Traveling Salesman Games with the Monge Property

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My interest

The relationship between

combinatorial optimization	$\leftrightarrow$	cooperative games
Shapley–Shubik, '72		
assignment problems (well-solvable)		assignment games (with nonempty core)
Deng–Nagamochi–Ibaraki, '99		
min. coloring problems (intractable)		min. coloring games (testing core-nonemptiness hard)

Contributions to traveling salesman games (TSG)

- It is $\mathcal{NP}$-hard to test the core non-emptiness of a TSG.
- The core of a TSG with the Monge property is always non-empty.

Traveling salesman problem (TSP)

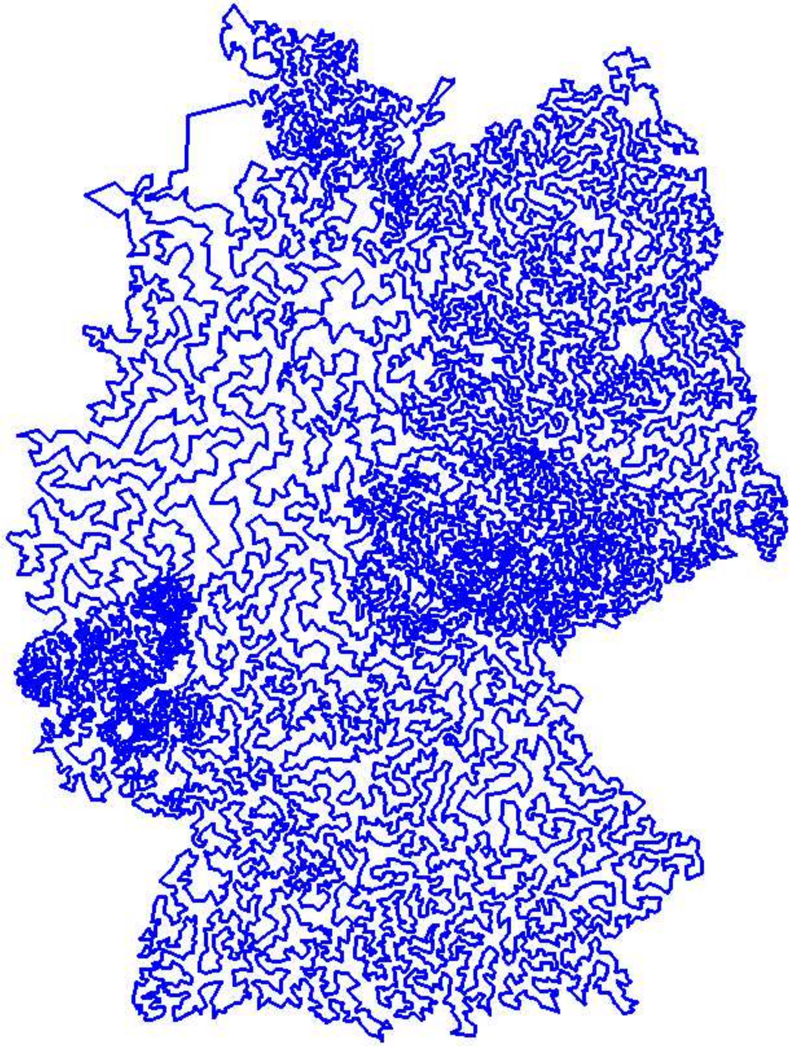
One of the most famous $\mathcal{NP}$-hard problems.

A salesman wants to travel along a tour of minimum cost.

$\Downarrow$

Find such a tour!

15,112 cities in Germany $\longrightarrow$



An optimal tour by Applegate–Bixby–Chvátal–Cook ('01)

TSG is a kind of cost allocation problems

- Asian countries invite a football team from Europe.
- The invited team visits each country.

⇒ How do these countries pay for the travel cost?

We consider how to allocate the total cost to each country.

Def: Traveling salesman game

Potters–Curiel–Tijs ('92)

$N = \{0, \dots, n\}$	Countries
$h \in N$	fixed (Home of an invited team)
C	$N \times N$ cost matrix
$N_h := N \setminus \{h\}$	players (Asian countries)
$S \subseteq N_h$	coalition
$v_C : 2^{N_h} \to \mathbb{R}$	characteristic function

$v_C(S) =$ the cost of a shortest tour of $S \cup \{h\}$

$=$ the optimal value of TSP on $S \cup \{h\}$

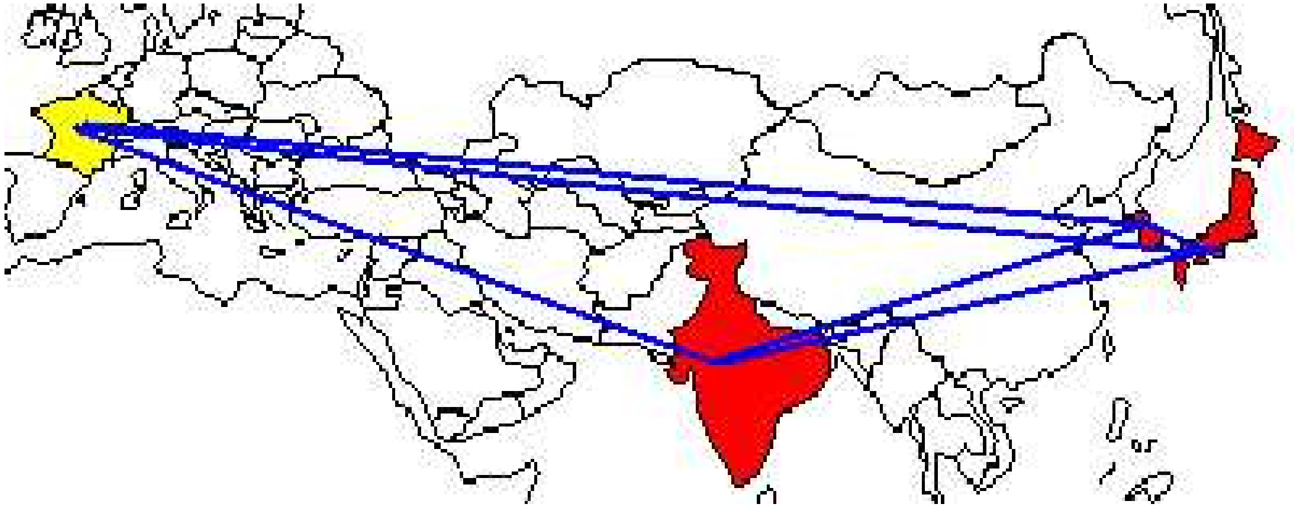
(determined by C)

(N_h, v_C)	traveling salesman game (TSG)

Goal: To allocate "fairly" the cost to each player

Example: Traveling salesman game

$N = \{\text{France}, \text{India}, \text{Japan}, \text{Korea}\}$, $h = \text{France}$



$$
\begin{array}{llll}
v_C(\emptyset) = & 0 & v_C(\{\mathsf{I},\mathsf{J}\}) = & 23 \\
v_C(\{\mathsf{I}\}) = & 14 & v_C(\{\mathsf{I},\mathsf{K}\}) = & 21 \\
v_C(\{\mathsf{J}\}) = & 20 & v_C(\{\mathsf{J},\mathsf{K}\}) = & 20 \\
v_C(\{\mathsf{K}\}) = & 18 & v_C(\{\mathsf{I},\mathsf{J},\mathsf{K}\}) = & 23 \ \ (\times 10^3\text{km})
\end{array}
$$

We want to allocate fairly $v_C(\{\mathsf{I},\mathsf{J},\mathsf{K}\}) = 23$ to $\{\mathsf{I},\mathsf{J},\mathsf{K}\}$.

Assumptions on the matrix

Some assumptions on the cost matrix C

- The diagonal entries of C are all 0.
- The non-diagonal entries of C are all positive.
- C is symmetric.

Remark that

★ C may not satisfy the triangle inequality.

Def: Core

$x \in \mathbb{R}^{N_h}$ is a core allocation of (N_h, v_C) if

$$\sum\{x[i] : i \in N_h\} = v_C(N_h),$$

(x is an allocation,)

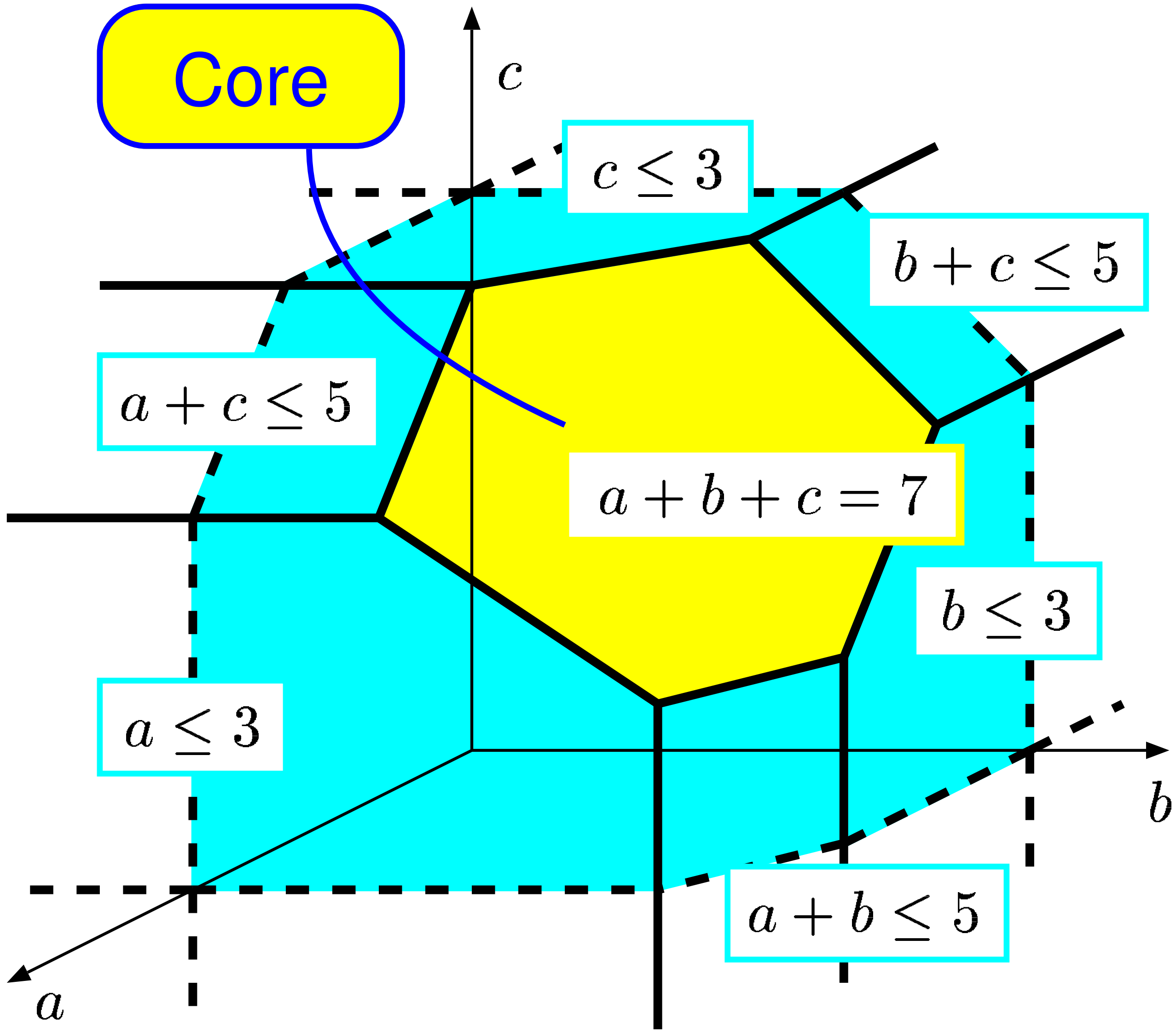
$$\sum\{x[i] : i \in S\} \leq v_C(S) \qquad \text{for all } S \subseteq N_h,$$

(each coalition $S \subseteq N_h$ is satisfied with x.)

- A core allocation represents a “fair” allocation.

The core of (N_h, v_C) is the set of all core allocations.

Example: Core



$N_h = \{a, b, c\}$

$v_C(\emptyset)$	0
$v_C(\{a\})$	3
$v_C(\{b\})$	3
$v_C(\{c\})$	3
$v_C(\{a, b\})$	5
$v_C(\{a, c\})$	5
$v_C(\{b, c\})$	5
$v_C(\{a, b, c\})$	7

[Remark] The core is possibly empty.

$\Longrightarrow$ There may not exist a core allocation!!

It's important to test the core non-emptiness!!

Is it easy?

$\Longrightarrow$ We consider the complexity of this problem.

The formal description of the problem:

Problem:

CORE NON-EMPTINESS OF TSG

Instance:

N	countries
$h \in N$	home
C	$N \times N$ symm. cost matrix

Question:

Is the core of the TSG (N_h, v_C) non-empty?

Theorem

CORE NON-EMPTINESS OF TSG is $\mathcal{NP}$-hard.

Idea of Proof

Reduction from HAMILTONIAN PATH PROBLEM.

This result implies that

- we're unlikely to have a polynomial time algorithm for CORE NON-EMPTINESS OF TSG.
- we're unlikely to have a good characterization of core non-emptiness of TSG.

In the literature

The core is non-empty

when C satisfies the triangle inequality and

- TSG $=$ a routing game (Potters, '89)
- $n \leq 4$ (Tamir, '89)
- $n = 5$ (Kuipers, '93)

We show

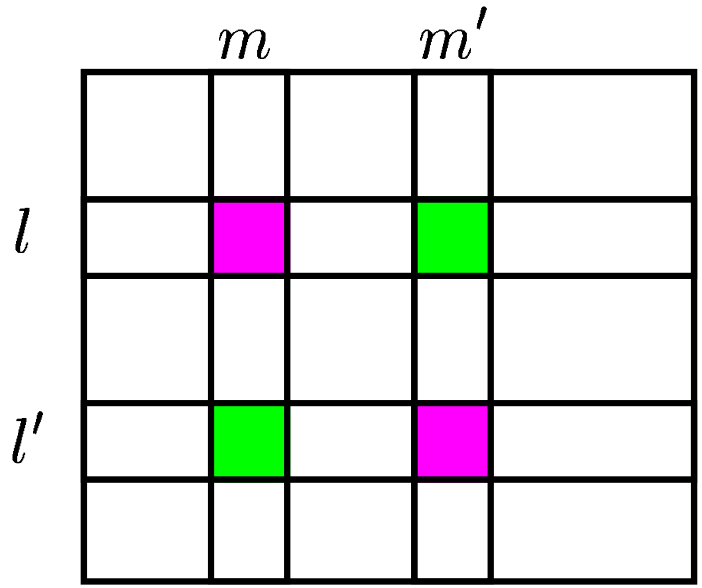
another condition based on the Monge property.

Def: Monge matrix

C is a Monge matrix if C satisfies

$$\boxed{c[l,m]+c[l',m']} \leq \boxed{c[l,m']+c[l',m]}$$

for all $l < l'$ and $m < m'$.



Fact TSP with Monge matrices can be solved in poly time.

(Klyaus, '76; Gilmore–Lawler–Shmoys, '85)

Theorem

TSG with a symmetric Monge cost matrix has the non-empty core.

This result is important since

- it's independent from the number of the players,
- it suggests the relationship between
  - the core non-emptiness of games
  - well-solved cases for optimization.

How about other well solvable cases??

C: a symmetric matrix

- symm. Monge matrices

  $$c[l,m] + c[l',m'] \le c[l,m'] + c[l',m] \qquad (l < l',\ m < m')$$

- Kalmanson matrices

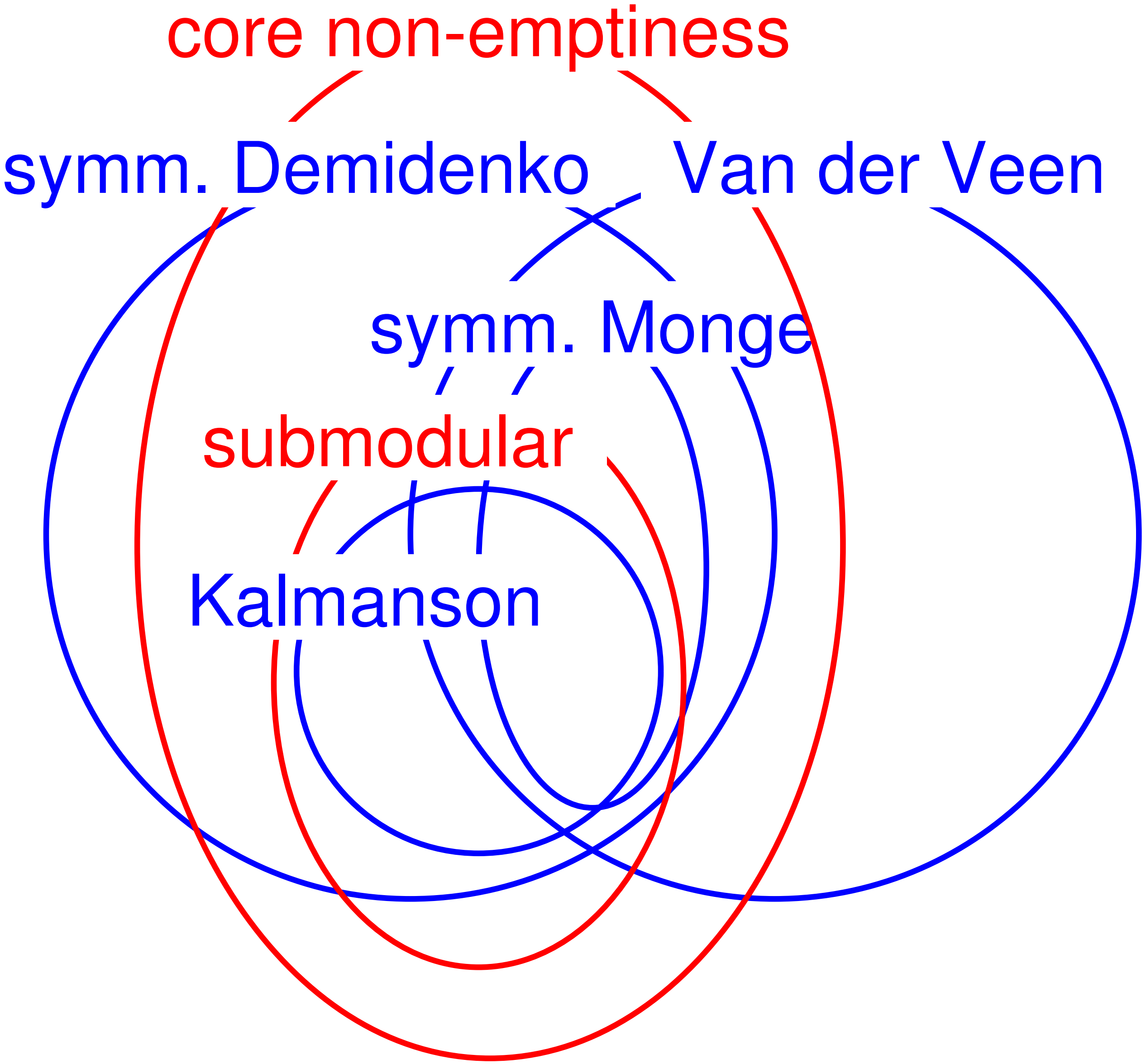
  $$c[i,j] + c[k,l] \le c[i,k] + c[j,l],$$
  $$c[i,l] + c[j,k] \le c[i,k] + c[j,l] \qquad (i < j < k < l)$$

- symm. Demidenko matrices

  $$c[i,j] + c[k,l] \le c[i,k] + c[j,l] \qquad (i < j < k < l)$$

- Van der Veen matrices

  $$c[i,j] + c[k,l] \le c[i,l] + c[j,k] \qquad (i < j < k < l)$$


core non-emptiness
symm. Demidenko
Van der Veen
symm. Monge
submodular
Kalmanson

- How about other solution concepts?

  (solution concepts $\approx$ criteria of fairness)
- How about asymmetric TSP?
- Can we obtain new well solvable cases from the core non-emptiness results for TSG?

  e.g.: facility location games (Goemans–Skutella, '00)
- How about other optimization problems?

Combinatorial optimization games
for which testing the core non-emptiness is intractable:

- Min coloring games (Deng–Nagamochi–Ibaraki, '99)
- Facility location games (Goemans–Skutella, '00)
- Traveling salesman games (Okamoto)

Combinatorial optimization games
with the non-empty cores:

- Assignment games (Shapley–Shubik, '72)
- Min-cost spanning tree games (Granot–Huberman, '81)
- Simple flow games (Kalai–Zemel, '82)

Combinatorial optimization games
with good characterizations of core non-emptiness:

- Min-cost base games on matroids (Nagamochi–Zeng–Kabutoya–Ibaraki, '97)
- Max matching games (Deng–Nagamochi–Ibaraki, '99)
- Min vertex cover games (Deng–Nagamochi–Ibaraki, '99)
- Min edge cover games (Deng–Nagamochi–Ibaraki, '99)
- Max independent set games (Deng–Nagamochi–Ibaraki, '99)
- Delivery games (Granot–Hamers–Tijs, '99)

Contribution to traveling salesman games

- It is $\mathcal{NP}$-hard to test the core non-emptiness of a traveling salesman game.
- The core of a traveling salesman game with the Monge property is always non-empty.

You can get my slides from

http://www.inf.ethz.ch/personal/okamotoy/.

- http://www.math.princeton.edu/tsp/
(World record of TSP, and more information on TSP)
- http://www.geographic.org/
(Map of Asia)