Some properties of the core on convex geometries

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Abstract

A game on a convex geometry was introduced by Bilbao as a model of partial cooperation. We investigate some properties of the core of a game on a convex geometry. First, we show that if a game is quasi-convex, then the core is stable. This result can be seen as an extension of a result by Shapley for traditional cooperative games. Secondly, we show the core on the class of balanced games on a convex geometry has a consistency property, called the reduced game property. Moreover, we axiomatize the core by means of consistency, as is analogous to a result by Peleg for traditional cooperative games.

Key words Consistency; Convex geometry; Cooperative game; Core; Stability

1 Introduction

The concept of the core forms a part of the main topics of cooperative game theory [11]. We consider a game on a convex geometry. A cooperative game on a convex geometry was introduced by Bilbao [1], and we can find some papers which discussed its properties, for example, axiomatization of the Shapley value [2], axiomatization of the Banzhaf value [3], relationship the core and the Weber set [4]. In this paper we investigate some properties of the core.

The organization of this paper is as follows. The next section is devoted to the definition of a game on a convex geometry. In Section 3, we show that the core of a quasi-convex game is the unique stable set, which is an extension of a result by Shapley [13] for traditional cooperative games. In Section 4, we show the core has a consistent property, called the reduced game property. Moreover, we axiomatize the core on the class of balanced games on an atomic convex geometry by means of consistency, as is analogous to a result by Peleg [12] for traditional cooperative games.

Now we briefly mention the notation in this paper. For sets A and B, we use $A \subset B$ for $A \subseteq B$ and $A \neq B$. The set of reals is denoted by $\mathbf{R}$. For a real vector x and a finite subset S of indices, we denote $\sum_{i \in S} x_i$ by $x(S)$. For the case in which $S = \emptyset$, set $x(S) = 0$.

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2 Games on convex geometries

Recall the definition of a convex geometry. Let $N = \{1, 2, \dots, n\}$ be a finite set. A *convex geometry* $\mathcal{L}$ on N is a family of subsets of N with the following properties:

$$\emptyset \in \mathcal{L}, N \in \mathcal{L}, \tag{1}$$

$$S, T \in \mathcal{L} \text{ imply } S \cap T \in \mathcal{L}, \tag{2}$$

$$S \in \mathcal{L} \setminus \{N\} \text{ implies } S \cup \{i\} \in \mathcal{L} \text{ for some } i \in N \setminus S. \tag{3}$$

We can replace the condition (3) with the following one: for any $S \in \mathcal{L}$, there exists a chain $\mathcal{C} : \emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_m = S$ such that $S_i \in \mathcal{L}$ and $|S_i| = i$ for $i = 0, \dots, m$, where $m = |S|$. For a convex geometry $\mathcal{L}$ on N, we call an element i of $S \in \mathcal{L}$ an *extreme point* of S if $S \setminus \{i\} \in \mathcal{L}$. The set of the extreme points of S is denoted by $\mathrm{ex}(S)$, called the *extreme set* of S. See [7, 9] for a further theory of convex geometries.

A *characteristic function* on a convex geometry $\mathcal{L}$ on N is a function $v : \mathcal{L} \to \mathbf{R}$ with $v(\emptyset) = 0$. A *cooperative game*, or a *game*, on a convex geometry $\mathcal{L}$ on N is a triple $(N, \mathcal{L}, v)$ where v is a characteristic function on $\mathcal{L}$. The *players* are the elements of N and the *coalitions* are the members of $\mathcal{L}$. We denote the set of all cooperative games on a convex geometry $\mathcal{L}$ by $\Gamma(\mathcal{L})$.

A cooperative game on a convex geometry is introduced by Bilbao [1] as a model of cooperation structures in which only certain coalitions are allowed to form. This includes some situations introduced formerly. Indeed, if $\mathcal{L} = 2^N$, then it is a game in which all the coalitions are allowed to form. Such games are called *traditional* in this paper. If $\mathcal{L}$ is defined via a partially ordered set $P = (N, \leq)$ as

$$\mathcal{L} = \{S \subseteq N : S \text{ is an order ideal of } P\}, \tag{4}$$

then $\mathcal{L}$ forms a convex geometry, which is a model introduced by Faigle–Kern [8]. Besides, let G be a connected graph with the vertex set N. Myerson [10] introduced a communication game. Define $\mathcal{L}$ as

$$\mathcal{L} = \{S \subseteq N : G[S] \text{ is connected}\}, \tag{5}$$

where $G[S]$ means the subgraph induced by S. If G is a block-graph, that is, every 2-connected component forms a complete graph, then $\mathcal{L}$ is a convex geometry.

3 Stability of the core

In this section, we introduce the core and investigate properties of the core.

Let $\mathcal{L}$ be a convex geometry on the set of players N, and $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ be a game on $\mathcal{L}$. The set of *pre-imputations* of the game $(N, \mathcal{L}, v)$ is

$$I^*(N, \mathcal{L}, v) = \left\{x \in \mathbf{R}^N : x(N) = v(N)\right\}, \tag{6}$$

and the set of *imputations* is

$$I(N, \mathcal{L}, v) = \{x \in I^*(N, \mathcal{L}, v) : x_i \geq v(\{i\}) \text{ for all } \{i\} \in \mathcal{L}\}. \tag{7}$$

We may say that a pre-imputation is *Pareto-optimal* or *efficient* and that an imputation is an *individually rational* pre-imputation. Note that the set of imputations can be empty and not necessarily bounded. Indeed, we have the following result. A convex geometry is *atomic* if $\{i\} \in \mathcal{L}$ for all $i \in N$.

Proposition 1 ([1]). *Let $\mathcal{L}$ be a convex geometry on N, and $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ be a game.*

1. *If $\mathcal{L}$ is not atomic, then $I(N, \mathcal{L}, v) \neq \emptyset$.*

2. *If $\mathcal{L}$ is atomic, then*

$$I(N, \mathcal{L}, v) \neq \emptyset \text{ if and only if } v(N) \geq \sum_{i \in N} v(\{i\}). \tag{8}$$

3. *$I(N, \mathcal{L}, v)$ is bounded if and only if $\mathcal{L}$ is atomic.*

Let $\Gamma_0(\mathcal{L}) \subseteq \Gamma(\mathcal{L})$ be a class of games. A *solution* on $\Gamma_0(\mathcal{L})$ is a function σ which maps a game $(N, \mathcal{L}, v) \in \Gamma_0(\mathcal{L})$ to a set $\sigma(N, \mathcal{L}, v) \subseteq I^*(N, \mathcal{L}, v)$ of pre-imputations. Various solution concepts are known: the Shapley value, the core, stable sets, the nucleolus, the kernel, etc. The *core* of a game $(N, \mathcal{L}, v)$ is defined as

$$\mathrm{Core}(N, \mathcal{L}, v) = \{x \in I^*(N, \mathcal{L}, v) : x(S) \geq v(S) \text{ for all } S \in \mathcal{L}\}. \tag{9}$$

We may also say that an imputation in the core is *coalitionally rational*. For the core, the next proposition is known.

Proposition 2 ([4]). *Let $\mathcal{L}$ be a convex geometry on N, and $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ be a game.*

1. *The core* $\mathrm{Core}(N, \mathcal{L}, v)$ *is either empty or a pointed polyhedron.*

2. *Moreover,* $\mathrm{Core}(N, \mathcal{L}, v)$ *is bounded if and only if $\mathcal{L}$ is atomic.*

A game with the non-empty core is called *balanced*.

We can consult [5, 11] for other solution concepts of traditional cooperative games.

Now, we investigate some properties of the core. First we show a relationship between the core and a stable set. Let $x, y \in I(N, \mathcal{L}, v)$ for a game $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$, and $S \in \mathcal{L}$. We say that y *dominates* x *via* S if

$$y(S) \leq v(S) \quad \text{and} \tag{10}$$

$$y_i > x_i \quad \text{for all } i \in S, \tag{11}$$

and denote it by $x \prec_S y$. A *stable set*, or a *von Neumann–Morgenstern solution* [14], of a game $(N, \mathcal{L}, v)$ is a set $K \subseteq I(N, \mathcal{L}, v)$ satisfying the following two conditions:

1. for all $x \in I(N, \mathcal{L}, v) \setminus K$, there exist $y \in K$ and $S \in \mathcal{L}$ such that $x \prec_S y$,

2. for all $x, y \in K$ and $S \in \mathcal{L}$, it holds that $x \not\prec_S y$ and $y \not\prec_S x$.

The first condition is called the *external stability*, while the second one is called the *internal stability*.

Remark 1. 1. *A stable set of a game is not necessarily unique and there exist some games which have no stable set.*

2. *If two collections of imputations are distinct stable sets, then they do not include each other.*

3. *If the core of a game is not empty and K is a stable set, then the core is included in K. In addition, if the core itself happens to be stable, then it is the unique stable set.*

A well-known fact on stable sets is that, if $\mathcal{L} = 2^N$ and a characteristic function v is convex, then the core is a unique stable set [13], where a characteristic function $v : 2^N \to \mathbf{R}$ is *convex* if we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \tag{12}$$

for all $S, T \in 2^N$. We also say that a traditional game is convex if its characteristic function is convex. In the rest of this section, we show that if a game on a convex geometry is quasi-convex, then the stable set is unique and coincides with the core.

A function $v : \mathcal{L} \to \mathbf{R}$ is *quasi-convex* if for all $S, T \in \mathcal{L}$ with $S \cup T \in \mathcal{L}$, we have

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T). \tag{13}$$

We also say that a game $(N, \mathcal{L}, v)$ is quasi-convex if the characteristic function v is quasi-convex. The quasi-convexity may be first defined in [4]. Obviously, if $\mathcal{L} = 2^N$, then a quasi-convex game is convex.

Let $\mathcal{L}$ be a convex geometry, $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ be a game on $\mathcal{L}$. Take a maximal chain of $\mathcal{L}$

$$\mathcal{C} : \emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset S_n = N, \tag{14}$$

which means there is no set S such that $S_{i-1} \subset S \subset S_i$ for each $i = 1, \ldots, n$. Then, the *marginal worth vector* $a^{\mathcal{C}} \in \mathbf{R}^N$ with respect to the maximal chain $\mathcal{C}$ in the game $(N, \mathcal{L}, v)$ is defined by

$$a_i^{\mathcal{C}} = v(S_{\mathcal{C}}(i)) - v(S_{\mathcal{C}}(i) \setminus \{i\}) \quad \text{for all } i \in N, \tag{15}$$

where $S_{\mathcal{C}}(i) = \bigcap\{S \in \mathcal{C} : i \in S\}$, that is, $S_{\mathcal{C}}(i)$ is the minimal set of $\mathcal{C}$ containing i. Note that we have $i \in \mathrm{ex}(S_{\mathcal{C}}(i))$ from the minimality of $S_{\mathcal{C}}(i)$.

Lemma 1 ([4, Theorem 5]). *A game $(N, \mathcal{L}, v)$ on a convex geometry $\mathcal{L}$ is quasi-convex if and only if $a^{\mathcal{C}}$ belongs to* $\mathrm{Core}(N, \mathcal{L}, v)$ *with respect to every maximal chain $\mathcal{C}$ of $\mathcal{L}$.*

This lemma implies that a quasi-convex game is balanced, that is, it has the non-empty core.

Now, we show the main result of this section.

Theorem 1. *Let $\mathcal{L}$ be a convex geometry on N, and $v : \mathcal{L} \to \mathbf{R}$ be quasi-convex. Then the game $(N, \mathcal{L}, v)$ has a unique stable set, which coincides with the core.*

Proof. First, we show that the core satisfies the internal stability. Suppose that $x, y \in \mathrm{Core}(N, \mathcal{L}, v)$ satisfy $x \prec_S y$ for some $S \in \mathcal{L}$. Namely, $y(S) \leq v(S)$ and $y_i > x_i$ for all $i \in S$. Then we have $x(S) < v(S)$, which contradicts the fact that x is in the core.

Next, we consider the external stability. Take an arbitrary $x \in I(N, \mathcal{L}, v) \setminus \mathrm{Core}(N, \mathcal{L}, v)$. Define $g : \mathcal{L} \setminus \{\emptyset\} \to \mathbf{R}$ as

$$g(S) = \frac{v(S) - x(S)}{|S|}. \tag{16}$$

Let S^* be a maximizer of g. Since $x \notin \mathrm{Core}(N, \mathcal{L}, v)$ and S^* is a maximizer of g, we have $g(S^*) > 0$. We pick up a maximal chain $\mathcal{C}$ of $\mathcal{L}$ such that $S^* \in \mathcal{C}$. Now, define y as

$$y_i = \begin{cases} x_i + g(S^*) & (i \in S^*), \\ a_i^{\mathcal{C}} & (i \in N \setminus S^*). \end{cases} \tag{17}$$

Remark that y is an imputation of $(N, \mathcal{L}, v)$. (In fact, it shall be shown later that y belongs to the core.) Here, we have $y_i > x_i$ for all $i \in S^*$. Moreover,

$$y(S^*) = x(S^*) + |S^*| g(S^*) = v(S^*). \tag{18}$$

Therefore, we have $x \prec_{S^*} y$.

It remains to show that $y \in \mathrm{Core}(N, \mathcal{L}, v)$. First,

$$y(N) = v(S^*) + a^{\mathcal{C}}(N \setminus S^*) = a^{\mathcal{C}}(S^*) + a^{\mathcal{C}}(N \setminus S^*) = a^{\mathcal{C}}(N) = v(N). \tag{19}$$

Next, take an arbitrary $T \in \mathcal{L}$. Then,

$$\begin{aligned} y(T) &= y(S^* \cap T) + y(T \setminus S^*) &(20)\\ &= x(S^* \cap T) + |S^* \cap T| g(S^*) + a^{\mathcal{C}}(T \setminus S^*) &(21)\\ &\geq x(S^* \cap T) + |S^* \cap T| g(S^* \cap T) + a^{\mathcal{C}}(T \setminus S^*) &(22)\\ &= v(S^* \cap T) + a^{\mathcal{C}}(T \setminus S^*). &(23) \end{aligned}$$

If $T \setminus S^* = \emptyset$, then $a^{\mathcal{C}}(T \setminus S^*) = 0$ and $S^* \cap T = T$. Therefore, we have $y(T) \geq v(T)$.

Let us consider that $T \setminus S^* \neq \emptyset$, and we write $T \setminus S^* = \{i_1, i_2, \ldots, i_m\}$ where the players are ordered in the order of incorporation in the chain $\mathcal{C}$, i.e., $S_{\mathcal{C}}(i_1) \subset S_{\mathcal{C}}(i_2) \subset \cdots \subset S_{\mathcal{C}}(i_m)$. Notice that $S^* \subseteq S_{\mathcal{C}}(i_1)$. We denote by $S_0 = S^* \cap T$ and by $S_j = S_0 \cup \{i_1, i_2, \ldots, i_j\}$ for all $j \in \{1, \ldots, m\}$. Then we have $S_j = S_{\mathcal{C}}(i_j) \cap T \in \mathcal{L}$ and $i_j \in \mathrm{ex}(S_j)$. Since $S_{\mathcal{C}}(i_j) = (S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \cup S_j$, $S_{j-1} = (S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \cap S_j$, and v is quasi-convex, we have

$$a^{\mathcal{C}}_{i_j} = v(S_{\mathcal{C}}(i_j)) - v(S_{\mathcal{C}}(i_j) \setminus \{i_j\}) \geq v(S_j) - v(S_{j-1}), \tag{24}$$

and we also have

$$a^{\mathcal{C}}(T \setminus S^*) \geq \sum_{j=1}^{m} (v(S_j) - v(S_{j-1})) = v(T) - v(S^* \cap T). \tag{25}$$

Hence,

$$v(S^* \cap T) + a^{\mathcal{C}}(T \setminus S^*) \geq v(S^* \cap T) + v(T) - v(S^* \cap T) = v(T). \tag{26}$$

It concludes that $y \in \mathrm{Core}(N, \mathcal{L}, v)$. Therefore, $\mathrm{Core}(N, \mathcal{L}, v)$ is a stable set, and from Remark 1 $\mathrm{Core}(N, \mathcal{L}, v)$ is the unique stable set. □

4 An axiomatization of the core of games on convex geometries

Now we investigate other properties of the core. Let $\mathcal{L}$ be a convex geometry on N and $S \in \mathcal{L}$. The *restriction* of $\mathcal{L}$ to S is

$$\mathcal{L}|_S = \{T \in \mathcal{L} : T \subseteq S\}. \tag{27}$$

Note that $\mathcal{L}|_S$ is a convex geometry on S. Let $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ be a game, $S \in \mathcal{L} \setminus \{\emptyset\}$ and $x \in I^*(N, \mathcal{L}, v)$. The *reduced game* with respect to S and x is a game $(S, \mathcal{L}|_S, v^x_S) \in \Gamma(\mathcal{L}|_S)$ where

$$v^x_S(T) = \begin{cases} 0 & (T = \emptyset), \\ v(N) - x(N \setminus T) & (T = S), \\ \max\{v(T \cup R) - x(R) : R \subseteq N \setminus S \text{ with } T \cup R \in \mathcal{L}\} & (T \in \mathcal{L}|_S \setminus \{\emptyset, S\}). \end{cases} \tag{28}$$

Note that $v(N) - x(N \setminus T) = x(T)$ and that $v^x_N = v$.

Let $\Gamma_0(\mathcal{L}) \subseteq \Gamma(\mathcal{L})$ and σ be a solution on $\Gamma_0(\mathcal{L})$. We say that σ satisfies the *reduced game property* or (RGP) if $x \in \sigma(N, \mathcal{L}, v)$ implies $x|_S \in \sigma(S, \mathcal{L}|_S, v^x_S)$ for any $(N, \mathcal{L}, v) \in \Gamma_0(\mathcal{L})$ and $S \in \mathcal{L} \setminus \{\emptyset\}$, where $x|_S$ means the restriction of x to a coalition S. We can see that the reduced game property (RGP) represents a consistency property of the game. We now show that the core satisfies (RGP).

Lemma 2. *The core on* $\Gamma(\mathcal{L})$ *satisfies the reduced game property* (RGP) *for any convex geometry* $\mathcal{L}$.

Proof. Let $S \in \mathcal{L} \setminus \{\emptyset\}$ and $x \in \mathrm{Core}(N, \mathcal{L}, v)$. Take an arbitrary $T \in \mathcal{L}|_S$. If $T = S$, then we have $v^x_S(T) = v(N) - x(N \setminus T) = x(T) = x(S)$. Otherwise,

$$\begin{aligned}
&v^x_S(T) - x(T) \\
&\quad = \max\{v(T \cup R) - x(R) : R \subseteq N \setminus S \text{ with } T \cup R \in \mathcal{L}\} - x(T) && (29)\\
&\quad = \max\{v(T \cup R) - x(T \cup R) : R \subseteq N \setminus S \text{ with } T \cup R \in \mathcal{L}\} && (30)\\
&\quad \leq 0. && (31)
\end{aligned}$$

The equality (30) is due to $(N \setminus S) \cap T = \emptyset$, and the inequality (31) is derived from the fact that $x \in \mathrm{Core}(N, \mathcal{L}, v)$. Hence, $x(T) \geq v^x_S(T)$. □

Also, the core has other properties described as follows. A solution σ on $\Gamma_0(\mathcal{L})$ is said to satisfy the *superadditivity property* (SAP) if, for all $(N, \mathcal{L}, v), (N, \mathcal{L}, w) \in \Gamma_0(\mathcal{L})$, it holds that

$$\sigma(N, \mathcal{L}, v) + \sigma(N, \mathcal{L}, w) \subseteq \sigma(N, \mathcal{L}, v + w), \tag{32}$$

where $(v + w)(S) = v(S) + w(S)$ for any $S \in \mathcal{L}$ and $\sigma(N, \mathcal{L}, v) + \sigma(N, \mathcal{L}, w) = \{x + y : x \in \sigma(N, \mathcal{L}, v) \text{ and } y \in \sigma(N, \mathcal{L}, w)\}$.

Lemma 3. *The core on* $\Gamma(\mathcal{L})$ *satisfies the superadditivity property* (SAP) *for any convex geometry* $\mathcal{L}$.

Proof. Let $z \in \mathrm{Core}(N, \mathcal{L}, v) + \mathrm{Core}(N, \mathcal{L}, w)$. Then z can be represented as $z = x + y$ for some $x \in \mathrm{Core}(N, \mathcal{L}, v)$ and some $y \in \mathrm{Core}(N, \mathcal{L}, w)$. Therefore, for all $S \in \mathcal{L}$, we have $x(S) \geq v(S)$ and $y(S) \geq w(S)$. This implies that $x(S) + y(S) \geq v(S) + w(S)$. Hence we have $z(S) \geq (v + w)(S)$. □

We say that a solution σ on $\Gamma_0(\mathcal{L})$ satisfies the *individual rationality* (IR) if $x \in \sigma(N, \mathcal{L}, v)$ and $\{i\} \in \mathcal{L}$ imply $x_i \geq v(\{i\})$ for any game $(N, \mathcal{L}, v) \in \Gamma_0$, We also say that a solution σ on $\Gamma_0(\mathcal{L})$ satisfies the *nonemptiness property* (NE) if $\sigma(N, \mathcal{L}, v) \neq \emptyset$ for any game $(N, \mathcal{L}, v) \in \Gamma_0(\mathcal{L})$. Obviously, the core on $\Gamma(\mathcal{L})$ satisfies (IR). Moreover, the core on the class of balanced games satisfies (NE). So we have shown that the core on the class of balanced games satisfies the four properties: (NE), (IR), (SAP), (RGP). In fact, these four properties characterize the core on the class of balanced games on an atomic convex geometry.

Theorem 2. *Let* $\mathcal{L}$ *be an atomic convex geometry on* N. *Then, the core is the unique solution on the class of balanced games on* $\mathcal{L}$ *which satisfies the properties* (NE), (IR), (SAP) *and* (RGP).

Peleg [12] showed Theorem 2 for traditional cooperative games. So this theorem is an extension of a result by Peleg [12].

The rest of this section is devoted to a proof of Theorem 2. For our proof, the converse reduced game property is useful. Let $\Gamma_0(\mathcal{L}) \subseteq \Gamma(\mathcal{L})$ and σ a solution on $\Gamma_0(\mathcal{L})$. A solution σ on $\Gamma_0(\mathcal{L})$ is said to satisfy the *converse reduced game property* (CRGP) if: for any $(N, \mathcal{L}, v) \in \Gamma_0(\mathcal{L})$ if $x \in I^*(N, \mathcal{L}, v)$ satisfies $x|_T \in \sigma(T, \mathcal{L}|_T, v^x_T)$ for all $T \in \mathcal{L}$ with $|T| = 2$, then $x \in \sigma(N, \mathcal{L}, v)$. Peleg [12] showed that the core on the class of traditional cooperative games, that is, $\mathcal{L} = 2^N$, satisfies (CRGP). We show that the core on the class of games on an atomic convex geometry $\mathcal{L} \subseteq 2^N$ also satisfies (CRGP).

To prove this, we need a lemma.

Lemma 4. *Let* $\mathcal{L}$ *be an atomic convex geometry on the players* N, $T \in \mathcal{L} \setminus \{\emptyset, N\}$, *and* $j \in N \setminus T$ *such that* $T \cup \{j\} \in \mathcal{L}$. *Then, there exists a player* $i \in T$ *such that* $\{i, j\} \in \mathcal{L}$.

Proof. From the assumption, we have $T \cup \{j\} \in \mathcal{L}$. Also, since $\mathcal{L}$ is atomic, we have $\{j\} \in \mathcal{L}$. Then, there exists a chain from $\{j\}$ to $T \cup \{j\}$. In this chain, there exists a set $\{i, j\} \in \mathcal{L}$ such that $\{j\} \subset \{i, j\} \subset T \cup \{j\}$. Therefore $i \in T$, which concludes the proof. $\square$

By using Lemma 4, we can see that the core on $\Gamma(\mathcal{L})$ satisfies (CRGP) when $\mathcal{L}$ is atomic.

Lemma 5. *The core on the class of games on an atomic convex geometry satisfies the converse reduced game property* (CRGP).

Proof. If $|N| \leq 2$, then the statement obviously holds. Assume that $|N| \geq 3$. Let $x \in I^*(N, \mathcal{L}, v)$ satisfy $x|_T \in \text{Core}(T, \mathcal{L}|_T, v_T^x)$ for all $T \in \mathcal{L}$ with $|T| = 2$, and $S \in \mathcal{L} \setminus \{\emptyset, N\}$. By Lemma 4, there exist two players $i \in S$ and $j \in N \setminus S$ such that $\{i, j\} \in \mathcal{L}$. Since $\mathcal{L}$ is atomic and $x|_{\{i,j\}} \in \text{Core}(\{i, j\}, \mathcal{L}|_{\{i,j\}}, v_{\{i,j\}}^x)$,

$$v_{\{i,j\}}^x(\{i\}) - x_i \leq 0. \tag{33}$$

Then,

$$\begin{aligned} &v_{\{i,j\}}^x(\{i\}) - x_i \\ &\quad = \max\{v(\{i\} \cup R) - x(R) : R \subseteq N \setminus \{i, j\} \text{ with } R \cup \{i\} \in \mathcal{L}\} - x_i & (34)\\ &\quad = \max\{v(\{i\} \cup R) - x(\{i\} \cup R) : R \subseteq N \setminus \{i, j\} \text{ with } R \cup \{i\} \in \mathcal{L}\} & (35)\\ &\quad \geq v(S) - x(S). & (36) \end{aligned}$$

Combining the inequalities (33) and (36), we have $x(S) \geq v(S)$. $\square$

The next is an immediate consequence of Lemma 5.

Corollary 1. *Let $\mathcal{L}$ be an atomic convex geometry on N, $(N, \mathcal{L}, v) \in \Gamma(\mathcal{L})$ and $x \in I^*(N, \mathcal{L}, v)$. If $x|_S \in \text{Core}(S, \mathcal{L}|_S, v_S^x)$ for all $S \in \mathcal{L} \setminus \{\emptyset, N\}$, then $x \in \text{Core}(N, \mathcal{L}, v)$.*

The following is the key lemma for our proof of Theorem 2.

Lemma 6. *Let $\mathcal{L}$ be an atomic convex geometry on N and σ a solution on $\Gamma_0(\mathcal{L})$ which satisfies* (IR) *and* (RGP)*, then we have $\sigma(N, \mathcal{L}, v) \subseteq \text{Core}(N, \mathcal{L}, v)$. Particularly, if σ satisfies* (NE)*,* (IR) *and* (RGP) *and the core of a game $(N, \mathcal{L}, v) \in \Gamma_0(\mathcal{L})$ consists of a unique point, then $\sigma(N, \mathcal{L}, v) = \text{Core}(N, \mathcal{L}, v)$.*

Proof. We prove it by induction on $n = |N|$, namely, the number of the players. If $n = 1$, then the statement holds from (IR).

Assume that the statement is true for all $n \leq k$ and let $n = k + 1$. Let $\mathcal{L}$ be a convex geometry on N and $x \in \sigma(N, \mathcal{L}, v)$. By (RGP), $x \in \sigma(N, \mathcal{L}, v)$ implies $x|_S \in \sigma(S, \mathcal{L}|_S, v_S^x)$ for all $S \in \mathcal{L}$. From the induction hypothesis, we have $x|_S \in \text{Core}(S, \mathcal{L}|_S, v_S^x)$ for all $S \in \mathcal{L} \setminus \{N\}$. By Corollary 1, we conclude that $x \in \text{Core}(N, \mathcal{L}, v)$. $\square$

Now it is time to prove Theorem 2.

Proof of Theorem 2. We have shown that the core on the class of balanced games satisfies these four properties. Now we show the uniqueness. Let σ be a solution on the class of balanced games on the atomic convex geometry $\mathcal{L}$ which satisfies (NE), (IR), (RGP) and (SAP) and let $(N, \mathcal{L}, v)$ be a balanced game. We distinguish two cases. If $|N| \in \{1, 2\}$ then $\sigma(N, \mathcal{L}, v) = \text{Core}(N, \mathcal{L}, v)$ from Peleg [12] since an atomic convex geometry must be 2^N.

Now we consider the case in which $|N| \geq 3$. Pick up an arbitrary balanced game $(N, \mathcal{L}, v)$ and fix $x \in \text{Core}(N, \mathcal{L}, v)$. Then, define $w : \mathcal{L} \to \mathbf{R}$ by

$$w(S) = \begin{cases} v(\{i\}) & (S = \{i\}), \\ x(S) & (\text{otherwise}). \end{cases} \tag{37}$$

We have $\mathrm{Core}(N, \mathcal{L}, w) = \{x\}$ since the core of an atomic convex geometry is bounded. By Lemma 6, we have $\sigma(N, \mathcal{L}, w) = \{x\}$. Let $u = v - w$. Here, we have $u(N) = 0$, $u(\{i\}) = 0$ for all $\{i\} \in \mathcal{L}$ and $u(S) \leq 0$ for all $S \in \mathcal{L} \setminus \{N\}$. Therefore, $\mathrm{Core}(N, \mathcal{L}, u) = \{0\}$. By Lemma 6, we also have $\sigma(N, \mathcal{L}, u) = \{0\}$. Since σ satisfies (SAP), $\sigma(N, \mathcal{L}, v) \supseteq \sigma(N, \mathcal{L}, u) + \sigma(N, \mathcal{L}, w) = \{x\}$. This means $\mathrm{Core}(N, \mathcal{L}, v) \subseteq \sigma(N, \mathcal{L}, v)$. On the contrary, we have $\sigma(N, \mathcal{L}, v) \subseteq \mathrm{Core}(N, \mathcal{L}, v)$ from Lemma 6. Hence, $\sigma(N, \mathcal{L}, v) = \mathrm{Core}(N, \mathcal{L}, v)$. □

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