mean? (2)

Consider the following decision problem.

Instance: a graph G and a positive integer k

Question: ?? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$

s.t. $\text{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$??

Thm 2 implies: This problem $\in \mathcal{NP}$.

In the rest of my talk

(1) clique complex
which is a matroid

- Key concept: partition matroid

(2) clique complex
which is the intersection of k matroids

- Key concept: stable-set partition

(3) other results
(five more pages!!!)

$\boxed{\text{Thm 3:}}$ G a graph

cliq(G) is the intersection of two matroids

$$\Updownarrow$$

The stable-set graph of G is bipartite.

Clique complexes represented by two matroids

Thm 3: G a graph

cliq(G) is the intersection of two matroids

$\Updownarrow$

The stable-set graph of G is bipartite.

Note: Combining $\begin{cases} \text{Theorem 3 and} \\ \text{Protti–Szwarcfiter ('02),} \end{cases}$

we can tell, in polynomial time, whether

the clique complex of a given graph
is the intersection of two matroids or not.

Consider the following decision problem.

Fix: a positive integer k

Instance: a graph G

Question: ? $\exists$ k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$

s.t. $\mathrm{cliq}(G) = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$?

What is the complexity of this problem for each k?

k	1	2	3	4	$\cdots$
	$\mathcal{P}$	$\mathcal{P}$	?	?	$\cdots$

A graph can be seen as an independence system:

$$X \in G \quad \Leftrightarrow \quad X = \begin{cases} \emptyset & \text{or} \\ \{v\} & \text{or} \\ \{u, v\} & \text{if it's an edge.} \end{cases}$$

$\boxed{\text{Thm 4:}}$ G a graph

G is the intersection of k matroids

$$\Updownarrow$$

cliq(G) is the intersection of k matroids.

Number of matroids we need

Let $k(G) = \min\{k : \text{cliq}(G) \text{ is the intersection of } k \text{ matroids}\}.$

$k(n) = \max\{k(G) : G \text{ has } n \text{ vertices}\}.$

$\boxed{\text{Thm 5:}}$ $k(n) = n - 1.$

Question

- How many matroids do we need to represent a given clique complex as their intersection??

Result

- cliq(G) is the intersection of k matroids

$$\Updownarrow$$

cliq(G) is the intersection of k partition matroids from stable-set partitions of G

[End of the talk]